General Equilibrium – Mathematical Appendix

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Contents

1 Notations and basic notions

- $\mathbb{R}^n := \{x = (x_1, \ldots, x_h, \ldots, x_n) : x_h \in \mathbb{R},\ \forall\ h = 1, ..., n\}$
- $x \in \mathbb{R}^n$ and $\overline{x} \in \mathbb{R}^n$,

$$x \geq \overline{x} \Longleftrightarrow x_h \geq \overline{x}_h,\ \forall\ h = 1, ..., n$$

$$x > \overline{x} \Longleftrightarrow x \geq \overline{x} \text{ and } x \neq \overline{x}$$

$$x \gg \overline{x} \Longleftrightarrow x_h > \overline{x}_h,\ \forall\ h = 1, ..., n$$

- $x \in \mathbb{R}^n$ and $\overline{x} \in \mathbb{R}^n$, $x \cdot \overline{x}$ denotes the scalar product of x and $\overline{x}$.
- A is a matrix with m rows and n columns and B is a matrix with n rows and l columns, AB denotes the matrix product of A and B.
- $x \in \mathbb{R}^n$ is treated as a row matrix.
- x^T denotes the transpose of $x \in \mathbb{R}^n$, x^T is treated as a column matrix.
- f is a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$,

  f is **non-decreasing (or weakly increasing)** on X if for all x and $\overline{x}$ in X,

$$x \leq \overline{x} \Longrightarrow f(x) \leq f(\overline{x})$$

  f is **increasing** on X if for all x and $\overline{x}$ in X,

$$x \ll \overline{x} \Longrightarrow f(x) < f(\overline{x})$$

  f is **strictly increasing** on X if for all x and $\overline{x}$ in X,

$$x < \overline{x} \Longrightarrow f(x) < f(\overline{x})$$

$$f \text{ strictly increasing on } X \Longrightarrow f \text{ increasing on } X$$

$$f \text{ strictly increasing on } X \Longrightarrow f \text{ non-decreasing (or weakly increasing) on } X$$

- $X \subseteq \mathbb{R}^n$ is an open set, f is a function from X to $\mathbb{R}$ and $x \in X$,

$$\mathrm{D}f(x) := \left(\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_h}(x), \ldots, \frac{\partial f}{\partial x_n}(x) \right)$$

denotes the **gradient** of f at x, and

$$\mathrm{D}^2 f(x) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1}(x) & \ldots & \frac{\partial^2 f}{\partial x_h \partial x_1}(x) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_h}(x) & \ldots & \frac{\partial^2 f}{\partial x_h \partial x_h}(x) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_h}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \ldots & \frac{\partial^2 f}{\partial x_h \partial x_n}(x) & \ldots & \frac{\partial^2 f}{\partial x_n \partial x_n}(x) \end{bmatrix}_{n \times n}$$

denotes the **Hessian matrix** of f at x.

- $X \subseteq \mathbb{R}^n$ is an open set, $g := (g_1, \ldots, g_j, \ldots, g_m)$ is a mapping from X to $\mathbb{R}^m$ and $x \in X$,

$$\mathrm{D}g(x) := \begin{bmatrix} \frac{\partial g_1}{\partial x_1}(x) & \ldots & \frac{\partial g_1}{\partial x_h}(x) & \ldots & \frac{\partial g_1}{\partial x_n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_j}{\partial x_1}(x) & \ldots & \frac{\partial g_j}{\partial x_h}(x) & \ldots & \frac{\partial g_j}{\partial x_n}(x) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \ldots & \frac{\partial g_m}{\partial x_h}(x) & \ldots & \frac{\partial g_m}{\partial x_n}(x) \end{bmatrix}_{m \times n} = \begin{bmatrix} \mathrm{D}g_1(x) \\ \vdots \\ \mathrm{D}g_j(x) \\ \vdots \\ \mathrm{D}g_m(x) \end{bmatrix}_{m \times n}$$

denotes the **Jacobian matrix** of g at x.

1.1 Norms

Definition 1 (Norm) *A norm on* $\mathbb{R}^n$ *is a function:*

$$||\cdot|| : x \in \mathbb{R}^n \to ||x|| \in \mathbb{R}$$

that satisfies the following four properties.

1. $\forall\, x \in \mathbb{R}^n$, $||x|| \geq 0$,
2. $\forall\, x \in \mathbb{R}^n$, $||x|| = 0$ *if and only if* $x = 0$,
3. $\forall\, x \in \mathbb{R}^n$ *and* $\forall\, t \in \mathbb{R}$, $||tx|| = |t|(||x||)$,
4. $\forall\, x \in \mathbb{R}^n$ *and* $\forall\, \overline{x} \in \mathbb{R}^n$, $||x + \overline{x}|| \leq ||x|| + ||\overline{x}||$.

Proposition 2 *Let* $||\cdot||$ *be a norm on* $\mathbb{R}^n$*, the function defined below is a distance on* $\mathbb{R}^n$*:*

$$\forall\, (x, \overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n, d(x, \overline{x}) = ||x - \overline{x}||.$$

Some basic norms on $\mathbb{R}^n$:

i) $||\cdot||_1 : \mathbb{R}^n \to \mathbb{R}$, defined by $||x||_1 = \sum_{h=1}^{n} |x_h|$,

ii) $||\cdot||_2 : \mathbb{R}^n \to \mathbb{R}$, defined by $||x||_2 = \sqrt{\sum_{h=1}^{n} (x_h)^2}$,

iii) $||\cdot||_\infty : \mathbb{R}^n \to \mathbb{R}$, defined by $||x||_\infty = \max\{|x_h| : h = 1, ..., n\}$.

Theorem 3 *All the norms on* $\mathbb{R}^n$ *are equivalent.*[1]

[1] Two norms $||\cdot||_a$ and $||\cdot||_b$ on $\mathbb{R}^n$ are equivalent if there exist $\alpha > 0$ and $\beta > 0$ such that $||x||_a \leq \alpha ||x||_b$ and $||x||_b \leq \beta ||x||_a$.

1.2 Continuity

f is a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$.

Definition 4 (Continuous function) *f is continuous at $\overline{x} \in X$ if*

$$\lim_{x \to \overline{x}} f(x) = f(\overline{x})$$

f is continuous on X if f is continuous at every point $\overline{x} \in X$.

Exercise 5

1. f is continuous at $\overline{x} \in X$ if and only if for every open ball J of center $f(\overline{x})$ there exists an open ball B of center $\overline{x}$ such that $f(B \cap X) \subseteq J$.

2. f is continuous at $\overline{x} \in X$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - \overline{x}\| < \delta$ and $x \in X \Longrightarrow |f(x) - f(\overline{x})| < \varepsilon$.

Proposition 6 (Sequentially continuous function) *f is continuous at $\overline{x} \in X$ if and only if f is sequentially continuous at $\overline{x}$, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to \overline{x}$, we have that*

$$f(x_n) \to f(\overline{x})$$

1.3 Differentiability

$X \subseteq \mathbb{R}^n$ is an **open** set, f is a function from X to $\mathbb{R}$.

Definition 7 (Differentiable function) *f is differentiable at $\overline{x} \in X$ if*

1. *all the partial derivatives of f at $\overline{x}$ exist,*

2. *there exists a function $E_{\overline{x}}$ defined in some open ball $B(0, \varepsilon) \subseteq \mathbb{R}^n$ such that for every $u \in B(0, \varepsilon)$,*

$$f(\overline{x} + u) = f(\overline{x}) + \mathrm{D}f(\overline{x}) \cdot u + \|u\| E_{\overline{x}}(u)$$

$$\text{where } \lim_{u \to 0} E_{\overline{x}}(u) = 0$$

f is differentiable on X if f is differentiable at every point $\overline{x} \in X$.

Exercise 8 If f is differentiable at $\overline{x}$, then f is continuous at $\overline{x}$.

Definition 9 (Directional derivative) *Let $v \in \mathbb{R}^n$, $v \neq 0$. The directional derivative $D_v f(\overline{x})$ of f at $\overline{x} \in X$ in the direction v is defined as*

$$\lim_{t \to 0^+} \frac{f(\overline{x} + tv) - f(\overline{x})}{t}$$

if this limit exists and it is finite.

Proposition 10 (Differentiable function/Directional derivative) *If f is differentiable at $\overline{x} \in X$, then for every $v \in \mathbb{R}^n$ with $v \neq 0$,*

$$D_v f(\overline{x}) = \mathrm{D} f(\overline{x}) \cdot v$$

1.4 Compactness

X is a subset of $\mathbb{R}^n$.

Proposition 11 (Compact set/Subsequences) *X is compact if and only if for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ there exists a subsequence $(x_{n_k})_{k \in \mathbb{N}}$ of the sequence $(x_n)_{n \in \mathbb{N}}$ such that $(x_{n_k})_{k \in \mathbb{N}}$ converges to some point $\overline{x} \in X$.*[2]

Proposition 12 (Compact set) *X is compact if and only if it is closed and bounded.*

Definition 13 (Closed set) *X is closed if its complement $\mathcal{C}(X) := \mathbb{R}^n \setminus X$ is open.*

Proposition 14 (Sequentially closed) *X is closed if and only if it is sequentially closed, that is, for every sequence $(x_n)_{n \in \mathbb{N}} \subseteq X$ such that $x_n \to \overline{x}$, we have*

$$\overline{x} \in X$$

Definition 15 (Bounded set) *X is bounded if it is included in some ball, that is, there exists $\varepsilon > 0$ such that for all $x \in X$, $\|x\| < \varepsilon$.*

[2] Let $(x_n)_{n \in \mathbb{N}}$ be a sequence and $(n_k)_{k \in \mathbb{N}}$ be a strictly increasing sequence of natural numbers. The composed sequence $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of the sequence $(x_n)_{n \in \mathbb{N}}$.

2 Extreme Value Theorem

Theorem 16 (Extreme Value Theorem/Weierstrass Theorem) *Let f be a function from $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$. If X is a non-empty compact set and f is continuous on X, then*

- $\exists\, x^* \in X$ *such that* $f(x^*) \geq f(x)$ *for all* $x \in X$*, and*
- $\exists\, x^{**} \in X$ *such that* $f(x^{**}) \leq f(x)$ *for al* $x \in X$*.*

3 Constrained Optimization Problems

In this section, we provide necessary and sufficient conditions in terms of first order conditions for solving a maximization problem with constraints. In Subsection 3.1, we focus on the case of inequality constraints. In Subsection 3.2, we extend the analysis to the case of both equality and inequality constraints.

3.1 The case of inequality constraints: Karush–Kuhn–Tucker Theorems

In this subsection, we assume that $C \subseteq \mathbb{R}^n$ is **convex and open**, and that the following functions f and g_j with $j = 1, ..., m$ are **differentiable** on C.

$$f : x \in C \subseteq \mathbb{R}^n \longrightarrow f(x) \in \mathbb{R} \text{ and}$$
$$g_j : x \in C \subseteq \mathbb{R}^n \longrightarrow g_j(x) \in \mathbb{R},\ \forall\, j = 1, ..., m$$

Maximization problem

$$\begin{array}{ll} \max & f(x) \\ x \in C & \\ \text{subject to} & g_j(x) \geq 0,\ \forall\, j = 1, ..., m \end{array} \tag{1}$$

where f is the *objective* function, and g_j with $j = 1, ..., m$ are the *constraint* functions.

The **Karush–Kuhn–Tucker conditions** associated with problem (1) are given below

$$\begin{cases} \mathrm{D}f(x) + \displaystyle\sum_{j=1}^{m} \lambda_j \mathrm{D}g_j(x) = 0 \\ \lambda_j \geq 0,\ \forall\, j = 1, ..., m \\ \lambda_j g_j(x) = 0,\ \forall\, j = 1, ..., m \\ g_j(x) \geq 0,\ \forall\, j = 1, ..., m \end{cases} \tag{2}$$

where for every $j = 1, ..., m$, $\lambda_j \in \mathbb{R}$ is called *Lagrange multiplier* associated with the inequality constraint g_j.

Definition 17 *Let $x^* \in C$, we say that the constraint j is* ***binding*** *at x^* if $g_j(x^*) = 0$. We denote*

1. *$J(x^*)$ the set of all binding constraints at x^*, that is*

$$J(x^*) := \{j = 1, ..., m : g_j(x^*) = 0\}$$

2. *$m^* \leq m$ the number of elements of $J(x^*)$ and*

3. *$g^* := (g_j)_{j \in J(x^*)}$ the following mapping*

$$g^* : x \in C \subseteq \mathbb{R}^n \longrightarrow g^*(x) = (g_j(x))_{j \in J(x^*)} \in \mathbb{R}^{m^*}$$

Theorem 18 (Karush–Kuhn–Tucker necessary conditions) *Let x^* be a solution to problem (1). Assume that* ***one*** *of the following conditions is satisfied.*

1. *For all $j = 1, ..., m$, g_j is a* ***linear or affine*** *function.*

2. **Slater's Condition :**

   - *for all $j = 1, ..., m$, g_j is a* ***concave*** *function* ***or*** *g_j is a* ***quasi-concave*** *function with $\mathrm{D}g_j(x) \neq 0$ for all $x \in C$, and*
   - *there exists $\overline{x} \in C$ such that $g_j(\overline{x}) > 0$ for all $j = 1, ..., m$.*

3. **Rank Condition :** $\operatorname{rank} \mathrm{D}g^*(x^*) = m^* \leq n$

Then, there exists $\lambda^* = (\lambda_1^*, ..., \lambda_j^*, ..., \lambda_m^*) \in \mathbb{R}_+^m$ such that (x^*, λ^*) satisfies the Karush–Kuhn–Tucker Conditions (2).

Theorem 19 (Karush–Kuhn–Tucker sufficient conditions) *Suppose that there exists $\lambda^* = (\lambda_1^*, ..., \lambda_j^*, ..., \lambda_m^*) \in \mathbb{R}_+^m$ such that $(x^*, \lambda^*) \in C \times \mathbb{R}_+^m$ satisfies the Karush–Kuhn–Tucker Conditions (2). Assume that*

1. *f is a* ***concave*** *function* ***or*** *f is a* ***quasi-concave*** *function with $\mathrm{D}f(x) \neq 0$ for all $x \in C$, and*

2. *g_j is a* ***quasi-concave*** *function for all $j = 1, ..., m$.*

Then, x^* is a solution to problem (1).

3.2 The case of both equality and inequality constraints: generalized Karush–Kuhn–Tucker Theorems

In this subsection, we assume that $C \subseteq \mathbb{R}^n$ is **convex and open**, and that the following functions f, g_j with $j = 1, ..., m$, and h_k with $k = 1, ..., \ell$ are **differentiable** on C.

$$f : x \in C \subseteq \mathbb{R}^n \longrightarrow f(x) \in \mathbb{R}$$
$$g_j : x \in C \subseteq \mathbb{R}^n \longrightarrow g_j(x) \in \mathbb{R},\ \forall\, j = 1, ..., m \text{ and}$$
$$h_k : x \in C \subseteq \mathbb{R}^n \longrightarrow h_k(x) \in \mathbb{R},\ \forall\, k = 1, ..., \ell$$

Maximization problem

$$\begin{array}{ll} \max & f(x) \\ x \in C & \\ \text{subject to} & \begin{cases} g_j(x) \geq 0, & \forall\, j = 1, ..., m \\ h_k(x) = 0, & \forall\, k = 1, ..., \ell \end{cases} \end{array} \tag{3}$$

where f is the *objective* function, g_j with $j = 1, ..., m$ are the *inequality constraint* functions, and h_k with $k = 1, ..., \ell$ are the *equality constraint* functions.

Remark 20 *We remark that all the results given below come from the simple observation that any equality constraint can be written as two inequality constraints. More precisely,*

$$h_k(x) = 0 \Longleftrightarrow h_k(x) \geq 0 \text{ and } -h_k(x) \geq 0$$

The generalized **Karush–Kuhn–Tucker conditions** associated with problem (3) are given below

$$\begin{cases} \mathrm{D}f(x) + \sum_{j=1}^{m} \lambda_j \mathrm{D}g_j(x) + \sum_{k=1}^{\ell} \mu_k \mathrm{D}h_k(x) = 0 \\ \lambda_j \geq 0,\ \forall\, j = 1, ..., m \\ \lambda_j g_j(x) = 0,\ \forall\, j = 1, ..., m \\ g_j(x) \geq 0,\ \forall\, j = 1, ..., m \\ h_k(x) = 0,\ \forall\, k = 1, ..., \ell \end{cases} \tag{4}$$

where for every $j = 1, ..., m$, $\lambda_j \in \mathbb{R}$ is the *Lagrange multiplier* associated with the inequality constraint g_j and for every $k = 1, ..., \ell$, $\mu_k \in \mathbb{R}$ is the *Lagrange multiplier* associated with the equality constraint h_k.

Theorem 21 (Karush–Kuhn–Tucker necessary conditions) *Let x^* be a solution to problem (3). Assume that* ***one*** *of the following conditions is satisfied.*

1. *For all $j = 1, ..., m$ and for all $k = 1, ..., \ell$, g_j and h_k are* ***linear or affine*** *functions.*

2. **Rank Condition :** $\text{rank} \begin{bmatrix} \mathrm{D}g^*(x^*) \\ \mathrm{D}h(x^*) \end{bmatrix} = m^* + \ell \leq n$

   where the mapping g^* is defined by point 3 of Definition 17 and h denotes the following mapping

$$h : x \in C \subseteq \mathbb{R}^n \longrightarrow h(x) = (h_1(x), ..., h_k(x), ..., h_\ell(x)) \in \mathbb{R}^\ell$$

Then, there exist $\lambda^* = (\lambda_1^*, ..., \lambda_j^*, ..., \lambda_m^*) \in \mathbb{R}_+^m$ and $\mu^* = (\mu_1^*, ..., \mu_k^*, ..., \mu_\ell^*) \in \mathbb{R}^\ell$ such that (x^*, λ^*, μ^*) satisfies the Karush–Kuhn–Tucker Conditions (4).

Theorem 22 (Karush–Kuhn–Tucker sufficient conditions) *Suppose that there exist $\lambda^* = (\lambda_1^*, ..., \lambda_j^*, ..., \lambda_m^*) \in \mathbb{R}_+^m$ and $\mu^* = (\mu_1^*, ..., \mu_k^*, ..., \mu_\ell^*) \in \mathbb{R}^\ell$ such that $(x^*, \lambda^*, \mu^*) \in C \times \mathbb{R}_+^m \times \mathbb{R}^\ell$ satisfies the Karush–Kuhn–Tucker Conditions (4). Assume that*

1. *f is a* ***concave*** *function* ***or*** *f is a* ***quasi-concave*** *function with $\mathrm{D}f(x) \neq 0$ for all $x \in C$,*

2. *g_j is a* ***quasi-concave*** *function for all $j = 1, ..., m$, and*

3. *h_k are* ***linear or affine*** *functions for all $k = 1, ..., \ell$.*

Then, x^* is a solution to problem (3).

4 The Implicit Function Theorem

$V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ are open sets, $F := (F_1, \ldots, F_i, \ldots, F_n)$ is a mapping from $V \times W$ to $\mathbb{R}^n$ and $(v^*, w^*) \in V \times W$.

$$\mathrm{D}_v F(v^*, w^*) := \begin{bmatrix} \frac{\partial F_1}{\partial v_1}(v^*, w^*) & \ldots & \frac{\partial F_1}{\partial v_h}(v^*, w^*) & \ldots & \frac{\partial F_1}{\partial v_n}(v^*, w^*) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_i}{\partial v_1}(v^*, w^*) & \ldots & \frac{\partial F_i}{\partial v_h}(v^*, w^*) & \ldots & \frac{\partial F_i}{\partial v_n}(v^*, w^*) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_n}{\partial v_1}(v^*, w^*) & \ldots & \frac{\partial F_n}{\partial v_h}(v^*, w^*) & \ldots & \frac{\partial F_n}{\partial v_n}(v^*, w^*) \end{bmatrix}_{n \times n}$$

denotes the $\underline{\text{partial Jacobian matrix of } F \text{ with respect to } v}$ at (v^*, w^*),

$$\mathrm{D}_w F(v^*, w^*) := \begin{bmatrix} \frac{\partial F_1}{\partial w_1}(v^*, w^*) & \ldots & \frac{\partial F_1}{\partial w_k}(v^*, w^*) & \ldots & \frac{\partial F_1}{\partial w_m}(v^*, w^*) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_i}{\partial w_1}(v^*, w^*) & \ldots & \frac{\partial F_i}{\partial w_k}(v^*, w^*) & \ldots & \frac{\partial F_i}{\partial w_m}(v^*, w^*) \\ \vdots & & \vdots & & \vdots \\ \frac{\partial F_n}{\partial w_1}(v^*, w^*) & \ldots & \frac{\partial F_n}{\partial w_k}(v^*, w^*) & \ldots & \frac{\partial F_n}{\partial w_m}(v^*, w^*) \end{bmatrix}_{n \times m}$$

denotes the $\underline{\text{partial Jacobian matrix of } F \text{ with respect to } w}$ at (v^*, w^*).

Theorem 23 (The Implicit Function Theorem) Let $V \subseteq \mathbb{R}^n$ and $W \subseteq \mathbb{R}^m$ be open sets. Let F be a mapping from $V \times W$ to $\mathbb{R}^n$. Assume that

- F is C^1 (i.e., F is continuously differentiable [3]),
- $(v^*, w^*) \in V \times W$,

$$F(v^*, w^*) = 0 \text{ and } \operatorname{rank}\, \mathrm{D}_v F(v^*, w^*) = n$$

Then, there exist open sets $V^* \subseteq V$, $W^* \subseteq W$ containing v^* and w^*, respectively, and a C^1 mapping f from W^* to V^* such that

$$(v, w) \in V^* \times W^* \text{ and } F(v, w) = 0 \iff v = f(w)$$

(so that, in particular $v^* = f(w^*)$), and

$$\mathrm{D}f(w^*) = -[\mathrm{D}_v F(v^*, w^*)]^{-1} \mathrm{D}_w F(v^*, w^*)$$

or equivalently, the directional derivative $\Delta v = \mathrm{D}f(w^*)\Delta w$ is the unique solution to the system of linear equations

$$\mathrm{D}_v F(v^*, w^*)\Delta v + \mathrm{D}_w F(v^*, w^*)\Delta w = 0$$

(given the direction $\Delta w \neq 0$).

[3] F is continuously differentiable if all the first order partial derivatives exist and are continuous.

5 Continuous correspondences

In this section S is a subset of $\mathbb{R}^m$, T is a **compact** subset of $\mathbb{R}^n$, and ϕ is a correspondence from S to T.

Notice that if the set T is not compact, one may still be able to replace T by some compact set without altering the problem, and then use the results below.

Definition 24 (Upper semicontinuity) *The correspondence ϕ is upper semicontinuous at $\overline{v} \in S$ if for any sequence $(v^n, w^n)_{n\in\mathbb{N}}$ such that $(v^n, w^n) \in S \times \phi(v^n)$ for every $n \in \mathbb{N}$ and $(v^n, w^n) \to (\overline{v}, \overline{w})$, then $\overline{w} \in \phi(\overline{v})$.*

Definition 25 (Lower semicontinuity) *The correspondence ϕ is lower semicontinuous at $\overline{v} \in S$ if for any sequence $(v^n)_{n\in\mathbb{N}} \subseteq S$ such that $v^n \to \overline{v}$,*

if $\overline{w} \in \phi(\overline{v})$, then there exists a sequence $(w^n)_{n\in\mathbb{N}}$ such that $w^n \in \phi(v^n)$ for every $n \in \mathbb{N}$ and $w^n \to \overline{w}$.

Definition 26 (Continuity) *The correspondence ϕ is continuous at $\overline{v} \in S$ if it is upper and lower semicontinuous at $\overline{v}$.*

Remark 27 *Assume that for every $v \in S$, $\phi(v)$ is non-empty and single-valued (i.e., ϕ is a function). Then,*

1. *the definition of lower semicontinuity at $\overline{v} \in S$ is obviously equivalent to the definition of sequential continuity at $\overline{v}$ for a function,*
2. *one can prove that, because of the compactness of T, the definition of upper semicontinuity at $\overline{v} \in S$ is equivalent to the definition of sequential continuity at $\overline{v}$ for a function.*

Let f be a function from $S \times T$ to $\mathbb{R}$. Given $v \in S$, consider the problem of maximizing $f(v, \cdot)$ over the set $\phi(v)$. Denote $\mu(v) \subseteq T$ the set of solutions of this problem, and $g(v)$ the value of the maximum of $f(v, \cdot)$ on $\phi(v)$.

Theorem 28 (Berge's Theorem) *If the function f is continuous on $S \times T$, and if the correspondence ϕ is continuous at $v \in S$, then the correspondence μ is upper semicontinuous at v, and g is continuous at v.*

6 Fixed Point Theorems

Theorem 29 (Brouwer's Fixed Point Theorem) *If S is a non-empty, compact, convex subset of $\mathbb{R}^n$, and if f is a continuous function from S to S, then f has a fixed point, i.e. there exists $p^* \in S$ such that $p^* = f(p^*)$.*

Theorem 30 (Kakutani's Fixed Point Theorem) *If S is a non-empty, compact, convex subset of $\mathbb{R}^m$, and if ϕ is an upper semicontinuous correspondence from S to S such that for all $v \in S$ the set $\phi(v)$ is non-empty and convex, then ϕ has a fixed point, i.e. there exists $v^* \in S$ such that $v^* \in \phi(v^*)$.*

7 Regular Values and Transversality

The theory of general economic equilibrium from a differentiable prospective is based on results from differential topology. The following results, as well as generalizations on these issues, can be found for instance in **?**, Mas-Colell (1985) and Villanacci et al. (2002).

M and N are two C^r manifolds of dimensions m and n respectively, $f : M \to N$ is a C^r mapping.

Definition 31 (Regular Value) *$y \in N$ is a regular value for f if for every $\xi^* \in f^{-1}(y)$, the differential mapping $Df(\xi^*)$ is onto.*

Theorem 32 (Regular Value Theorem) *Let M, N be C^r manifolds of dimensions m and n, respectively. Let $f : M \to N$ be a C^r function. Assume $r > \max\{m - n, 0\}$. If $y \in N$ is a regular value for f, then*

1. *if $m < n$, $f^{-1}(y) = \emptyset$,*
2. *if $m \geq n$, either $f^{-1}(y) = \emptyset$, or $f^{-1}(y)$ is an $(m - n)$-dimensional submanifold of M.*

Corollary 33 *Let M, N be C^r manifolds of the same dimension. Let $f : M \to N$ be a C^r function. Assume $r \geq 1$. Let $y \in N$ a regular value for f such that $f^{-1}(y)$ is non-empty and compact. Then, $f^{-1}(y)$ is a finite subset of M.*

The following results is a consequence of Sard's Theorem for manifolds.

Theorem 34 (Transversality Theorem) *Let M, Ω and N be C^r manifolds of dimensions m, p and n, respectively. Let $f : M \times \Omega \to N$ be a C^r function. Assume $r > \max\{m-n, 0\}$. If $y \in N$ is a regular value for f, then there exists a full measure subset Ω^* of Ω such that for any $\omega \in \Omega^*$, $y \in N$ is a regular value for f_ω, where*

$$f_\omega : \xi \in M \to f_\omega(\xi) := f(\xi, \omega) \in N$$

Definition 35 *Let (X, d) and (Y, d') be two metric spaces. A function $\pi : X \to Y$ is proper if it is continuous and one among the following conditions holds true.*

1. *π is closed and $\pi^{-1}(y)$ is compact for each $y \in Y$,*

2. *if K is a compact subset of Y, then $\pi^{-1}(K)$ is a compact subset of X,*

3. *if $(x^n)_{n\in\mathbb{N}}$ is a sequence in X such that $(\pi(x^n))_{n\in\mathbb{N}}$ converges in Y, then $(x^n)_{n\in\mathbb{N}}$ has a converging subsequence in X.*

The conditions above are equivalent.

8 Homotopy Theorem

Theorem 36 (Homotopy Theorem) *Let M and N be C^2 manifolds of the same dimension contained in euclidean spaces. Let $y \in N$ and $f, g : M \to N$ be two functions such that*

1. *f is continuous,*
2. *g is C^1, y is a regular value of g and $\#g^{-1}(y)$ is odd.*

Let H be a continuous homotopy from g to f such that $H^{-1}(y)$ is compact. Then, $f^{-1}(y)$ is compact and $f^{-1}(y) \neq \emptyset$.

9 Appendix: Concavity and Quasi-concavity

In this section, we assume that C is a **convex** subset of $\mathbb{R}^n$ and f is a function from C to $\mathbb{R}$.

9.1 Concavity

Definition 37 (Concave function) *f is concave if for all $t \in [0,1]$ and for all x and $\overline{x}$ in C,*

$$f(tx + (1-t)\overline{x}) \geq tf(x) + (1-t)f(\overline{x})$$

Proposition 38 *f is concave **if and only if** the set*

$$\{(x, \alpha) \in C \times \mathbb{R} : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^{n+1}$ (the set above is called hypograph of f).

Proposition 39 *C is **open** and f is **differentiable** on C. f is concave **if and only if** for all x and $\overline{x}$ in C,*

$$f(x) \leq f(\overline{x}) + \mathrm{D}f(\overline{x}) \cdot (x - \overline{x})$$

Proposition 40 *C is **open** and f is **twice continuously differentiable** on C.[4] f is concave **if and only if** for all $x \in C$ the Hessian matrix $\mathrm{D}^2 f(x)$ is negative semidefinite, that is for all $x \in C$*

$$v\mathrm{D}^2 f(x) v^T \leq 0, \ \forall\, v \in \mathbb{R}^n$$

Definition 41 (Strictly concave function) *f is strictly concave if for all $t \in]0,1[$ and for all x and $\overline{x}$ in C with $x \neq \overline{x}$,*

$$f(tx + (1-t)\overline{x}) > tf(x) + (1-t)f(\overline{x})$$

Proposition 42 *C is **open** and f is **differentiable** on C. f is strictly concave **if and only if** for all x and $\overline{x}$ in C with $x \neq \overline{x}$,*

$$f(x) < f(\overline{x}) + \mathrm{D}f(\overline{x}) \cdot (x - \overline{x})$$

[4] f is twice continuously differentiable if all the second order partial derivatives exist and are continuous. A very useful property of a twice continuously differentiable function is that its Hessian matrix is a symmetric matrix.

Proposition 43 *C is* ***open*** *and f is* ***twice continuously differentiable*** *on C.* ***If*** *for all $x \in C$ the Hessian matrix $\mathrm{D}^2 f(x)$ is negative definite, that is for all $x \in C$*

$$v\mathrm{D}^2 f(x) v^T < 0,\ \forall\, v \in \mathbb{R}^n,\ v \neq 0$$

then *f is strictly concave.*

9.2 Quasi-concavity

Definition 44 (Quasi-concave function) *f is quasi-concave if and only if for all $\alpha \in \mathbb{R}$ the set*

$$\{x \in C : f(x) \geq \alpha\}$$

is a convex subset of $\mathbb{R}^n$ (the set above is called upper contour set of f at α).

Proposition 45 *f is quasi-concave* ***if and only if*** *for all $t \in [0,1]$ and for all x and $\overline{x}$ in C,*

$$f(tx + (1-t)\overline{x}) \geq \min\{f(x), f(\overline{x})\}$$

Proposition 46 *C is* ***open*** *and f is* ***differentiable*** *on C. f is quasi-concave* ***if and only if*** *for all x and $\overline{x}$ in C,*

$$f(x) \geq f(\overline{x}) \Longrightarrow \mathrm{D}f(\overline{x}) \cdot (x - \overline{x}) \geq 0$$

Proposition 47 *C is* ***open*** *and f is* ***differentiable*** *on C.* ***If*** *f is quasi-concave $\underline{\text{and } \mathrm{D}f(x) \neq 0 \text{ for all } x \in C}$,* ***then*** *for all x and $\overline{x}$ in C with $x \neq \overline{x}$,*

$$f(x) > f(\overline{x}) \Longrightarrow \mathrm{D}f(\overline{x}) \cdot (x - \overline{x}) > 0$$

Proposition 48 *C is* ***open*** *and f is* ***twice continuously differentiable*** *on C.* ***If*** *f is quasi-concave,* ***then*** *for all $x \in C$ the Hessian matrix $\mathrm{D}^2 f(x)$ is negative semidefinite on $\mathrm{Ker}\ \mathrm{D}f(x)$, that is for all $x \in C$*

$$v \in \mathbb{R}^n \text{ and } \mathrm{D}f(x) \cdot v = 0 \Longrightarrow v\mathrm{D}^2 f(x) v^T \leq 0$$

Definition 49 (Strictly quasi-concave function) *f is strictly quasi-concave if and only if for all $t \in]0,1[$ and for all x and $\overline{x}$ in C with $x \neq \overline{x}$,*

$$f(tx + (1-t)\overline{x}) > \min\{f(x), f(\overline{x})\}$$

Proposition 50 *C is **open** and f is **differentiable** on C.*

1. ***If** for all x and $\overline{x}$ in C with $x \neq \overline{x}$,*

$$f(x) \geq f(\overline{x}) \Longrightarrow \mathrm{D}f(\overline{x}) \cdot (x - \overline{x}) > 0$$

***then** f is strictly quasi-concave.*

2. ***If** f is strictly quasi-concave and $\mathrm{D}f(x) \neq 0$ for all $x \in C$, **then** for all x and $\overline{x}$ in C with $x \neq \overline{x}$,*

$$f(x) \geq f(\overline{x}) \Longrightarrow \mathrm{D}f(\overline{x}) \cdot (x - \overline{x}) > 0$$

Proposition 51 *C is **open** and f is **twice continuously differentiable** on C. **If** for all $x \in C$ the Hessian matrix $\mathrm{D}^2 f(x)$ is negative definite on $\mathrm{Ker}\, \mathrm{D}f(x)$, that is for all $x \in C$*

$$v \in \mathbb{R}^n,\ v \neq 0 \text{ and } \mathrm{D}f(x) \cdot v = 0 \Longrightarrow v\mathrm{D}^2 f(x) v^T < 0$$

***then** f is strictly quasi-concave.*

A function which satisfies the property stated in Proposition 51 is called *differentiably strictly quasi-concave*. This assumption is often used in Consumer Theory. Note that the definition of differentiably strictly quasi-concave function is not the same that f is strictly quasi-concave plus f is differentiable.

Remark 52 *We remark that*

f linear or affine	$\Rightarrow$	*f concave*	$\Leftarrow$	*f strictly concave*
		$\Downarrow$		$\Downarrow$
		f quasi-concave	$\Leftarrow$	*f strictly quasi-concave*

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