Logic and Sets
Mid-term exam 2023 (1h)

Name: QEM/MMEF

Exercise 1 (9pts)

Indicate for each of the following assertions if they are true (T) or false (F).

1. If Roma is in France, then $2 + 2 = 4$
2. Dogs are blue if and only if cats are red
3. The negation of $p \to q$ is $\neg p \to \neg q$
4. The negation of $(p \vee q) \wedge (r \vee s)$ is $(\neg p \wedge \neg q) \vee (\neg r \wedge \neg s)$
5. The negation of $[\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, f(x) = g(y) \text{ and } x \geq y]$ is $[\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, f(x) \neq g(y) \text{ or } x < y]$
6. The negation of $[\forall n$ in the set of even numbers, $\exists p \in \mathbb{N}, \frac{1}{2}n = p]$ is $[\exists n$ in the set of odd numbers, $\forall p \in \mathbb{N}, \frac{1}{2}n \neq p]$.
7. $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, y = 2.$
8. $\exists z \in \mathbb{R}, \forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x^2 + y^2 = z^2$
9. $(A \setminus B) \setminus (A \cap B) = \emptyset$
10. $A \Delta A = \emptyset$
11. $A \setminus (B \cup C) = (A \setminus B) \cup (A \setminus C)$
12. $\mathcal{P}(\{1, \{1\}\}) = \{\emptyset, \{1\}, \{\{1\}\}, \{1, \{1\}\}\}$
13. $A \cap B = \emptyset \Leftrightarrow A \setminus B = A$
14. $\{1, 2, \{3\}\} \cup \{3\} = \{1, 2, 3\}$
15. $\{a, \{a, b\}\} \cap \{a\} = a$
16. $\mathcal{P}(\emptyset) = \emptyset$
17. The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ is surjective.
18. The function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2 + y^2$ is injective.

Exercise 2 (8pts)

1. Show that $n^2 + 2n + 1$ is even if and only if n is odd.

2. Show by contradiction that there is no $n \in \mathbb{N}$ which [is divisible by 8] and [is the square of a prime number].

3. Show by induction that for any $n \geq 1$, $\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

4. Show that if $f : X \to Y$ and A, B are subsets of X, then $f(A) \setminus f(B) \subseteq f(A \setminus B)$. Is the converse true (give a counterexample if not)?

Exercise 3 (3pts)

Write the truth tables of $((p \wedge q) \vee \neg p) \to q$ and $((p \to q) \to r) \wedge p) \vee r)$.