Microeconomics 1A. September 18, 2023.

Exercise 1: Assume $\succsim$ is rational.
Show that:

(i) $\succsim$ is reflexive.
(ii) $\sim$ reflexive, transitive and symmetric.
(iii) $\succ$ irreflexive, transitive
(iv) $x \succ y \succsim z \longrightarrow x \succ z$.

(i)
Start by recalling each definition.

* $\succsim$ rational $\Longrightarrow$ $\succsim$ transitive and complete.

transitive $\Updownarrow$ $\forall x, y, z$, $x \succsim y \succsim z \Rightarrow x \succsim z$.

complete $\Updownarrow$ $\forall x, y$. either $x \succsim y$ or $y \succsim x$ or both.

* $\succsim$ reflexive $\Updownarrow$ $\forall x \quad x \succsim x$.

Assume $x \not\succsim x$, then, by completeness of $\succsim$ (because $\succsim$ is rational), we must have $x \precsim x$, which is equivalent to $x \succsim x$ (read backward).
$\longrightarrow$ CONTRADICTION.

hence we must have $x \succsim x \ \forall x$. $\square$

(ii) (a) show $\sim$ reflexive :

$\gtrsim$ is reflexive (by (i)), so $\forall x$, $x \gtrsim x$.

read backward, that's $x \lesssim x$.

so we have $x \gtrsim x$ <u>and</u> $x \lesssim x$ $\Longrightarrow$ $x \sim x$.

recalling that the definition of $\sim$ was :

$$x \sim y \iff \begin{matrix} x \gtrsim y \\ \underline{\text{and}}\ x \lesssim y \end{matrix}$$

(b) show $\sim$ transitive :

Assume $x \sim y$ and $y \sim z$, we want to show that $x \sim z$.

Explicitly the definitions :

$x \sim y \iff x \gtrsim y$ and $x \lesssim y$

and

$y \sim z \iff y \gtrsim z$ and $y \lesssim z$

these two together imply that $x \gtrsim z$ (by transitivity of $\gtrsim$, because $\gtrsim$ is rational)

likewise these two imply $x \lesssim z$ (by trans. of $\gtrsim$).

hence we have ($x \gtrsim z$ and $x \lesssim z$) $\Longrightarrow$ $x \sim z$. $\square$

(c) show $\sim$ symmetric, i.e. $x \sim y \iff y \sim x$.

$$x \sim y \iff \begin{pmatrix} x \gtrsim y \\ \text{and} \\ x \lesssim y \end{pmatrix} \Longleftrightarrow \begin{pmatrix} y \gtrsim x \\ \text{and} \\ y \lesssim x \end{pmatrix} \Longleftrightarrow y \sim x \quad \square$$

read backward... and interchanging the first and second...

(iii) (a) show $\succ$ is irreflexive. i.e. $x \not\succ x$.

Def of $\succ$: $\boxed{x \succ y \iff x \succsim y \text{ and } x \not\precsim y}$

Assume $x \succ x \iff x \succsim x$ and $x \not\precsim x$

$\Updownarrow$

contradiction $\rightarrow$ $x \not\succsim x$.

hence we cannot have $x \succ x$.

so $x \not\succ x$.

(b) show $\succ$ transitive

let $x \succ y$ and $y \succ z$, we want to show that it implies $x \succ z$.
ie $x \succsim z$ and $x \not\precsim z$.

$\Updownarrow$ (o) $\qquad \Updownarrow$ (#)

$\begin{pmatrix} x \succsim y \\ \text{and} \\ x \not\precsim y \end{pmatrix} \qquad \begin{bmatrix} y \succsim z \\ \text{and} \\ y \not\precsim z \end{bmatrix}$ $\longrightarrow$ by transitivity of $\succsim$, those two imply $x \succsim z$. (*)

BUT. we cannot deduce $x \not\precsim z$ from $x \not\precsim y$ and $y \not\precsim z$ ...

So ..., assume $x \not\succ z \iff$ NOT ($x \succsim z$ and $x \not\precsim z$).

$\iff x \not\succsim z$ or $x \precsim z$

this one is not possible, see (*)

3/7

... is $x \lesssim z$ possible?

if $x \lesssim z$ was true, together with $z \lesssim y$ (consequence of $y \succ z$, ($\square$)),

then, by transitivity of $\succsim$, we would have $x \lesssim y$, but this <u>contradicts</u> $x \succ y$ (see ($\circ$)).

(iv) show $x \succ y \succsim z \implies x \succ z$.

$$\underbrace{x \succ y}_{(\circ\circ)} \qquad \underbrace{y \succsim z}_{(\square\square)}$$

$(\circ\circ) \iff$ $x \succsim y$ and $x \not\precsim y$

these two together, by transitivity of $\succsim$, imply $x \succsim z$. ($\ast\ast$).

can we show that $x \not\precsim z$ directly?
no, we cannot. Let's try by contradiction again ...

Suppose $x \not\succ z \iff$ NOT ($x \succsim z$ and $x \not\precsim z$).

$\iff \underbrace{x \not\succsim z}$ or $\underbrace{x \precsim z}$.

that one contradicts ($\ast\ast$).

and that one, together with ($\square\square$) $y \succsim z$,

imply (by trans. of $\succsim$) that $x \precsim y$.

which contradicts the hypothesis that $x \succ y$ (see ($\circ\circ$)) $\square$

Exercice 2: The lexicographic preference is defined by: (on $\mathbb{R}^2_+$)

$$x \succsim y \iff \begin{cases} x_1 > y_1 \\ \text{or} \\ (x_1 = y_1 \text{ and } x_2 \geq y_2) \end{cases}$$

$(x = (x_1, x_2))$.

$\longrightarrow$ show that $\succsim$ is rational.

Def: $\succsim$ rational $\iff$ $\succsim$ transitive and complete.

* let's show completeness first:

completeness means that $\forall x, y$, either $x \succsim y$ or $x \precsim y$ or both.

so, if $x \not\succsim y$, we must have $x \precsim y$!

$\Longrightarrow$ Assume $x \not\succsim y$, we want to show that $x \precsim y$ (where $\succsim$ is the lexicographic pref.)

... $x \not\succsim y \iff \text{NOT} \begin{cases} x_1 > y_1 \\ \text{or} \\ (x_1 = y_1 \text{ and } x_2 \geq y_2) \end{cases}$

$\iff (x_1 \leq y_1)$ and $(x_1 \neq y_1$ or $x_2 < y_2)$

"and" behaves like a multiplication.
and "or" like an addition.

("and" is distributive w.r.t. "or"),

so:

$$(x_1 \leq y_1) \text{ and } (x_1 \neq y_1 \text{ or } x_2 < y_2)$$

$$\iff \underbrace{(x_1 \leq y_1 \text{ and } x_1 \neq y_1)}_{\Updownarrow \; x_1 < y_1} \text{ or } (x_1 \leq y_1 \text{ and } x_2 < y_2)$$

removing the redundancy, because $x_1 \leq y_1$ means $x_1 < y_1$ or $x_1 = y_1$

falls in the lhs.

$$\iff (x_1 < y_1) \text{ or } (x_1 = y_1 \text{ and } x_2 < y_2).$$

$$\iff y \gtrsim x \text{ if } x \neq y.$$

because of the strict inequality there...

But if $x = y$, then $x_1 = y_1$ and $x_2 = y_2$.

$$\Downarrow$$

$$x_1 = y_1 \text{ and } x_2 \leq y_2.$$

$$\Updownarrow$$

so we also have

$$y \gtrsim x \quad !$$

from this we also deduced the reflexivity of the lexicograph. prof!

$\square$

* transitiveness: NOT DONE IN CLASS.

Let $x \succsim y$ and $y \succsim z$.

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{cases} x_1 > y_1 \\ \text{or} \\ x_1 = y_1 \text{ and } x_2 \geq y_2 \end{cases} \text{ and } \begin{cases} y_1 > z_1 \\ \text{or} \\ y_1 = z_1 \text{ and } y_2 \geq z_2 \end{cases}$$

$\Updownarrow$ distributing "and" w.r.t. "or".

$$\begin{cases} (x_1 > y_1 \text{ and } y_1 > z_1) \Longrightarrow x_1 > z_1 \text{ (trans.)} \\ \text{or} \\ (x_1 > y_1 \text{ and } y_1 = z_1 \text{ and } y_2 \geq z_2 \nearrow \text{also} \\ \text{or} \\ (x_1 = y_1 \text{ and } x_2 \geq y_2 \text{ and } y_1 > z_2 \nearrow \text{also} \end{cases}$$

or

$$\underbrace{(x_1 = y_1 \text{ and } x_2 \geq y_2 \text{ and } y_1 = z_1 \text{ and } y_2 \geq z_2)}$$

$$\Downarrow$$

$x_1 = z_1$ and $x_2 \geq z_2$ (trans.).

hence $x \succsim z$. hence $\succsim$ transitive $\square$.

$\Longrightarrow$ $\succsim$ is transitive and complete, so it is rational! $\square$.

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EXTRA.

It appeared that many of you were confused by the notation $\succsim$ to refer to a preference relation.

Note that $\succsim$ was calligraphic, not to be confused with $\geq$, defined in the usual way on $\mathbb{R}^N$ as $x \geq y$.
$\iff \forall i = 1, \dots, N.\ x_i \geq y_i$.

If you know $\geq$ by heart, especially on $\mathbb{R}$, the real line, you should not extrapolate your knowledge of $\geq$ (a very specific case) to the broader concept of a preference relation, $\succsim$.

To avoid the confusion and more easily understand the difference between the two, let's note our generally defined preference relation $\mathcal{P}$.

Using that notation, let's rewrite the definition of a preference relation and prove one of our proposals.

Def: * $\mathcal{P}$ is a rational preference relation iff it is transitive and complete.

*a preference relation $\mathcal{P}$ is a binary relation allowing the comparison of two elements of a set.

P is defined on a set X. In most of the course, we assume $X \subseteq \mathbb{R}^N$, but it could be anything, which is another reason why you should not extrapolate the properties of $\geq$ on $\mathbb{R}^N$ to P on X.

$\boxed{\begin{array}{l} \underline{\text{Def}}: \ast\ P\ \underline{\text{transitive}} \text{ iff } \forall x,y,z \in X; \\ \qquad x P y \text{ and } y P z \Longrightarrow x P z. \\ \ast\ P\ \underline{\text{complete}} \text{ iff } \forall x,y \in X; \\ \qquad \text{either } x P y \text{ or } y P x \text{ (or both)}. \\ \qquad (\text{i.e every two elements in } X \text{ can be compared.}) \end{array}}$

Show $\boxed{\underline{\text{Prop}}\ (\text{i}): \ast\ P \text{ rational} \Longrightarrow P \text{ reflexive}}$

where $\boxed{\underline{\text{Def}}: \ast\ P\ \underline{\text{reflexive}} \text{ iff } \forall x \in X \quad x P x.}$

proof by contradiction: assume P irreflexive, i.e.

$x \not{P} x$, (I use the slash as a notation for "NOT $(x P x)$" or $\neg (x P x)$).

then, because of completeness, we must have

$x P x$ (where we have "exchanged the entries"),

2

... but the two entries are the same and we cannot have $x \not P x$ and $x P x$ at the same time $=$ CONTRADICTION.

Thus P reflexive. $\square$.

Let's also define the strict preference relation and the indifference relation using less ambiguous notations. (than $\succsim$ and $\sim$).

Def.
- we say that x is <u>strictly preferred</u> to y, and we note $x \mathcal{S} y$, iff. $(x P y$ and $y \not P x)$
- we say that x is <u>indifferent</u> to y, and we note $x \mathcal{I} y$, iff $(x P y$ and $y P x)$.

Now, let's prove again e.g.

<u>Prop</u> (iv) : $(x \mathcal{S} y$ and $y P z) \Longrightarrow x \mathcal{S} z$.

Now, hopefully you can no longer use your intuition to claim that the result is obvious (because it is not).

... let's prove it.

Assume ($x \mathcal{S} y$ and $y P z$.)

$\Updownarrow$ by def.

($x P y$ and $\underbrace{y \not P x}_{(*)}$).

these two together imply (by transitivity of P, because P is rational) that $x P z$.

now, to prove that our two starting hypotheses imply $x \mathcal{S} y$,

we further have to prove that $z \not P x$.

We cannot find any mean to prove it directly

let's see if we can more easily prove that the opposite statement cannot be true.

i.e., assume. NOT ($z \not P x$) i.e. $z P x$.

then, from transitivity of P, these two imply

($y P z$ <u>and</u> $z P x$) $\Longrightarrow$ $y P x$

which is a CONTRADICTION with (4) $y \not P x$.

$\square$

4/

Thus we've proven:

$$x \mathcal{S} y \text{ and } y \mathcal{P} z$$
$$\Longrightarrow (x \mathcal{P} z \text{ and } z \not\mathcal{P} x)$$
$$\text{i.e. } x \mathcal{S} z. \quad \square$$

Lastly, we often use (and that's what we'll do in the course) the notations

$\succsim$ for $\mathcal{P}$

$\succ$ for $\mathcal{S}$

and $\sim$ for $\mathcal{I}$.

One good reason for this is that:

$$\boxed{\underline{\text{Prop}}: \underset{(\mathcal{P})}{\succsim} \text{ is equivalent to } (\underset{(\mathcal{S})}{\succ} \ \underline{\underline{\text{or}}} \ \underset{(\mathcal{I})}{\sim}).}$$

<u>Proof</u>:

$\longrightarrow$

Assume $(x \mathcal{S} y \quad \text{or} \quad x \mathcal{I} y)$.

$$\Big((x P y \text{ and } y \not P x) \text{ or } (x P y \text{ and } y P x) \Big).$$

$$\Updownarrow$$

"factoring" by $x P y$. $\qquad x P y \text{ and } \underbrace{(y \not P x \text{ or } y P x)}$

which is equivalent to saying nothing. ((A or not A) is always true since A is either true or false).

$$\Updownarrow$$

$x P y$. $\square$.

The notations $\succsim$ and $\succ$ are also convenient because we can note

$$x \precsim y \text{ for } y P x$$

$$\text{and } x \prec y \text{ for } y \mathcal{S} x \quad \ldots$$

If needed, please try going through the exercises once again alone, but this time using the unambiguous notations P, $\mathcal{S}$ and $\mathcal{I}$.
It should help you lift the confusion you seemed to make with $\geq$, $>$ and $=$.