Macroeconomics: Economic Growth (Licence 3)
Lesson 3: The Solow Model (Part 2)

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Acknowledges: some slides and figures are taken or adapted from the supplemental ressources to the textbook "Introduction to Economic Growth" by Charles I. Jones and Dietrich Vollrath, Third Edition, Norton W.W. Company Inc.

Comparative statics of Solow model

- We can perform some comparative statics exercises,
- Analyze the response of the model to changes in parameters

A change in the population n

Suppose the economy is at the steady-state, what happens if n changes?

- In the long run, $gy = gk = 0$ a change in n does not have an impact on the long run growth of y and k (per capita)
- In the long run, $gY = gK = n$ a change in n has an impact on the long run growth of Y and K (aggregate)
- A change in n has an impact on the long run levels of y and k

$$k^* = \left(\frac{s}{\delta + n}\right)^{1/(1-\alpha)}$$

$$y^* = \left(\frac{s}{\delta + n}\right)^{\alpha/(1-\alpha)}$$

A change in the population n

Suppose the economy is at the steady-state, what happens if n changes?

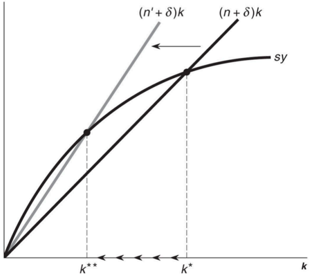
- A change in n has an impact on the growth rates of k (and y) during the transition dynamics:

$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta + n)$$

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$$

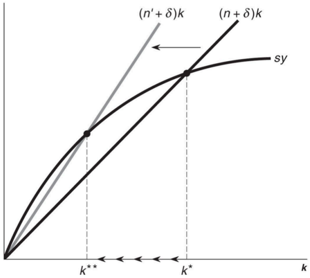
→ A change in n has only temporary effects on the growth rates of k and y , and permanent effects on the long run levels of k and y .

An increase in population rate n



- Increase population growth rate due to immigration
- The $(\delta + n)k$ curve rotates up and to the left to the new curve $(\delta + n')k$
- Investment per worker is no longer enough to keep the capital labor ratio constant
- due to increase in population

An increase in population rate n



- so, the capital-labor ratio begins to fall
- Until the point at which $sy = (\delta + n')k$
- the economy has less capital per worker and it is poorer

A change in investment (savings) s

Suppose the economy is at the steady-state, what happens if s changes?

- In the long run, $gy = gk = 0$ a change in s does not have an impact on the long run growth of y and k (per capita)

- A change in s has an impact on the long run levels of y and k

$$k^* = \left(\frac{s}{\delta + n}\right)^{1/(1-\alpha)}$$

$$y^* = \left(\frac{s}{\delta + n}\right)^{\alpha/(1-\alpha)}$$

A change in investment (savings) s

Suppose the economy is at the steady-state, what happens if s changes?

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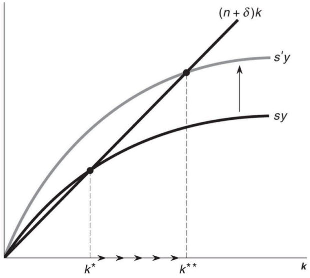
$$\frac{\dot{k}}{k} = sk^{\alpha-1} - (\delta + n)$$

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k}$$

$\rightarrow$ A change in s has only temporary effects on the growth rates of k and y , and permanent effects on the long run levels of k and y .

An increase in investment (savings) s



- Increase in investment rate shift the sy curve upward till $s'y$
- investment per worker now ($s'y$ exceeds the amount required to keep capital per worker constant.
- Capital deepening after an increase in s (investment rate)
- Until $s'y = (\delta + n)k$

Golden rule

- An important question is whether all possible steady state values of k^* (and y^*) maximize the steady state per capita consumption or not
- per capita consumption summarizes the welfare.
- Higher k^* (and y^*) with higher s
- and of $c^* = y^* - sy^*$
- Does any increase in s ensure higher consumption?

Golden rule

- Does any increase in s ensure higher consumption?
  - Not necessarily as when s increases, steady-state level of output per worker increases, and so potentially more resources for consumption
  - While, on the other hand, when s increases, the share of output for consumption decreases

Golden rule

- In steady state: $c^* = y^* - sy^*$ and $sy^* = (\delta + n)k^*$
- $c^* = y^* - sy^* = k^{*\alpha} - (\delta + n)k^*$
- To find the value of k^* that maximizes c^*, we differentiate respect to k^* and set equal to 0:

$$\frac{dc^*}{dk^*} = 0$$

Golden rule

- $c^* = y^* - sy^* = k^{*\alpha} - (\delta + n)k^*$
- To find the value of k^* that maximizes c^*, we differentiate respect to k^* and set equal to 0:

$$\frac{dc^*}{dk^*} = 0$$

$$\alpha k^{*\alpha - 1} - (\delta + n) = 0$$

$$\alpha k^{*\alpha} = (\delta + n)k^*$$

$$k^{*1-\alpha} = \frac{\alpha}{\delta + n}$$

$$k^{gr} = (\frac{\alpha}{\delta + n})^{1/(1-\alpha)}$$

Where *gr* states for the Golden Rule where c is maximized.

Golden rule

- Given that $k^* = (\frac{s}{\delta+n})^{1/(1-\alpha)}$ and $k^{gr} = (\frac{\alpha}{\delta+n})^{1/(1-\alpha)}$
- The level of capital per worker of the steady state coincides of that of the
- Golden rule: $k^* = k^{gr}$ only if $s = \alpha$
- the level of s that maximizes c is equal to α

Golden rule

- Given $y = k^\alpha$, the marginal product of capital is:

$$MPK = \alpha k^{\alpha-1}$$

- Replacing the steady state the value of $k^* = (\frac{s}{\delta+n})^{1/(1-\alpha)}$

$$MPK^* = \alpha(\frac{s}{\delta+n})^{\alpha-1/(1-\alpha)} = \alpha(\frac{\delta+n}{s})$$

- In the Golden rule equilibrium $s = \alpha$ and so

$$MPK^{gr} = \delta + n$$

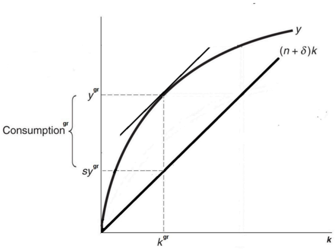
Golden rule

- Since $\delta + n$ is the slope of the line representing the break-even investment per worker at the steady state of golden rule k^{gr}

- The slope of the production function equals the slope of $(\delta + n)k$ line

- At the steady state of golden rule a tangent to the production function is parallel to the $(\delta + n)k$ line.

Golden rule



- The slope of the production function equals the slope of $(\delta + n)k$ line
- At the steady state of golden rule a tangent to the production function is parallel to the $(\delta + n)k$ line.

Golden rule

- Given that s and α are exogenous parameters
- There is no reason for $k^* = k^{gr}$
- The steady state equilibrium in the Solow model tends to be dynamically inefficient
- $\rightarrow$ the economy does not tend to reach $k^* = k^{gr}$ spontaneously where c is maximized
- Policy implications: the government needs to implement policies to change the investment/savings rate s

Implications: Differences in per capita income levels

The simple Solow model predicts: long-term disparities in per capita income levels across countries

- In the steady state: $y^* = (\frac{s}{\delta+n})^{\alpha/(1-\alpha)}$
- Differences in long-term levels of per capita income (y) across countries are possible is countries differ in the parameters s, n, δ, α
- However, differences in saving rates s across countries as the main determinant of
- $\rightarrow$ per capita income differences is not convincing.

Implications: Differences in per capita income levels

For instance

- Suppose two countries share the same production functions and same parameters except s
- The ratio of income per capita of these two countries at the state state is

$$y_1 = \left(\frac{s_1}{\delta + n}\right)^{\frac{\alpha}{(1-\alpha)}}$$

$$y_2 = \left(\frac{s_2}{\delta + n}\right)^{\frac{\alpha}{(1-\alpha)}}$$

$$\frac{y_1}{y_2} = \left(\frac{s_1}{s_2}\right)^{\frac{\alpha}{(1-\alpha)}}$$

Implications: Differences in per capita income levels

- Suppose that $s_1 = 2s_2$ and $\alpha = 0.33$ (close to the US estimated value)

$$\frac{y_1}{y_2} = 2^{0.5} = 1.41$$

- Even a huge difference in saving rates across countries leads to an income per capita difference of about 40 percent
- If we $s_1 = 4s_2$, this would imply an income per capita twice as large
- These numbers are far way from the differences illustrated in lecture 1: income ratios of 20 and 30 times
- There must be some other factors not present in the simple model of Solow that explain long-term differences in income per capita across countries as technical progress

Implications: Differences in per capita income levels

Is the simple model of Solow able to predict long-term differences in income per capita?

- In steady state: $g_y = 0$
- The model predicts long term growth rates equal to zero
- Countries empirically differ in long run trend growth of per capita income
- There must be some other factors not present in the simple model of Solow that explain long-term differences in income per capita across countries as technical progress

Implications: Differences in per capita income levels

Is the simple model of Solow able to predict long-term differences in income per capita?

- $\dot{y}/y = \alpha\dot{k}/k$ and $\dot{k}/k = sy/k - (\delta + n)$
- The model predicts that shocks to parameters and the stock of capital lead to l temporary changes in growth rates
- the model can partially help to explain differences across countries in short term growth of per capita income

The Solow model

- Extensions of the simple version of the Solow growth model
- A version of the model with technological progress

To allow for trend growth in output per worker in the steady state in the Solow model, we'll incorporate productivity improvements.

- Let production function be

$$Y = K^{\alpha}(AL)^{1-\alpha} \quad (1)$$

where A is labor-augmenting technological progress. In per worker terms this is

$$y = Y/L = k^{\alpha}A^{1-\alpha} \quad (2)$$

Growth in A will allow there to be trend growth in output per worker even in

steady state.

We can express the production function in terms **of units of effective labor (dividing both sides by AL)** or in efficiency units:

$$Y/AL = (K/AL)^{\alpha} \tag{3}$$

$$\hat{y} = \hat{k}^{\alpha} \tag{4}$$

Where $\hat{y} = Y/AL$ and $\hat{k} =$ represents units of effective labor

Exogenous Technological Change: We assume that

$$A(t) = A(0)e^{\gamma t} \tag{5}$$

which implies that taking the log and derivative respect to time to obtain the growth rate of A

$$\frac{\dot{A}}{A} = \gamma \tag{6}$$

- As in the model without technological progress, population grows at a constant and exogenous rate n

$$\frac{\dot{L}}{L} = n \tag{7}$$

- Accumulation of capital stock is as in the previous model

$$\frac{\dot{K}}{K} = s\frac{Y}{K} - \delta \tag{8}$$

- Defining $\hat{k} = \frac{K}{AL}$ and taking the log and derivative with respect to time we obtain the accumulation of capital per unit of effective labor

- Defining $\hat{k} = \frac{K}{AL}$ and taking the log and derivative with respect to time we obtain the accumulation of **capital per unit of effective labor**

$$ln\hat{k} = lnK - lnA - lnL \tag{9}$$

$$\frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{K}}{K} - \frac{\dot{A}}{A} - \frac{\dot{L}}{L} \tag{10}$$

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\frac{Y}{K} - \delta - \gamma - n \tag{11}$$

Dividing and multiplying by AL we get:

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\frac{\hat{y}}{\hat{k}} - (\delta + \gamma + n) \tag{12}$$

Replacing $\hat{y} = \hat{k}^{\alpha}$

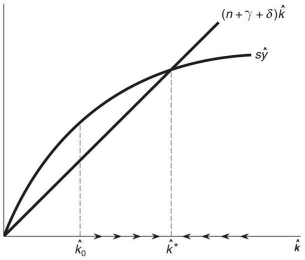
$$\frac{\dot{\hat{k}}}{\hat{k}} = s\frac{\hat{k}^{\alpha}}{\hat{k}} - (\delta + \gamma + n) \tag{13}$$

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\frac{\hat{k}^{\alpha}}{\hat{k}} - (\delta + \gamma + n) \tag{14}$$

Rearranging terms, we get:

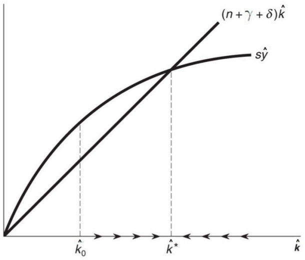
$$\dot{\hat{k}} = s\hat{k}^{\alpha} - (\delta + \gamma + n)\hat{k} \tag{15}$$

The Solow Diagram



- The Solow diagram with technological progress: expressed in **per unit of effective labor (in terms of AL)**
- Plotting the two main equations: **production function with A**: $\hat{y} = \hat{k}^{\alpha}$
- **Capital accumulation function:** $\dot{\hat{k}} = s\hat{k}^{\alpha} - (\delta + \gamma + n)\hat{k}$

The Solow Diagram



- The diagram shows us that capital per worker will stabilize at some value $\hat{k}^*$ (steady-state),
- where the creation of new capital per worker due to investment ($s\hat{y} = s\hat{k}^{\alpha}$)
- just offsets the capital per worker "lost" due to depreciation and population growth
- Capital deepening when $s\hat{k}^{\alpha} > (\delta + n + \gamma)\hat{k}$
- Capital widening when $s\hat{k}^{\alpha} < (\delta + n + \gamma)\hat{k}$

Implications:

- If $\hat{k} < \hat{k}^*$, then $s\hat{k}^\alpha > (\delta + n + \gamma)\hat{k}$, and $\dot{\hat{k}} > 0$
- If $\hat{k} > \hat{k}^*$, then $s\hat{k}^\alpha < (\delta + n + \gamma)\hat{k}$, and $\dot{\hat{k}} < 0$
- If $\hat{k} = \hat{k}^*$, then $s\hat{k}^\alpha = (\delta + n + \gamma)\hat{k}$, and $\dot{\hat{k}} = 0$

- The Solow model predicts that capital per worker will stabilize at some value $\hat{k}^*$
- where investment sk^α just offsets depreciation and population growth $(\delta + n + \gamma)\hat{k}$.
- We refer to $\hat{k}^*$ as the *steady state* of the Solow Model.

Economic Growth

- In the **Solow model with technological progress, the dynamics is equivalent to the model without** technological progress change

- as long as we express the model in efficiency units (in terms of AL):

- **The growth rate of capital expressed in units of effective labor is**

- $\hat{k} = k/A, \rightarrow \frac{\dot{\hat{k}}}{\hat{k}} = s\frac{\hat{y}}{\hat{k}} - (\delta + \gamma + n)$

- Replacing $\hat{y} = \hat{k}^{\alpha}$

$$\frac{\dot{\hat{k}}}{\hat{k}} = s\hat{k}^{\alpha-1} - (\delta + \gamma + n) \tag{16}$$

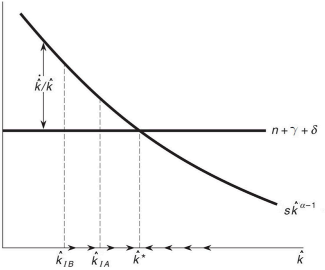
$$\frac{\dot{\hat{y}}}{\hat{y}} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} \tag{17}$$

Finally recall $y = k^{\alpha} A^{1-\alpha}$, thus:

$$\frac{\dot{y}}{y} = \alpha \frac{\dot{k}}{k} + (1-\alpha)\frac{\dot{A}}{A} \tag{18}$$

Transition dynamics

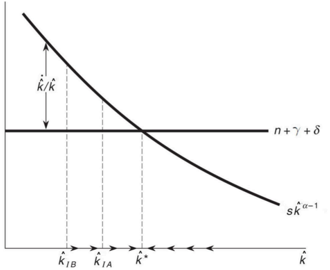
Transition dynamics based on growth rate of capital in effective units of labor



- $\frac{\dot{\hat{k}}}{\hat{k}} = s\hat{k}^{\alpha-1} - (\delta + \gamma + n)$
- The first curve is $s\hat{k}^{\alpha-1}$: the higher the level of capital per worker, the lower the average product of capital (diminishing returns to capital since $\alpha < 1$)
- The second term $(\delta + \gamma + n)$ does not depend on capital so it is a horizontal line.
- The difference between the two lines is the growth rate of capital stock $\frac{\dot{\hat{k}}}{\hat{k}}$.

Transition dynamics

Transition dynamics based on growth rate of capital in effective units of labor



- An economy starting at $k_0 < \hat{k}_*$ will experience a growing capital per effective units of labor (AL)
- **The further to the left of the steady state (the lower the capital per effective units of labor), the higher is the growth rate of the capital per effective units of labor;**
- Growth rates of the capital and income per worker in effective units of labor are proportional to the vertical distance between the $s\hat{k}^{\alpha-1}$ curve and the $(\delta+\gamma+n)$

Long run levels: variables per unit of effective labor and per capita

- Solving the model we find **the steady state equilibrium levels of capital in effective units of labor**
- $\hat{k}_*$ is the value of $\hat{k}$ such that $\dot{\hat{k}} = 0$. So

$$\dot{\hat{k}} = s\hat{k}^{\alpha} - (\delta + \gamma + n)\hat{k} = 0 \tag{19}$$

This solves to

$$(\delta + \gamma + n)\hat{k} = s\hat{k}^{\alpha} \tag{20}$$

$$\hat{k}^{1-\alpha} = \frac{s}{\delta + \gamma + n} \tag{21}$$

$$\hat{k} = \left(\frac{s}{\delta + \gamma + n}\right)^{1/(1-\alpha)} \tag{22}$$

which is the capital per worker at steady state, or

$$\hat{k}^* = \left(\frac{s}{\delta + \gamma + n}\right)^{1/(1-\alpha)} \tag{23}$$

Long run levels: variables per unit of effective labor and per capita

- Solving the model we find **the steady state equilibrium levels of capital and income in effective units of labor**
- which depends now also **on the growth rate of the exogenous technological progress (γ)**

$$\hat{k}^* = \frac{k^*}{A} = \left(\frac{s}{\delta + \gamma + n}\right)^{\frac{1}{1-\alpha}} \tag{24}$$

$$\hat{y}^* = \frac{y^*}{A} = \left(\frac{s}{\delta + \gamma + n}\right)^{\frac{\alpha}{1-\alpha}} \tag{25}$$

Long run growth (variables per unit of effective labor)

- We have seen that in the long run $\dot{\hat{k}} = 0$, thus:

$$\frac{\dot{\hat{k}}}{\hat{k}} = 0 \tag{26}$$

- Since $\frac{\dot{\hat{y}}}{\hat{y}} = \alpha \frac{\dot{\hat{k}}}{\hat{k}}$

$$\frac{\dot{\hat{y}}}{\hat{y}} = 0 \tag{27}$$

Long run growth (per worker variables)

- We have defined: $\hat{k} = \frac{k}{A} \rightarrow \frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{k}}{k} - \frac{\dot{A}}{A}$
- Since in the log run $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ and $\frac{\dot{\hat{A}}}{\hat{A}} = \gamma$
- In the long run the growth rate of capital per worker is

$$\frac{\dot{k}}{k} = \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A} = \gamma \tag{28}$$

Long run growth (per worker variables)

- Since $\hat{y} = \frac{y}{A} \rightarrow y = \hat{y}A = \hat{k}^{\alpha}A$
- Thus $\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A}$
- Thereby the growth rate of income per worker in the long run is:

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A} = \gamma \tag{29}$$

- Since in the log run $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ and $\frac{\dot{\hat{A}}}{\hat{A}} = \gamma$
- **In the Solow model with technological progress in the long run we have:**
- $g_{\hat{k}} = g_{\hat{y}} = 0$ and $g_k = g_y = g_A = \gamma$

Long run growth (per worker variables)

- **Main difference of Solow model with technological progress and without:**
- **produces a balanced growth path (BGP): where everything's growth rate remains constant**
- Respect to the version of the model without technological progress,
- **here the growth in A allows trend growth in output per worker and capital per worker in the long-run!**

Balanced Growth Path: The steady state of the Solow model will be exactly a situation where growth in y is constant at g. Further, y and k and A will all grow at the same rate.

What is the steady state when there is technology? We saw above that

$\dot{y}/y = \gamma$ is the constant growth rate. This also implies $\dot{k}/k = \gamma$.

Steady State with Technology: We said that

$$\hat{k} = \frac{k}{A} = \left(\frac{s}{\delta + n + \gamma}\right)^{1/(1-\alpha)} \tag{30}$$

in steady state. **The ratio k/A is constant because k and A grow at the rate γ.**

Output per worker in this steady state is

$$y = k^{\alpha} A^{1-\alpha} \tag{31}$$

$$= A\left(\frac{k}{A}\right)^{\alpha} \tag{32}$$

$$= A\left(\frac{s}{\delta + n + \gamma}\right)^{\alpha/(1-\alpha)} \tag{33}$$

$$y(t) = A(t)\left(\frac{s}{\delta + n + \gamma}\right)^{\alpha/(1-\alpha)} \tag{34}$$

where the $y(t)$ and $A(t)$ are there to be explicit that both are growing over time.

Level Effects versus Growth Effects

$$y(t) = A(t)\left(\frac{s}{\delta + n + \gamma}\right)^{1/(1-\alpha)} \tag{35}$$

- **The growth rate of y is equal to γ along the balanced growth path - which is the steady state** of the Solow model.

- The parameters for savings, s, and population growth, n, have effects on the **level** of output per worker,
- but don't alter the **growth rate** of output per worker.

- Changes in those parameters will induce a temporary changes in growth
- as the economy shifts to the new trend line.

Example: Consider an increase in savings from s to s'. You can use the old Solow model without technology to consider what happens in the transition – output per worker grows rapidly for a while until we reach the new steady state. Then just add trend growth to have the model of exogenous technological progress.

Economic Growth