MMEF – Final exam

Ex 1 :

1 There are 3 possible outcomes:

A : working in the financial industry

B : ______ movie industry

C : ______ car industry.

The decision maker compare 3 acts:

Choose school 1 : $(0, 0, 1)$ prob that C happens

probability that A happens   prob that B happens

Choose school 2 : $(1, 0, 0)$

Choose school 3 : $(0.9, 0.1, 0)$

We have:

$$\begin{cases} (0,0,1) \sim (0.9, 0.1, 0) \\ (1,0,0) \precsim (0,0,1) \end{cases} \quad (*)$$

We assume that the decision maker follows expected utility model.

We denote : $A := (1, 0, 0)$

$B := (0, 1, 0)$

$C := (0, 0, 1)$

$$(*) \iff \begin{cases} C \sim 0{,}9 . A + 0.1 B \\ A \precsim C \end{cases}$$

By independence axiom, we have:

$$0.9 A + 0.1 C \precsim 0.9 C + 0.1 C$$

$$\iff 0.9 A + 0.1 C \precsim C$$

By transitivity,

$$0.9 A + 0.1 C \precsim 0.9 A + 0.1 B$$

$$\Rightarrow C \precsim B$$

$$\Rightarrow \boxed{A \precsim C \precsim B}$$

We can normalize : $u(A) = 0$ and $u(B) = 1$

$u(C) = U(0.9A + 0.1B)$

where u is the Bernoulli utility function

and U is the "total" utility function

$\Rightarrow u(C) = 0.9\,u(A) + 0.1\,u(B)$

$= 0.1$

$$\Rightarrow \begin{cases} U(\text{school } 1) = u(C) = 0.1 \\ U(\text{school } 2) = u(A) = 0 \\ U(\text{school } 3) = 0.1 \end{cases}$$

2) School 4 : $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

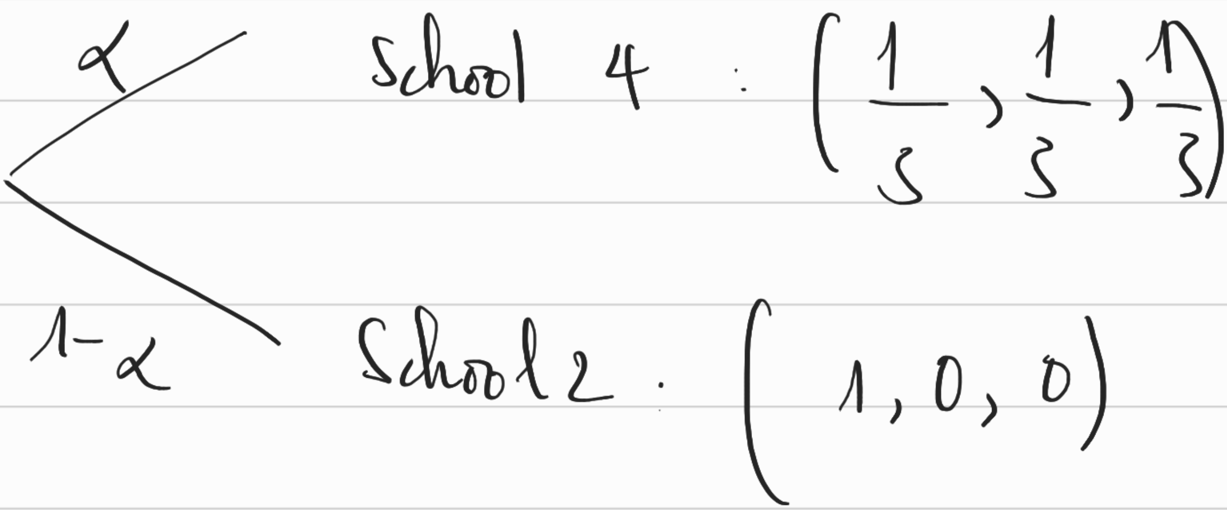
$\Rightarrow U(\text{School } 4) = \frac{1}{3}\left(u(A) + u(B) + u(C)\right)$

$$= \frac{1}{3}(0 + 1 + 0.1) = \frac{1{,}1}{3}$$

$\Rightarrow$ The new school is the best choice among 4 possible acts

3)

The new lottery:



$\Rightarrow$ The new lottery is indifferent to

$$\left(\frac{1}{3}\alpha + 1 - \alpha,\ \frac{1}{3}\alpha,\ \frac{1}{3}\alpha\right)$$

$\Rightarrow$ U (the new lottery)

$$= \left(1 - \frac{2}{3}\alpha\right) u(A) + \frac{1}{3}\alpha\, u(B)$$

$$+ \frac{1}{3}\alpha\, u(C)$$

$$= \frac{1}{3}\alpha + \frac{1}{3}\alpha \cdot 0{,}1$$

$U(\text{school } 1) = 0,1$

School 1 $\succsim$ the new lottery

$\Leftrightarrow \quad 0,1 \geq \frac{1}{3}\alpha + \frac{1}{3}\alpha \, 0,1$

$\Leftrightarrow \quad 0,1 \geq \frac{1}{3}\alpha \cdot 1,1$

$\Leftrightarrow \quad \frac{3}{11} \geq \alpha$

$\Rightarrow \quad \alpha \in \left[0, \frac{3}{11}\right]$

Ex 2 :

1) - If the firm uses the first technology, the highest quantity of output is: $\sqrt{0,5}$

- If the firm uses the second technology, the highest quantity of output is: $0,5$

$\Rightarrow$ The firm should use the 1^{st} technology to get higher maximum quantity of output

2)

The highest quantity of output

$$= \begin{cases} \sqrt{2} & \text{if using } 1^{st} \text{ technology} \\ 2 & \text{if using } 2^{nd} \text{ technology} \end{cases}$$

$\Rightarrow$ Should use the 2^{nd} technology

3) Let $q(z)$ be the maximal quantity of output the firm can produce using z unit of input.

Assume that the firm spend $0 \le z_1 \le z$ on the first technology and $(z - z_1)$ on the second technology.

$$\Rightarrow \quad q(z) = \max_{z_1 \in [0, z]} \{ f_1(z_1) + f_2(z - z_1) \}$$

$$\Leftrightarrow q(z) = \max_{z_1 \in [0, z]} \{ \sqrt{z_1} + z - z_1 \}$$

let $f(z_1) = \sqrt{z_1} + z - z_1$, $z_1 \in [0,z]$

$$f'(z_1) = \frac{1}{2\sqrt{z_1}} - 1$$

$$f'(z_1) = 0 \iff 2\sqrt{z_1} = 1 \iff z_1 = \frac{1}{4}.$$

*) If $z \leq \frac{1}{4} \Rightarrow z_1 \leq \frac{1}{4}$

$\Rightarrow f'(z_1) \geq 0$

$\Rightarrow$ f is increasing.

$\Rightarrow f(z_1) \leq f(z) = \sqrt{z}$

$\Rightarrow g(z) = \sqrt{z}$

*) If $z > \frac{1}{4}$

z_1	0		$1/4$		z
		$+$		$-$	
$f(z_1)$		$\nearrow$		$\searrow$	

$\Rightarrow f(z_1) \leq \sqrt{\frac{1}{4} + z - \frac{1}{4}} = z + \frac{1}{4}$

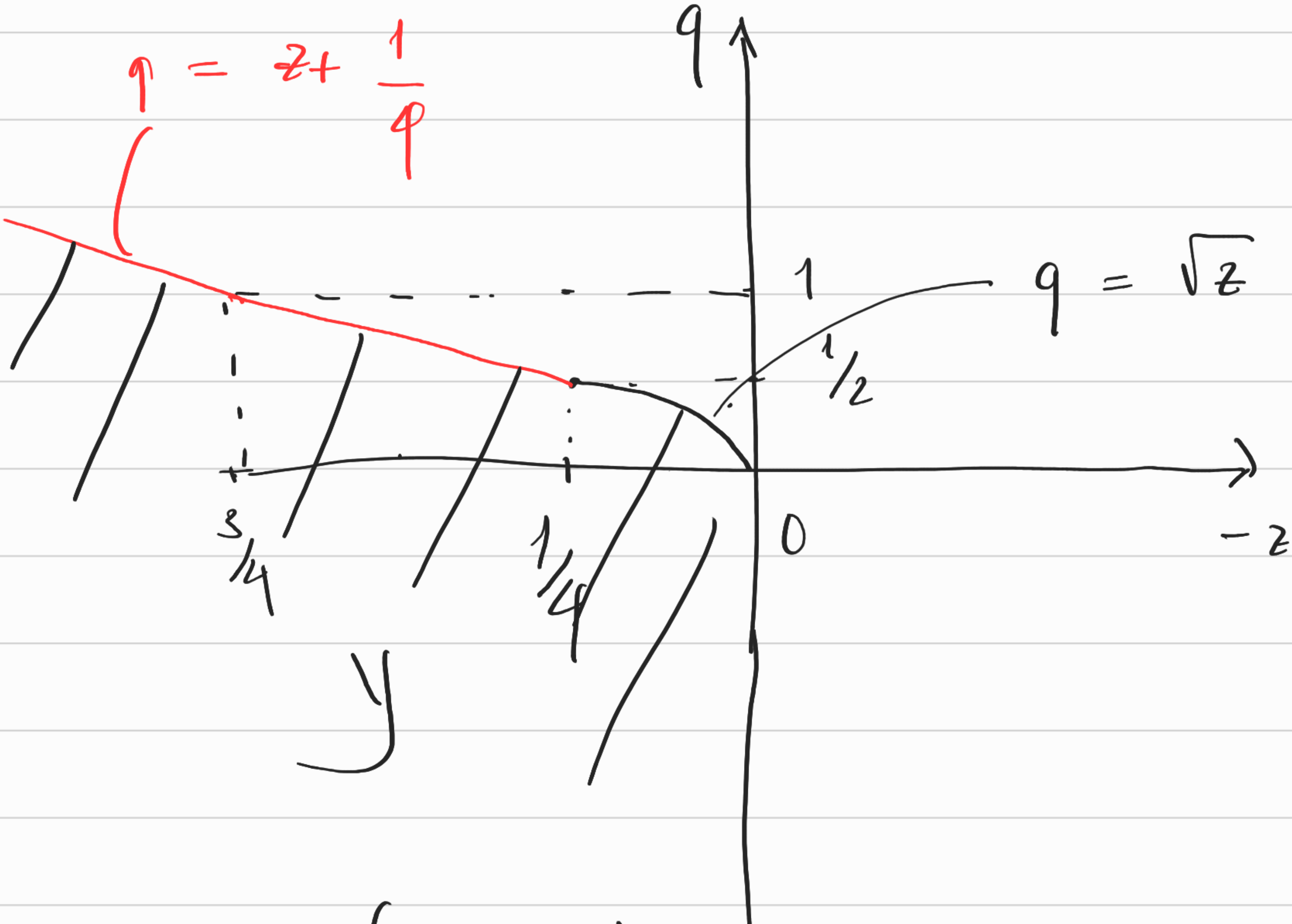
$\Rightarrow q(z) = z + \frac{1}{4}.$

Finally, we have:

$$q(z) = \begin{cases} \sqrt{z} & \text{if } z \leq \frac{1}{4} \\ z + \frac{1}{4} & \text{if } z > \frac{1}{4} \end{cases}$$

$\Rightarrow \mathcal{Y} = \{ (-z, q) \mid z \geq 0 \text{ and } q \leq q(z) \}$

$\Leftrightarrow \mathcal{Y} = \{ (-z, q) \mid z \in [0, \frac{1}{4}] \text{ and } q \leq \sqrt{z} \} \cup \{ (-z, q) \mid z > \frac{1}{4} \text{ and } q \leq z + \frac{1}{4} \}$



4) Let $(-z, q) \in Y$ and $(\alpha_1, \alpha_2) \in \mathbb{R}^2_+$

We have to prove that

$$(-(z+\alpha_1), q-\alpha_2) \in Y$$

$$\Leftrightarrow \begin{cases} z+\alpha_1 \geqslant 0 & (1) \\ q-\alpha_2 \leq q(z+\alpha_1) & (2) \end{cases}$$

(1) is trivially true

Now, I will prove ②.
It is sufficient to prove that $q(\cdot)$ is an increasing function of z.
let $0 \leq z_1 \leq z_2$.

Recall:

$$q(z) = \begin{cases} \sqrt{z} & \text{if } z \leq \frac{1}{4} \\ z + \frac{1}{4} & \text{if } z > \frac{1}{4} \end{cases}$$

If $0 \leq z_1 \leq z_2 \leq \frac{1}{4}$.

$\Rightarrow q(z_1) \leq q(z_2)$

If $0 \leq z_1 \leq \frac{1}{4} \leq z_2$

$\Rightarrow q(z_1) \leq \frac{1}{2} \leq q(z_2)$

If $\frac{1}{4} \leq z_1 \leq z_2$

$\Rightarrow q(z_1) \leq q(z_2)$

In all cases, $q(z_1) \leq q(z_2)$

$\Rightarrow q(\cdot)$ is increasing.

$$\Rightarrow \quad q - \alpha_2 \leq q \leq q(z)$$
$$\leq q(z + \alpha_1)$$

$\Rightarrow$ (2) is true

So y is free disposal.

Ex 3 :

1. $\mathcal{Y} = \{(-z, q) \mid z \geq 0$ and $q \leq f(z)\}$

$\Leftrightarrow \mathcal{Y} = \{(-z, q) \mid z \geq 0$ and $q \leq \alpha\sqrt{z}\}$

2)
- The supply function $s : \mathbb{R}^2_{++} \longrightarrow \mathcal{Y}$

$p \longrightarrow s(p) \in \mathcal{Y}$

$s(p)$ is the solution of the following problem

$$\underset{(-z,q) \in \mathcal{Y}}{\text{Max}} \; p \cdot (-z, q)$$

- The profit function : $\pi : \mathbb{R}_{++} \longrightarrow \mathbb{R}$

$p \longrightarrow \pi(p)$

$$\pi(p) = \underset{(-z,q) \in \mathcal{Y}}{\text{Max}} \; p \cdot (-z, q)$$

If $s(p)$ is single-valued, we can write :

$$\pi(p) = p \cdot s(p)$$

3) (p^*, x^*, y^*) is a competitive equilibrium of the economy iff :

(i) $p^* . y^* \geq p^* . y \quad \forall y \in Y$

(ii) x^* solves the following problem :

$$\text{Max } u(x)$$
$$\text{st : } x \geq 0$$
$$p^* . x \leq p^* . e + p^* . y^*$$

(iii) Market clearing condition :

$$x^* = e + y^*$$

4) let $(p, 1)$ is a price vector.

(PMP)
$$\text{Max } q - p.z.$$
$$\text{st : } z \geq 0$$
$$q \leq \alpha \sqrt{z}$$

We have :

$$q - p.z \leq \alpha\sqrt{z} - p.z$$

Let $g(z) = \alpha\sqrt{z} - p.z.$, $z \in [0, +\infty)$

$$g'(z) = \frac{\alpha}{2\sqrt{z}} - p$$

$$g'(z) = 0 \iff z = \left(\frac{\alpha}{2p}\right)^2$$

z	0		$\left(\alpha/2p\right)^2$		$+\infty$
$g'(z)$		$+$	0	$-$	
$g(z)$		$\nearrow$		$\searrow$	

$$\Rightarrow \quad g(z) \le g\left(\left(\frac{\alpha}{2p}\right)^2\right)$$

$$\iff \quad g(z) \le \alpha \cdot \frac{\alpha}{2p} - p \cdot \left(\frac{\alpha}{2p}\right)^2$$

$$\iff \quad g(z) \le \frac{\alpha^2}{4p}$$

$$\Rightarrow \quad q - p \cdot z \le \frac{\alpha^2}{4p}$$

"=" happens when $q = \alpha\sqrt{z}$

and $z = \left(\frac{\alpha}{2p}\right)^2$

$$\rightarrow s(p) = \left\{\left(-\left(\frac{\alpha}{2p}\right)^2, \frac{\alpha^2}{2p}\right)\right\}$$

and $$\pi(p) = \frac{\alpha^2}{4p}$$

You can solve it by using KKT.

5)

$$(UMP) \quad \text{Max} \quad x_1 \cdot x_2$$
$$\text{st}: \quad x_1 \geq 0, \; x_2 \geq 0$$
$$p x_1 + x_2 \leq p \cdot 12 + \pi(p)$$

$$\Leftrightarrow \quad \text{Max} \quad x_1 \cdot x_2$$
$$\text{st}: \quad x_1 \geq 0, \; x_2 \geq 0$$
$$p x_1 + x_2 \leq 12p + \frac{\alpha^2}{4p}$$

Let $x^* = (x_1^*, x_2^*)$ be a solution of (UMP).

If $x_1^* = 0$ or $x_2^* = 0$ then
$u(x^*) = 0$
But we can choose
$x = (\varepsilon, \varepsilon)$
such that $\varepsilon > 0$ and
$$p\varepsilon + \varepsilon \leq 12p + \frac{\alpha^2}{4p}$$
In particular, $\varepsilon \in \left(0, \frac{1}{p+1}\left(12p + \frac{\alpha^2}{4p}\right)\right]$

$\Rightarrow 0 = u(x^*) < u(x) = \varepsilon^2$
(Contradictory !)

$\Rightarrow \boxed{x^* \gg 0}$

KKT:

$$\nabla u(x_1^*, x_2^*) = \lambda \cdot (p, 1)$$

$$\Leftrightarrow \begin{cases} x_2^* = \lambda p \\ x_1^* = \lambda \end{cases} \Rightarrow x_2^* = p x_1^*$$

We also have:

$$p x_1^* + x_2^* = 12p + \frac{\alpha^2}{4p}$$

$$\Leftrightarrow 2 p x_1^* = 12p + \frac{\alpha^2}{4p}$$

$$\Rightarrow x_1^* = 6 + \frac{\alpha^2}{8p^2}$$

$$\Rightarrow x_2^* = 6p + \frac{\alpha^2}{8p}$$

b) Market clearing condition:

$$x_1^* = e_1 + y_1^*$$

$$\Leftrightarrow 6 + \frac{\alpha^2}{8p^2} = 12 - \frac{\alpha^2}{4p^2}$$

$$\Leftrightarrow \frac{3\alpha^2}{8p^2} = 6$$

$\Leftrightarrow \quad p = \dfrac{\alpha}{4}$

$\Rightarrow \quad x^* = \left(6 + \dfrac{\alpha^2}{8 \cdot \frac{\alpha^2}{16}} , 6 \cdot \dfrac{\alpha}{4} + \dfrac{\alpha^2}{8 \cdot \frac{\alpha}{4}} \right)$

$\Leftrightarrow \quad x^* = \left(8 , 2\alpha \right)$

$y^* = \left(- \dfrac{\alpha^2}{4 \cdot \frac{\alpha^2}{16}} , \dfrac{\alpha^2}{2 \cdot \frac{\alpha}{4}} \right)$

$\Leftrightarrow \quad y^* = \left(-4 , 2\alpha \right)$

$$\Rightarrow \quad \Big(\underbrace{\tfrac{\alpha}{4}}_{p^*} , \underbrace{(8, 2\alpha)}_{x^*} , \underbrace{(-4, 2\alpha)}_{y^*} \Big)$$

is the unique C. E of the economy

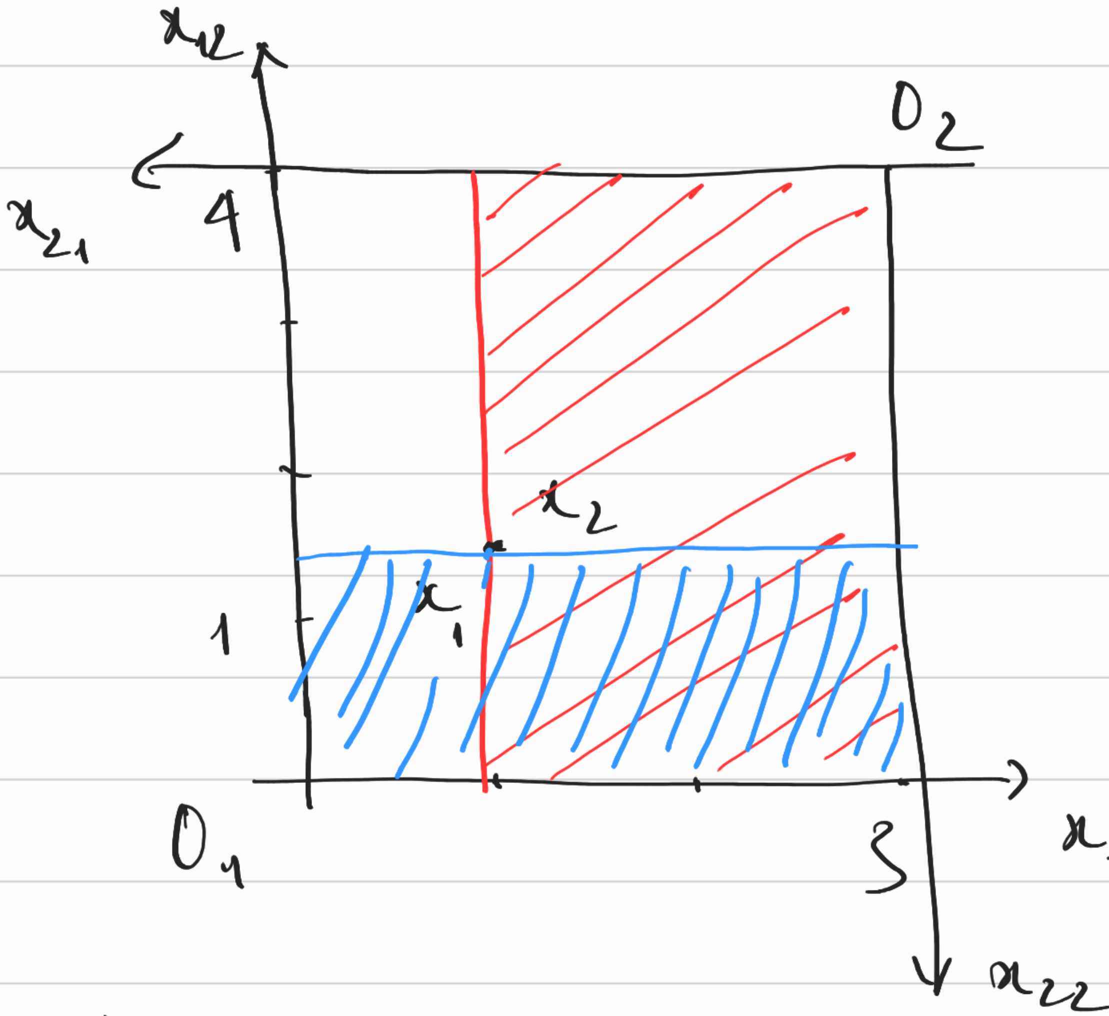
7) $x_1^* = 8$ does not depend on α.

$u(x^*) = 8 \cdot 2\alpha = 16\alpha$

It is a increasing function of α

Ex 4 :

1



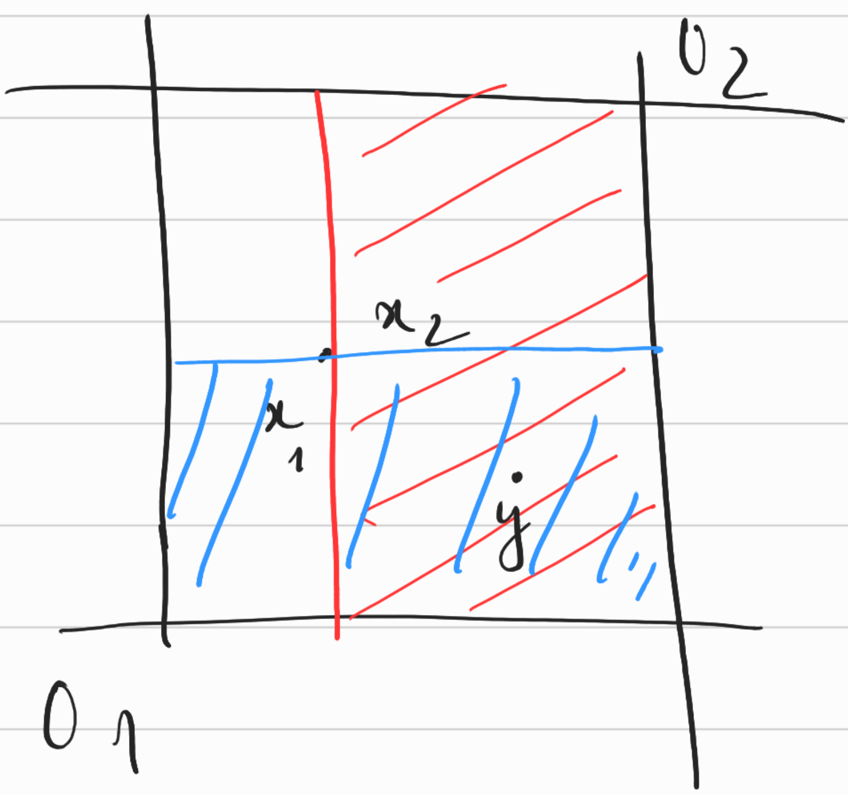
Consider $x \in$ E.B.

- the red curve is the indifferent curve at x of consumer 1
- the red area is the upper contour set at x of consumer 1.

- the blue cure is the indifferent curve at x of consumer 2.
- the blue area is the upper contour set at x of consumer 2.

2) See the definition again (I did it)

3)
- Consider $x \in$ interior of E.B

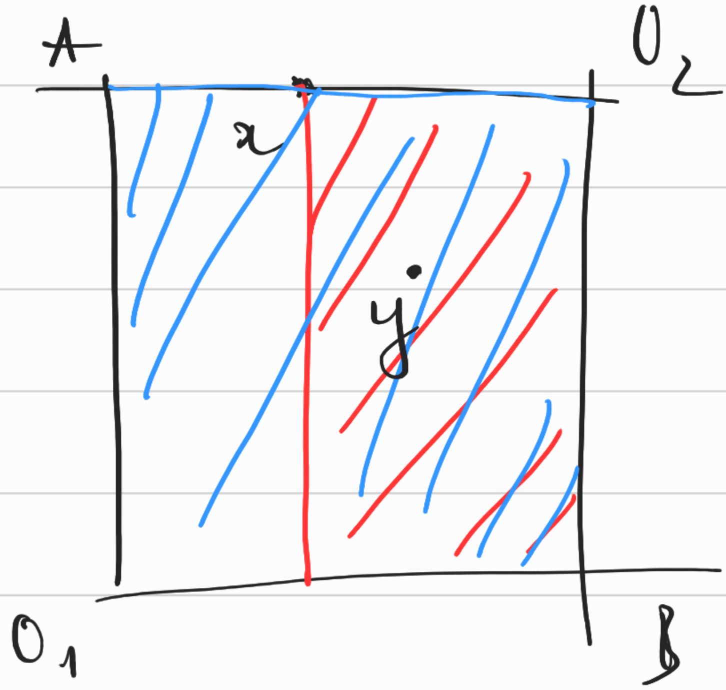


$y \in$ E.B and y pareto dominates x

$\Rightarrow$ x is NOT a POA.

— Consider $x \in$ boundary of E.B

Case 1: $x \in AO_2$

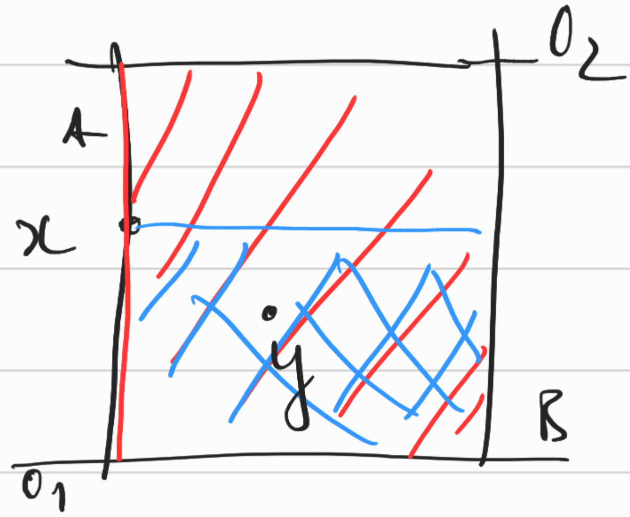


y Pareto dominates x

$\Rightarrow$ x is not a POA.

Case 2:

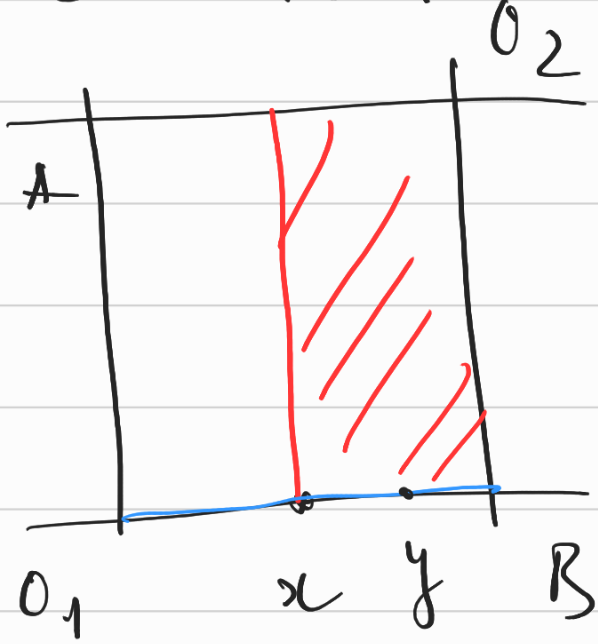
$x \in AO_1$



y Pareto dominates x

$\Rightarrow$ x is NOT a POA

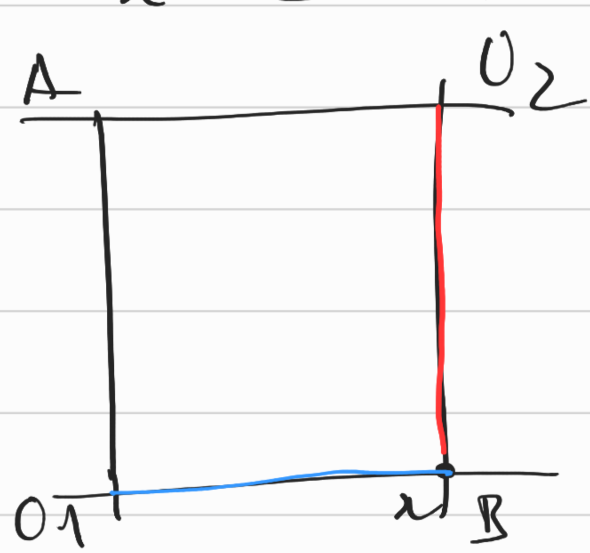
Case 3 : $x \in O_1 B \setminus \{B\}$



y Pareto dominates x

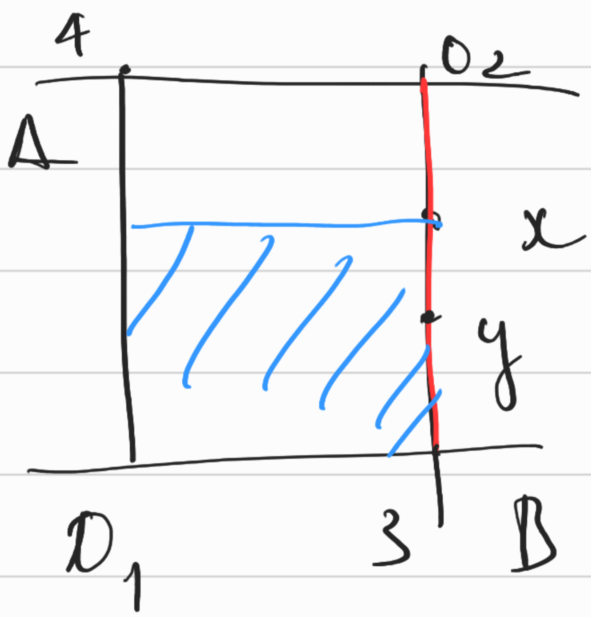
$\Rightarrow$ x is not a POA

Case 4 : $x \equiv B$



$\Rightarrow$ B is a POA.

Case 5: $x \in O_2B / \{B\}$



y Pareto dominates x

$\Rightarrow$ x is not a POA

$\Rightarrow$ The unique POA is B.

or $\big((3,0), (0,4)\big)$