Macroeconomics: Economic Growth (Licence 3)
Lesson 2: The Solow Model (Part 1)

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Acknowledges: some slides and figures are taken or adapted from the supplemental ressources to the textbook "Introduction to Economic Growth" by Charles I. Jones and Dietrich Vollrath, Third Edition, Norton W.W. Company Inc.

Lesson 2

- **Aim** find the main causes explaining long-run economic growth of countries
- Based on the the seminal work is Robert Solow (1956)
- Robert Solow is an American Neo-Keynesian economist (MIT) that was awarded the Nobel Prize in 1987 for his important contributions to theories of economic growth.
- In the 1950s, he developed a mathematical model illustrating how various factors can contribute to sustained national economic growth.
- From the 1960s on, Solow's studies helped persuade governments to channel funds into technological research and development to spur economic growth.
- https://www.nobelprize.org/prizes/economic-sciences/1987/solow/facts/

Lesson 2

- **Aim** find the main causes explaining long-run economic growth of countries
- The seminal work is Solow (1956), although the model is often called the Solow-Swan model, as Swan published an article with a similar model the same year as Solow
- Solow, R., 1956"A contribution to the theory of economic growth", Quarterly Journal of Economics, vol. 70, pp. 65-94
- Swan, T.W., 1956, "Economic growth and capital accumulation", Economic Record, vol. 32, pp. 334-361
- We begin studying a basic version of the Solow growth model (i.e., without technological progress) ;
- then we will add the technological progress to the model;
- At the end, we will look at a version of the model including human capital

Lesson 2

- The baseline growth model of Solow
- Main assumptions
- Main predictions

Main assumptions of the Solow Model

- Perfect competition
- Only one single, homogeneous good is produced (output, Y)
- No international trade
- Technology is exogenous
- Consumer behavior is totally passive
- Economic growth relates to a dynamic process, which in Solow is summarized by investment in physical capital (and exogenous technological change)

Main assumptions of the Solow Model

- Two key equations:
  - (1) A production function;
  - (2) A capital accumulation equation;
- Prices:
  - the price of the produced good is numeraire
  - w is the wage paid by the firms to a unit of labor for one period
  - r is the amount paid by the firms to rent a unit of capital for one period
- No currency, no unemployment

The Solow Model

Production function:
Real output, Y, is produced according to a function like this

$$Y = F(K, L) \tag{1}$$

where K is the stock of physical capital and L is the number of workers.

Very often, we use a very specific function, $F()$, called the Cobb-Douglas.

$$Y = K^{\alpha} L^{1-\alpha} \tag{2}$$

The Solow Model Production function:

The Cobb-Douglas has constant returns to scale; if you double the amount of each input, you double output.

$$(zK)^{\alpha}(zL)^{1-\alpha} = z^{\alpha}z^{1-\alpha}K^{\alpha}L^{1-\alpha} = zK^{\alpha}L^{1-\alpha} = zY \tag{3}$$

Supply side: Firms

Perfect competition assumptions:

- There are many firms
- All producing the same homogeneous output
- Firms enter and exit freely
- They all produce using a similar Cobb-Douglas function
- They are all price-takers for the use of labor and capital

For the representative firm, **profits** are:

$$\Pi = K^{\alpha} L^{1-\alpha} - rK - wL \tag{4}$$

- where r is the market rate for renting capital
- w is the wage of a worker.

The firm's profit-maximizing decision **first-order conditions (FOC)** are

$$MPL = (1-\alpha)\frac{Y}{L} \quad = \quad w \tag{5}$$

$$MPK = \alpha\frac{Y}{K} \quad = \quad r \tag{6}$$

→ Firm sets the marginal product of a factor equal to its marginal cost.

The firm's profit-maximizing decision **first-order conditions (FOC)** are

$$\Pi = K^{\alpha} L^{1-\alpha} - rK - wL$$

$$FOC(d\Pi/dL = 0) : (1-\alpha)K^{\alpha} L^{-\alpha} = w$$

$$MPL = F'_L = (1-\alpha)K^{\alpha} L^{-\alpha}$$

$$MPL = w = (1-\alpha)K^{\alpha} L^{-\alpha} L/L$$

$$MPL = w = (1-\alpha)Y/L$$

The firm's profit-maximizing decision **first-order conditions (FOC)** are

$$\Pi = K^{\alpha}L^{1-\alpha} - rK - wL$$

$$FOC(d\Pi/dK = 0) : \alpha K^{\alpha-1}L^{1-\alpha} = r$$

$$MPK = F'_K = \alpha K^{\alpha-1}L^{1-\alpha}$$

$$MPK = r = \alpha K^{\alpha-1}L^{1-\alpha}K/K$$

$$MPK = r = \alpha Y/K$$

Factor Income: From the firm first-order conditions:

$$MPL = (1-\alpha)\frac{Y}{L} = w \quad (7)$$

$$MPK = \alpha\frac{Y}{K} = r \quad (8)$$

Note that total **payments to factors are equal to total output:**

$$wL + rK = (1-\alpha)\frac{Y}{L}L + \alpha\frac{Y}{K}K = (1-\alpha)Y + \alpha Y = Y. \quad (9)$$

Factor Income: This means that firms all have zero profits:

$$\Pi = Y - rK - wL = 0. \tag{10}$$

- This is consistent with our assumption that firms are perfectly competitive with each other,
- firms enter and exit freely.
- If there were profits,
- more firms would enter and compete them away.

Factor Shares

Calculate the fraction of total output that is paid to each factor:

$$\frac{wL}{Y} = (1-\alpha)\frac{Y}{L}\frac{L}{Y} = (1-\alpha) \tag{11}$$

and

$$\frac{rK}{Y} = \alpha\frac{Y}{K}\frac{K}{Y} = \alpha \tag{12}$$

Capital factor share

Calculate the fraction of total output that is paid to capital ($\frac{rK}{Y}$), using FOC $MPK = r$:

$$r = \alpha Y/K$$

We multiply K on both sides

$$rK = \alpha Y$$

We divide by Y on both sides and get:

$$\frac{rK}{Y} = \alpha \frac{Y}{Y} = \alpha \qquad (13)$$

Labor factor share

Calculate the fraction of total output that is paid to labor ($\frac{wL}{Y}$), using FOC $MPL = w$:

$$w = (1 - \alpha)Y/L$$

We multiply L on both sides

$$wL = (1 - \alpha)Y$$

We divide by Y on both sides and get:

$$\frac{wL}{Y} = (1 - \alpha)$$

Factor Shares

- Factor shares of output are thus constant when we use the Cobb-Douglas,
- regardless of the amount of K or L.
- Consistent with the stylized facts on factor shares.
- Those facts suggest that $\alpha = 1/3$
- $(1 - \alpha) = 2/3$, roughly.

Output per Worker We typically care about output *per worker*, not just total output. Thereby, we divide total output by L to get:

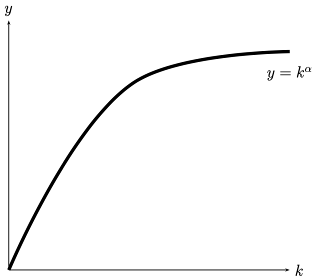
$$y = \frac{Y}{L} = \frac{K^{\alpha}L^{1-\alpha}}{L} = \frac{K^{\alpha}}{L^{\alpha}} = k^{\alpha} \tag{14}$$

where we are using $y = Y/L$ and $k = K/L$ to represent per-worker values.

Output per Worker

- Captial per worker has a diminishing marginal output ($f_k' > 0$ and $f_k'' < 0$).
- If k rises, output per worker rises,
- but the size of the increase falls as k increases.

Economic Growth



- With more capital per worker firms produce more output per worker
- But there are diminishing returns to capital per worker
- each additional unit of capital we give to a single worker increases output of that worker by less and less.

Capital Accumulation

- Output depends on capital and labor,
- so we need to know how those two things accumulate over time.
- It is often more convenient to work in continuous time
- as we will work with time derivatives rather than differences

Growth rate in continuous time

- Reviews an useful math tool to work in continuous time
- Suppose a variable y evolving over time: y(t) (or y_t)
- In continuous time, **the derivative of y_t respect to time indicates the instantaneous variation of y_t at time t indicate by the variable with a dot above :**

$$\frac{dy_t}{dt} = \dot{y}_t$$

Growth rate in continuous time

- The **growth rate** g_y of this variable y_t is defined as this temporal variation y_t over the value of the function y_t at instant t:

$$g_y = \frac{\frac{dy_t}{dt}}{y_t} = \frac{\dot{y}_t}{y_t}$$

- The growth rate g_y of a variable changing over time is the derivative respect to time of the log of this variable:

$$g_y = \frac{d\ln y_t}{dt} = \frac{1}{y_t}\frac{dy_t}{dt} = \frac{\dot{y}_t}{y_t}$$

Capital Accumulation

- The stock of K accumulates over time in the Solow model according to this equation:

$$\dot{K} = sY - \delta K \tag{15}$$

- **$\dot{K}$ is the instantaneous change in the capital stock ($\dot{K} = \frac{dK_t}{dt}$).**
- It is the continuous time version of $K_{t+1} - K_t$.
- **$sY = I$ is gross investment where s is savings rate**
- We know that total income $(wL + rK)$ is equal to total output (Y).
- We assume that individuals - who work and own the capital - save a constant fraction, $0 < s < 1$, of their income.

$$sY = S = I$$

$$(1 - s)Y = C$$

Capital Accumulation

- δK is depreciation rate of capital stock.
- A fixed fraction, $0 < \delta < 1$, of the existing capital stock, K, breaks down at any given moment.
- **growth rate of capital**

Capital Accumulation

$$\dot{K} = sY - \delta K$$

It will be useful to write this as the **growth rate of capital** , or divide through by K,

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta. \tag{16}$$

Capital per Worker Accumulation

- What happens to capital *per worker*?
- Take logs and derivatives:

$$k = \frac{K}{L} \tag{17}$$

$$\ln k = \ln K - \ln L \tag{18}$$

$$\frac{\partial}{\partial t} \ln k = \frac{\partial}{\partial t} \ln K - \frac{\partial}{\partial t} \ln L \tag{19}$$

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \tag{20}$$

Capital per Worker Accumulation From the prior slide, we know that

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \quad (21)$$

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta \quad (22)$$

so we get:

$$\frac{\dot{k}}{k} = s\frac{Y}{K} - \delta - \frac{\dot{L}}{L}. \quad (23)$$

Population Growth

- The last piece of information we need is the growth rate of the number of workers, $\dot{L}/L$.
- We assume that the number of workers grows at the same rate as the population.
- **the number of workers exhibits exponential growth.** We assume that the number of workers grow at a **constant rate** n**.**Thus:

$$L(t) = L(0)e^{nt} \tag{24}$$

The growth rate of the number of workers is thus

$$\frac{\dot{L}}{L} = n, \tag{25}$$

which you can find by taking logs and derivatives of the equation for $L(t)$.

Population Growth

- If we take the log and time derivatives of $L(t) = L(0)e^{nt}$,
- we find that $\frac{\dot{L}}{L} = n$

$$lnL(t) = lnL(0) + nt$$

$$\frac{dlnL(t)}{t} = \frac{dlnL(0)}{t} + \frac{dnt}{t}$$

$$\frac{\dot{L}}{L} = 0 + n = n$$

The Solow Equation
Plugging in the growth rate of workers $\frac{\dot{L}}{L} = n$ into the equation of accumulation of capital per worker $\frac{\dot{k}}{k} = s\frac{Y}{K} - \delta - \frac{\dot{L}}{L}$., we now have:

$$\frac{\dot{k}}{k} = s\frac{Y}{K} - \delta - n. \tag{26}$$

- which says the growth rate of capital per worker depends on the capital/output ratio (K/Y)
- and the parameters s, δ, and n.

We can re-write this into what is commonly called *the Solow Equation* divinding by L/L,

$$\frac{\dot{k}}{k} = s\frac{Y/L}{K/L} - \delta - n \quad (27)$$

$$= s\frac{y}{k} - \delta - n. \quad (28)$$

and then multiply both sides by k to get

$$\dot{k} = sy - (\delta + n)k. \quad (29)$$

$$\dot{k} = sy - (\delta + n)k.$$

Finally, recall that we can write output per worker in terms of capital per worker using the per capita production function, $y = k^{\alpha}$, so that we have

$$\dot{k} = sk^{\alpha} - (\delta + n)k \tag{30}$$

- **Solving the Model**
- First, what do we mean by "solve"?
- We mean that we want to be able to express the endogenous variables in terms of *only* exogenous ones.

Endogenous variables - things we are trying to explain

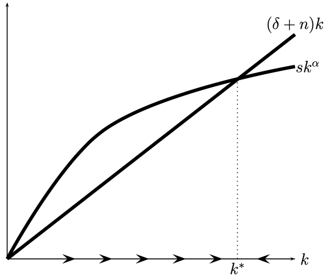
- (1) **From Production Function Equation:** Output Y and/or output per worker $y = k^{\alpha}$
- (2) **From Capital Accumulation equation:** Capital K and/or capital per worker $k \rightarrow \dot{k} = sk^{\alpha} - (\delta + n)k$

Exogenous variables - things we take as given

- α, capital's share in output
- s, the savings rate
- n, the population growth rate
- δ, the depreciation rate
- k_0, the initial level of capital per worker

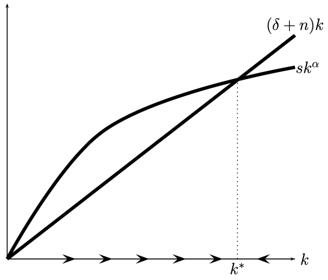
Getting a precise equation for k or y is not terribly easy. However, we can see the solution in some simple diagrams.

The basic Solow diagram



- Plot output per worker against capital per worker
- First curve: is the amount of investment per person $sy = sk^\alpha$
- it has the same shape as the production function but translated down by the factor s
- Second curve is the line $(\delta + n)k$: the amount of investment per person required to keep the amount of capital per worker constant.
- The difference between both curves is: the change in the amount of capital per worker. When $sk^\alpha = (\delta + n)k$ then $\dot{k} = 0$

The basic Solow diagram



- Steady state: $k^* \rightarrow \dot{k} = 0$ and $sy = (\delta + n)k$
- The diagram shows us that capital per worker will stabilize at some value k^* (steady-state),
- where the creation of new capital per worker due to investment ($sy = sk^{\alpha}$)
- just offsets the capital per worker "lost" due to depreciation and population growth
- Capital deepening when $sk^{\alpha} > (\delta + n)k$
- Capital widening when $sk^{\alpha} < (\delta + n)k$

Implications:

- With this graphical analysis, we have “solved” the Solow model in a general sense:
- if you tell me a value of $k0$ (i.e., the initial starting point for capital per worker),
- and the parameters δ, s, n, α
- I can tell you whether capital per worker will grow or shrink

Implications:

- If $k < k^*$, then $sk^\alpha > (\delta + n)k$, and $\dot{k} > 0$
- If $k > k^*$, then $sk^\alpha < (\delta + n)k$, and $\dot{k} < 0$
- The Solow model predicts that capital per worker will stabilize at some value k^*
- where investment sk^α just offsets depreciation and population growth $(\delta + n)k$.
- We refer to k^* as the *steady state* of the Solow Model.

Implications:

- The Solow model predicts that **growth is faster, the farther away from steady state** is an economy
- Recall $\frac{\dot{k}}{k} = s\frac{Y}{K} - \delta - n$
- and $y = k^{\alpha}$ and taking log $lny = \alpha lnk$ and the derivatives respect to time, we get:
- $\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k}$
- As k rises, the growth rates of both k and y fall

The Steady State

No matter what, k eventually ends up at k^*. We can give a precise answer to what determines k^{ast}. It is the value of k such that $\dot{k} = 0$. Or

$$0 = sk^\alpha - (\delta + n)k. \tag{31}$$

This solves to

$$(\delta + n)k = sk^\alpha \tag{32}$$

$$k^{1-\alpha} = \frac{s}{\delta + n} \tag{33}$$

$$k = \left(\frac{s}{\delta + n}\right)^{1/(1-\alpha)} \tag{34}$$

which is the capital per worker at steady state, or

$$k^* = \left(\frac{s}{\delta + n}\right)^{1/(1-\alpha)} \tag{35}$$

Implications: The steady state is:

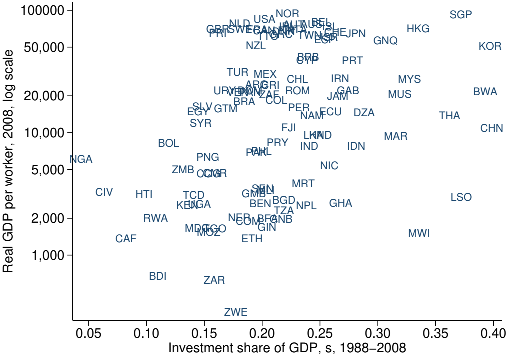
$$k^* = \left(\frac{s}{\delta + n}\right)^{1/(1-\alpha)} \tag{36}$$

which depends only on the parameters of the model. Note that it does *not* depend on k_0.

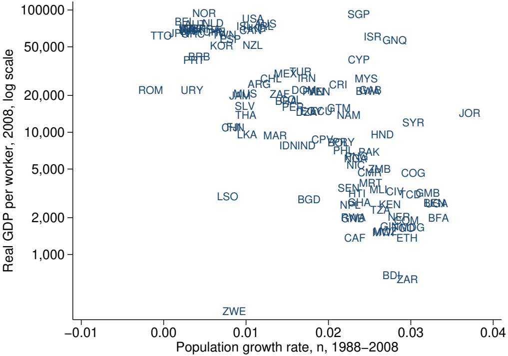
This implies that

- k^* (and so y^*) is rising with the savings rate, s
- k^* (and so y^*) is declining with the population growth rate, n

Savings and Output per Worker



Population and Output per Worker



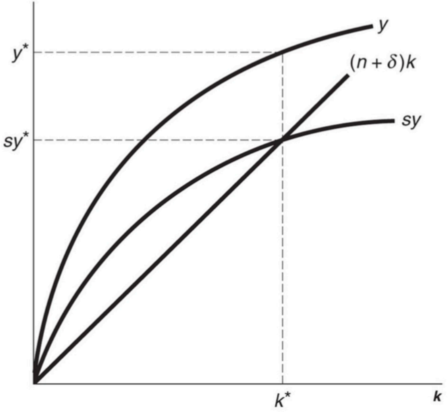
Economic Growth

- Since $k^* = \left(\frac{s}{\delta+n}\right)^{1/(1-\alpha)}$ and
- $y = k^\alpha$

$$y^* = k^{*\alpha} = \left(\frac{s}{\delta + n}\right)^{\alpha/(1-\alpha)}$$

- y^* does not depend on k_0 (or y_0)
- The higher is s, the higher is the level of y^*
- The higher is n, the lower is the level of y^*

Steady-state output per worker y^*

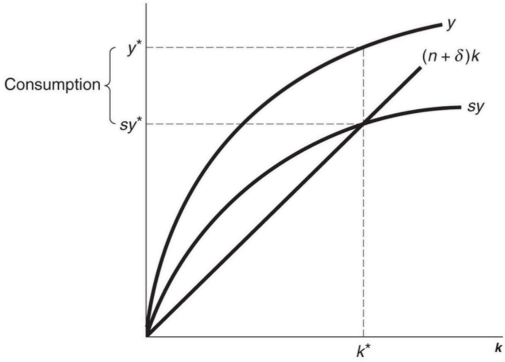


- sy is a downward translation of y as s is a constant < 1
- We can add to the Solow diagram the curve depicting $y = f(k)$
- where we can read y^* .

Steady-state consumption per worker y^*

- We have assumed that non-invested (non-saved) income is consumed: $C = (1 - s)Y$
- Thus, in per capita terms, we have
- $c = (1 - s)y$
- Consumption c^* at the steady state is given by:
- the vertical difference between the curve representing y^* and the curve representing sy
- $c^* = y^* - sy^* = (1 - s)y^*$

Steady-state consumption per worker y^*



- Consumption c^* at the steady state is given by:
- the vertical difference between the curve representing y^* and the curve representing sy
- $c^* = y^* - sy^* = (1 - s)y^*$

Steady-sate growth of per worker variables

- $\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$
- Thus in the steady state where $dotk = 0$:

$$\frac{\dot{k}}{k} = 0$$
$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} = 0$$

- In steady state, growth rates of income and capital per worker go to zero
- in the simple model of Solow without technological progress.

Steady-sate growth of aggregate variables

- Since $k = K/L$:

$$lnk = lnK - lnL$$

$$lnK = lnk + lnL$$

Remember deriving relative to time t, we get:

$$\frac{\dot{K}}{K} = \frac{\dot{k}}{k} + \frac{\dot{L}}{L} = 0 + n = n$$

$$\frac{\dot{K}}{K} = \frac{\dot{k}}{k} + \frac{\dot{L}}{L} = 0 + n = n$$

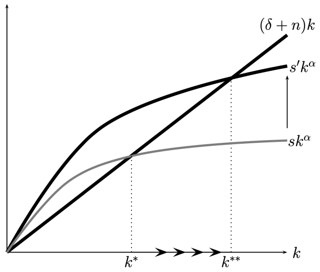
$$\frac{\dot{Y}}{Y} = \alpha\frac{\dot{K}}{K} + (1-\alpha)\frac{\dot{L}}{L} = \alpha n + (1-\alpha)n = n$$

- In steady state, growth rates of output, capital and labor (aggregate)
- are equal to the growth rate of population in the Solow model without technical progress

Savings and Output per Worker

Statics: What if the savings rate, s, changes? Could be a policy change, or some difference in individuals willingness to save for the future. Let s go to $s' > s$.

An increase in the investment rate s

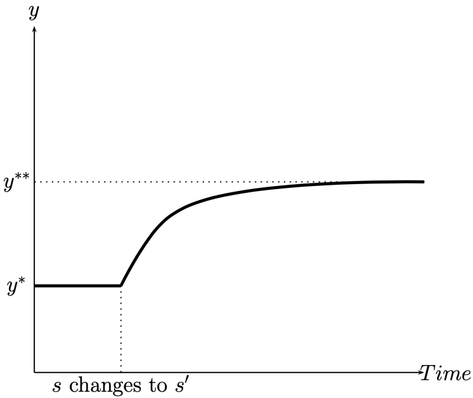


- Increase in investment rate shift the sy curve upward till $s'y$
- investment per worker now ($s'y$ exceeds the amount required to keep capital per worker constant.
- Capital deepening after an increase in s (investment rate)
- Until $s'y = (\delta + n)k$

Savings Increase: What is the effect of s rising to s'?

- The steady state rises to k^{**}. The economy will be richer *eventually*
- Immediately after the change, $k < k^{**}$, so $\dot{k} > 0$, capital starts growing
- Output per worker grows until the economy reaches the new steady state

Economic Growth



What is the policy implication?

Growth Rates: The Solow model predicts that growth is faster, the farther away from steady state is an economy. Look at the growth rate of k

$$\dot{k} = sk^{\alpha} - (\delta + n)k \tag{37}$$

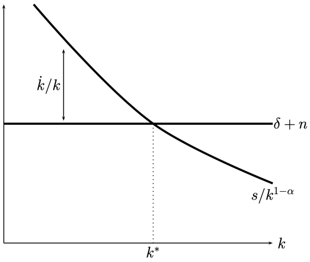
Dividing both sides by k, we get

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta + n) \tag{38}$$

$$\frac{\dot{k}}{k} = \frac{s}{k^{1-\alpha}} - (\delta + n). \tag{39}$$

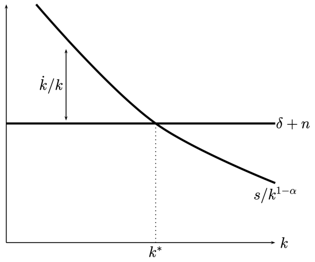
As k rises, the growth rate of k falls.

Transition dynamics based on capital accumulation



- $\frac{\dot{k}}{k} = \frac{s}{k^{1-\alpha}} - (\delta + n)$.
- The first curve is $\frac{s}{k^{1-\alpha}}$: the higher the level of capital per worker,
- the lower the average product of capital (diminishing returns to capital since $\alpha < 1$)
- The second term $(\delta + n)$ does not depend on capital so it is a horizontal line.
- The difference between the two lines is the growth rate of capital stock $\frac{\dot{k}}{k}$.

Transition dynamics



- An economy starting at $k_0 < k_*$ will experience a growing capital per worker
- The further to the left of the steady state (the lower the capital per worker), the higher is the growth rate of the capital per worker
- Growth rates of the capital and income per worker are proportional to the vertical distance between the sy/k curve and the $(\delta + n)$

Trend Growth: Growth rates go to zero in the Solow model. y does not grow in steady state. To see this, take logs and derivatives of $y = k^{\alpha}$ to find

$$\ln y = \alpha \ln k \quad (40)$$

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}. \quad (41)$$

Because $\dot{k} = 0$ in steady state, it must be that $\dot{y}/y = 0$ in steady state.