Microeconomics 1A. September 20, 2023.

Exercice 1: Show that. **Prop:**

STRONG MONOTONICITY.
$\Longrightarrow$ MONOTONICITY
$\Longrightarrow$ LOCAL NONSATIATION

Reminder: **Def:** $\succsim$ is monotonous iff.

$$x \gg y \Longrightarrow x \succ y$$

defined in $\mathbb{R}^l_+$ as

$$\boxed{x \gg y \Longleftrightarrow \forall i \; x_i > y_i} \quad (1)$$

our preference relations $\succsim$ will always be defined on $X \subseteq \mathbb{R}^l_+$

Def: $\succsim$ is strongly monotonous iff.

$$x > y \Longrightarrow x \succ y$$

defined in $\mathbb{R}^l_+$ as.

$$\boxed{x > y \Longleftrightarrow \forall i \; x_i \geq y_i \text{ and } \exists j \; x_j > y_i} \quad (2)$$

Note: $\gg$ and $>$ as I defined them in (1) and (2) are the notations used in Jean-Marc and Elena's class notes and Debreu's Theory of Value. However Mas-Colell and Antoine use a different notation.

They use $>$ for $\gg$, and for $>$, they instead write: $x \geq y$ and $x \neq y$.

where we agree commonly on the notation. $\boxed{x \geq y \iff \forall i \; x_i \geq y_i}$

But in $\mathbb{R}^l_+$, if we have $\forall i \; x_i \geq y_i$ and $x \neq y$, then it is equivalent to saying $\forall i \; x_i \geq y_i$ and $\exists x_j > y_j$.

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* We show STRONG. MONO $\Longrightarrow$ MONO:

Suppose $\succsim$ is strong. mono, i.e.

$$x > y \Longrightarrow x \succ y.$$

Let $x \gg y \iff \forall i \; x_i > y_i$
$\Longrightarrow \forall i \; x_i \geq y_i$ and $\exists_j \; x_j > y_j$
$\iff x > y$.

but $x > y \Longrightarrow x \succ y$, since we've assumed that $\succsim$ was strong. mono.

so. $x \gg y \Longrightarrow x \succ y$.

i.e. $\succsim$ is monotonous $\square$.

* Let's now show MONO $\Longrightarrow$ LOCAL NONSATIATION:

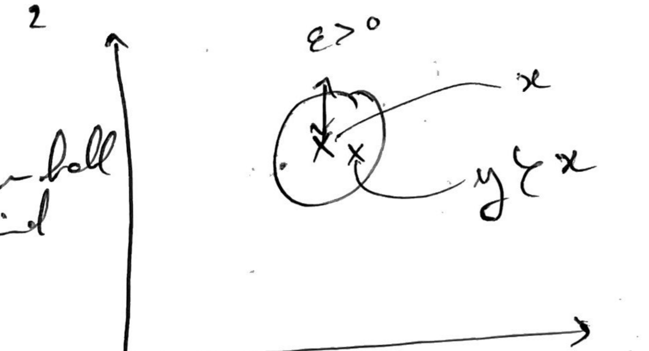
Reminder: $\boxed{\begin{array}{l}\text{Def: } \succsim \text{ is \underline{locally nonsatiated} iff} \\ \forall x, \forall \varepsilon > 0, \exists y \text{ s.th. } \underbrace{\|x - y\|} < \varepsilon \\ \text{and } y \succ x\end{array}}$

as small as we want!

that's the distance btw x and y according to some norm $\|.\|$.

The meaning of that definition is that, taking any x in the consumption set X, you can always find another consumption bundle y infinitely close to x that is strictly preferred to x.

graphically: in any open ball around any x, you can find an $y \succ x$.



Now for the demonstration ...

Assume $\succsim$ monotonous, i.e. $x \gg y \Rightarrow x \succ y$.
Take x.
Let $\varepsilon > 0$, then, for instance, the bundle

$$y = x + \begin{pmatrix} \varepsilon \\ | \\ \varepsilon \end{pmatrix} \frac{1}{2K}$$

is in the open ball of diameter ε around x, because it verifies

$$\| x - y \| < \varepsilon .$$

(if e.g. $\| \cdot \|$ was $\| \cdot \|_2$ defined as $\forall x$, $\| x \|_2 = \sqrt{\sum_i x_i^2}$).

but y also verifies $y \gg x$, since

$$\forall i \quad y_i > x_i$$

$$y_i = x_i + \underbrace{\frac{\varepsilon}{2K}}_{> 0} .$$

So, since $\succsim$ was assumed monotonous, we have $x \gg y \Rightarrow x \succ y$.

So $\succsim$ is locally nonsatiated. $\square$

Exercice C6 : LEXICOGRAPHIC Pref.

defined as. (on $\mathbb{R}^2_+$)
$$x \succsim y \iff \begin{cases} x_1 > y_1 \\ \text{or} \\ (x_1 = y_1 \text{ and } x_2 \geq y_2) \end{cases}.$$

Let y,

a) Draw the upper countour set $U(y)$, lower contour set $L(y)$ and indifference set $I(y)$.

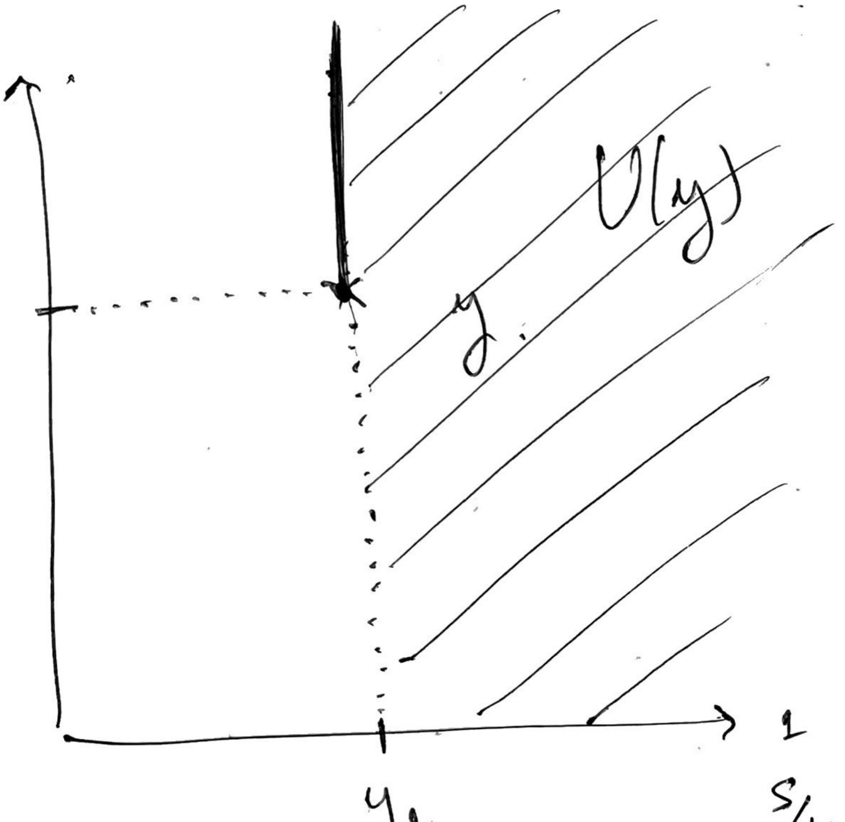
b). Show that $\succsim$ is strongly monotone and strictly convex but not continuous.

On Monday, we showed that $\succsim$ was RATIONAL. (i.e transitive and complete).

a) Pick a $y = (y_1, y_2)$ randomly.

$x \succsim y$ iff.

$x_1 > y_1$ (that is all points strictly at the right of the vertical line passing through y.)



or $(x_1 = y_1$ and $x_2 \geq y_2)$.

meaning x is on the vertical line passing through y

and above it! (including y).

Hence the upper contour set, defined as

Def: $\boxed{U(y) := \{x \, ; \, x \succsim y\}}$ is the dashed area draw on the previous page.

The other definitions are, taking y,

$L(y) := \{x ; \, x \precsim y\}$ is the lower contour set

and

$I(y) := \{x, \, x \sim y\}$ is the indifference set.

It is easy to show, because as we've proven on Monday, both $\succsim$ and $\sim$ are reflexive that.

$$y \in U(y), L(y), I(y).$$

it is also easy to see, straight from the definitions, that.

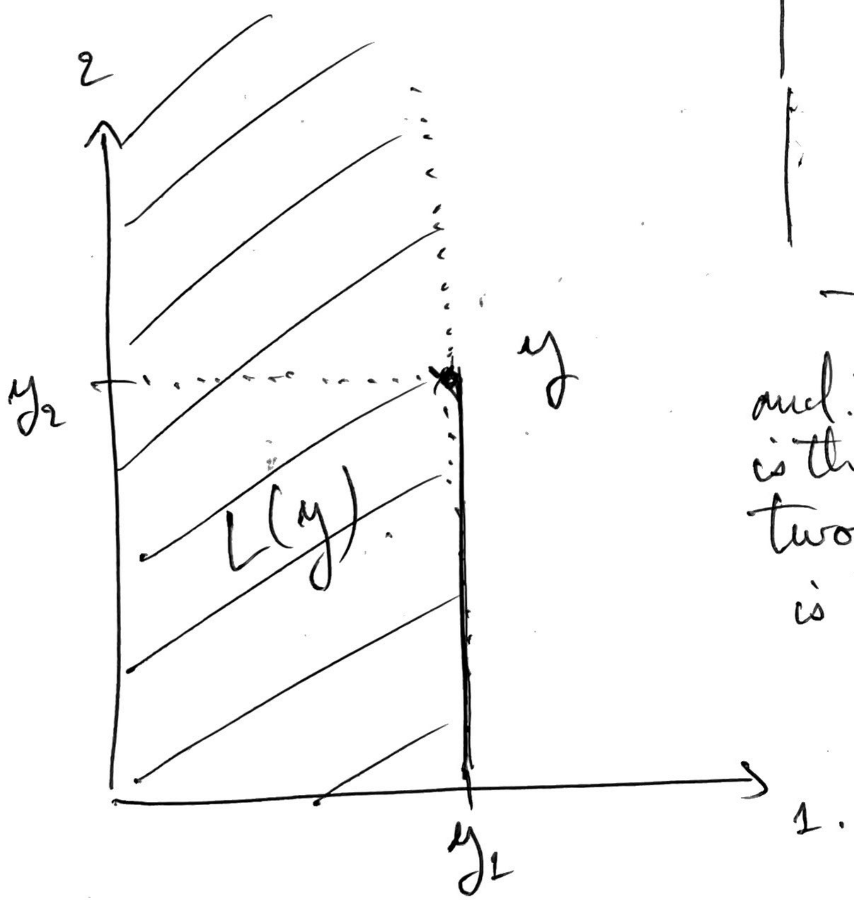
$$\boxed{I(y) = U(y) \cap L(y)},$$

and, e.g.,
$$\boxed{\begin{aligned}&\{x\,;\ x > y\} \\ &= U(y) \cap \bar{L}(y).\end{aligned}}$$

where $\bar{L}(y)$ denotes the complement of $L(y)$ in X, i.e.

$$\bar{L}(y) = \{x\,;\ x \nleq y\}$$

going back to the drawings... the lower contour set is



and the indifference set is the intersection of the two, which as we see is reduced to y.

We can show it formally.

Suppose $x \in I(y)$, i.e. $x \sim y$.

$\Longleftrightarrow$ $x \succsim y$ and $y \succsim x$.

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\left(\begin{array}{c} x_1 > y_1 \\ \text{or} \\ (x_1 = y_1 \text{ and } x_2 \geq y_2). \end{array} \right) \text{ and } \left(\begin{array}{c} y_1 > x_1 \\ \text{or} \\ (y_1 = x_1 \text{ and } y_2 \geq x_2) \end{array} \right).$$

$\Longleftrightarrow$ ... the only possible combination. (when you develop the above using that "and" is distributive w.r.t "or").

is.

$$\underbrace{x_1 = y_1 \text{ and } (\underbrace{x_2 \geq y_2 \text{ and } y_2 \geq x_2}_{\Longleftrightarrow\ x_2 = y_2}).}$$

$\Longleftrightarrow$ $x = y$.

Hence $I(y) = \{y\}$ is a singleton reduced to y. $\square$

b) * Show that $\succsim$ is strongly monotone.

$$(\text{ie. } x > y \implies x \succ y).$$

NOT DONE IN CLASS

let $x > y$, ie. $\left(\begin{array}{l}\forall i \;\; x_i \geq y_i \\ \text{and } \exists j \;\; x_j > y_j\end{array}\right)$.

$$\Updownarrow$$

$$\left(\text{either } \; x_1 > y_1 \; \text{ or } \; \left(\begin{array}{l} x_1 = y_1 \\ \text{and. } x_2 > y_2 \end{array}\right)\right).$$

$$\Updownarrow$$

$x \succsim y$ and. $x \not\precsim y$. since that one would require $x_2 \leq y_2$.

$$\Longleftrightarrow \; x \succ y \; . \qquad \square.$$

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* Show that. $\succsim$ is strictly convex.

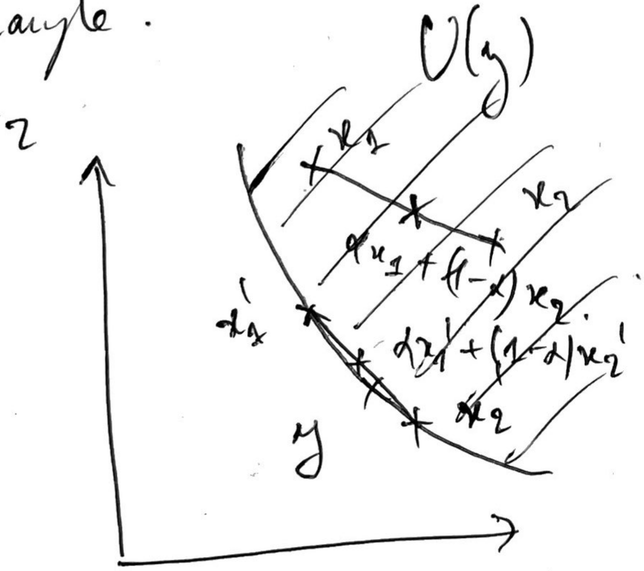
Reminder.

<u>Def</u>: * $\succsim$ is <u>CONVEX</u> iff.
$\forall\ x \succsim z$ and $y \succsim z$, we have that
$\alpha x + (1-\alpha) y \succsim z \quad \forall \alpha \in [0,1]$.

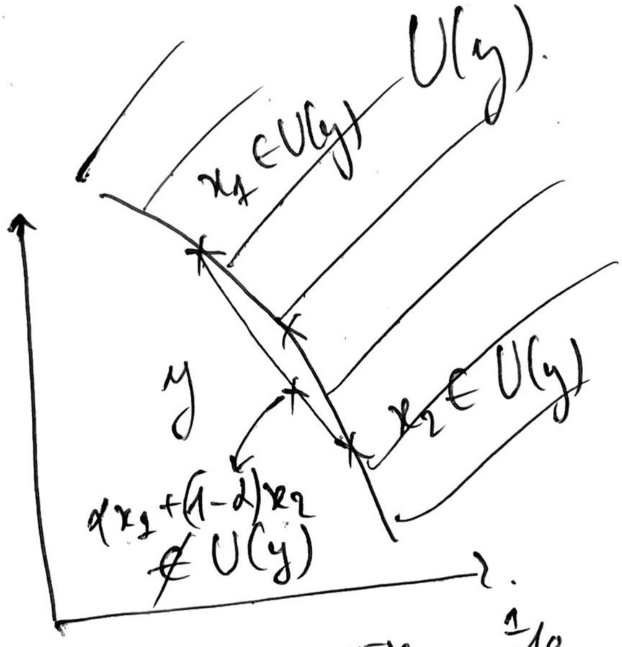
The definition is equivalent to saying that the <u>upper contour set is convex</u>.

since X convex set iff
$\forall x, y \in X,\ \forall \alpha \in [0,1]$,
$\alpha x + (1-\alpha) y \in X$.

example.



is <u>CONVEX</u>



is <u>NOT</u> CONVEX

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i.e. a set is convex iff any point of the segment linking two of its elements are also included in the set.

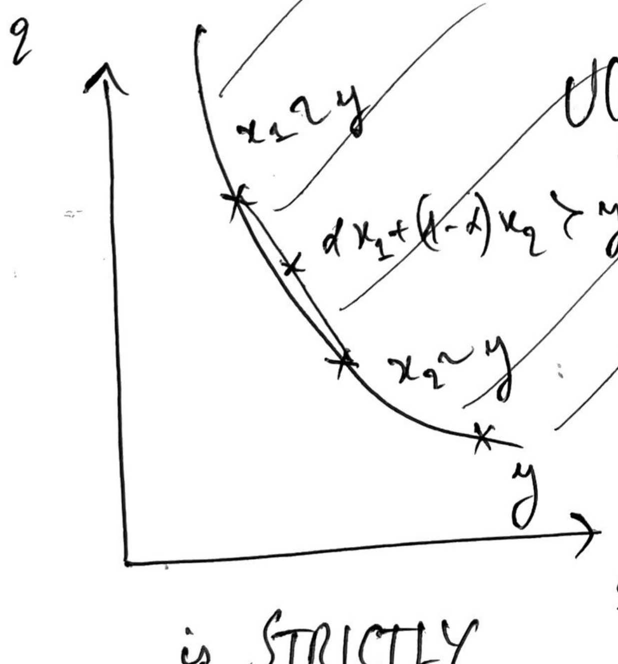
Def: $\succsim$ is STRICTLY CONVEX iff.

$$x \succsim z \text{ and } y \succsim z \text{ and } x \neq z$$

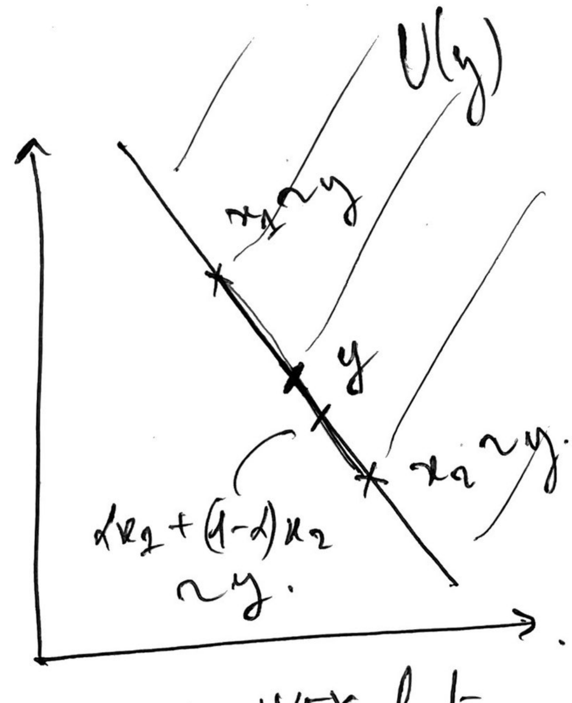
$$\Longrightarrow \alpha x + (1-\alpha) y \succ z \quad \forall \alpha \in]0,1[$$

$\Longleftrightarrow U(y)$ (the upper contour set) is strictly convex $\forall y$.

example.



is STRICTLY CONVEX.



is CONVEX but NOT strictly convex.

* we wanted to show that the lexicographic pref. is strictly convex.

Assume. $x \succsim z$ and $y \succsim z$. with $x \neq z$.

ie.

$\left(\begin{array}{c} x_1 > z_1. \\ \text{or.} \\ (x_1 = z_1 \text{ and. } x_2 \geq z_2) \end{array} \right)$ and. $\left(\begin{array}{c} y_1 > z_1. \\ \text{or.} \\ (y_1 = z_1 \text{ and } y_2 \geq z_2 \end{array} \right.$

$\Longleftrightarrow$
(distributing "and" w.r.t. "or")

$(x_1 > z_1 \text{ and } y_1 > z_1.)$

or.

$(x_1 > z_1 \text{ and } y_1 = z_1.)$

or.

$(x_1 = z_1 \text{ and } y_1 > z_1)$

or

$\left(\begin{array}{l} x_1 = z_1 \text{ and. } y_1 = z_1. \\ \text{and } x_2 \geq z_2 \text{ and } y_2 \geq z_2 \end{array} \right)$.

$\Longrightarrow$ in which case. $\alpha x_1 + (1-\alpha) y_1 > z_1$. $\forall \alpha \in]0,1[$.

.. in that last case, we cannot have. both $x_2 = z_2$ and. $y_2 = z_2$ since $x \neq y$, by hypothesis,

...

... so the last case is equivalent to.

$$x_1 = y_1 = z_1$$

and $\left\{ \begin{pmatrix} x_2 \geq z_2 \\ \text{and} \\ y_2 > z_2 \end{pmatrix} \text{ or } \begin{pmatrix} x_2 > y_2 \\ \text{and} \\ y_2 \geq z_2 \end{pmatrix} \right.$.

$$\Downarrow$$

$\forall \alpha \in]0,1[$, $\quad \alpha x_1 + (1-\alpha) y_1 = z_1$.

<u>and</u>

$$\alpha x_2 + (1-\alpha) y_2 > z_2.$$

<u>STRICT</u> inequality.

Hence we cannot have.

$$\alpha x + (1-\alpha) y \precsim z.$$

so. $\alpha x + (1-\alpha) y \succ z$. $\square$.

Thus $\succsim$ is <u>strictly convex</u>.

* Lastly, we wanted to show that $\succsim$ was <u>not</u> continuous.

Reminder. <u>Def</u>:

$\succsim$ is <u>continuous</u> iff it is <u>preserved under limits</u>:

$\iff \forall \{x_n\}, \{y_n\} \in X^{\mathbb{N}}$ s.th. $x_n \succsim y_n$,

$$\begin{matrix} x_n \to x \\ y_n \to y \end{matrix} \implies x \succsim y.$$

$\iff \forall y$, both $U(y)$ and $L(y)$ are <u>closed</u>.

with the extra reminder that a closed set is defined as:

$$\boxed{X \text{ is } \underline{\text{closed}} \text{ if } \forall \{x_n\} \in X^{\mathbb{N}}, \quad x_n \to x \implies x \in X.}$$

i.e., for all converging sequence of elements of X, its limit is also in X.

The opposite of closed is <u>open</u>.

$\therefore$ To show that $\succsim$ is not continuous, we just need to find a counter example.

that is, we need to find two sequences $\{x_n\}$ and $\{y_n\}$ s.th. $x_n \succsim y_n$ $\forall n$.

both converging $x_n \longrightarrow x$, $y_n \longrightarrow y$ but s.th.

$$x \not\succsim y.$$

For instance, take

$$x_n = \begin{pmatrix} \frac{1}{n} + a \\ b \end{pmatrix} \quad \text{and} \quad y_n = \begin{pmatrix} a \\ 2b \end{pmatrix}. \quad \text{with } a, b > 0.$$

we have that $x_{n1} = a + \underbrace{\frac{1}{n}}_{>0} > y_{n1} = a.$

so $x_n \succsim y_n$ $\forall n.$

BUT $x_n \longrightarrow x = \begin{pmatrix} a \\ b \end{pmatrix}.$

$y_n \longrightarrow y = \begin{pmatrix} a \\ 2b \end{pmatrix}.$ with $x \prec y$.

since $b < 2b$.

$\Rightarrow$ So $\succsim$ is NOT continuous. $\square$.

Please work on exercises C7 and C8 at home,

I will try to send you a written correction for those later.

(C7) Linear pref. $\forall x, y \in \mathbb{R}^2_+$. $a, b > 0$.

$$x \succsim y \iff a x_1 + b x_2 \geq a y_1 + b y_2.$$

a) $\forall y$, draw $I(y)$ and $U(y)$.

b). Show. $\succsim$ is continuous, convex, strongly mono. but not strictly convex.

a). $I(y) = \{x ; x \sim y\}$.

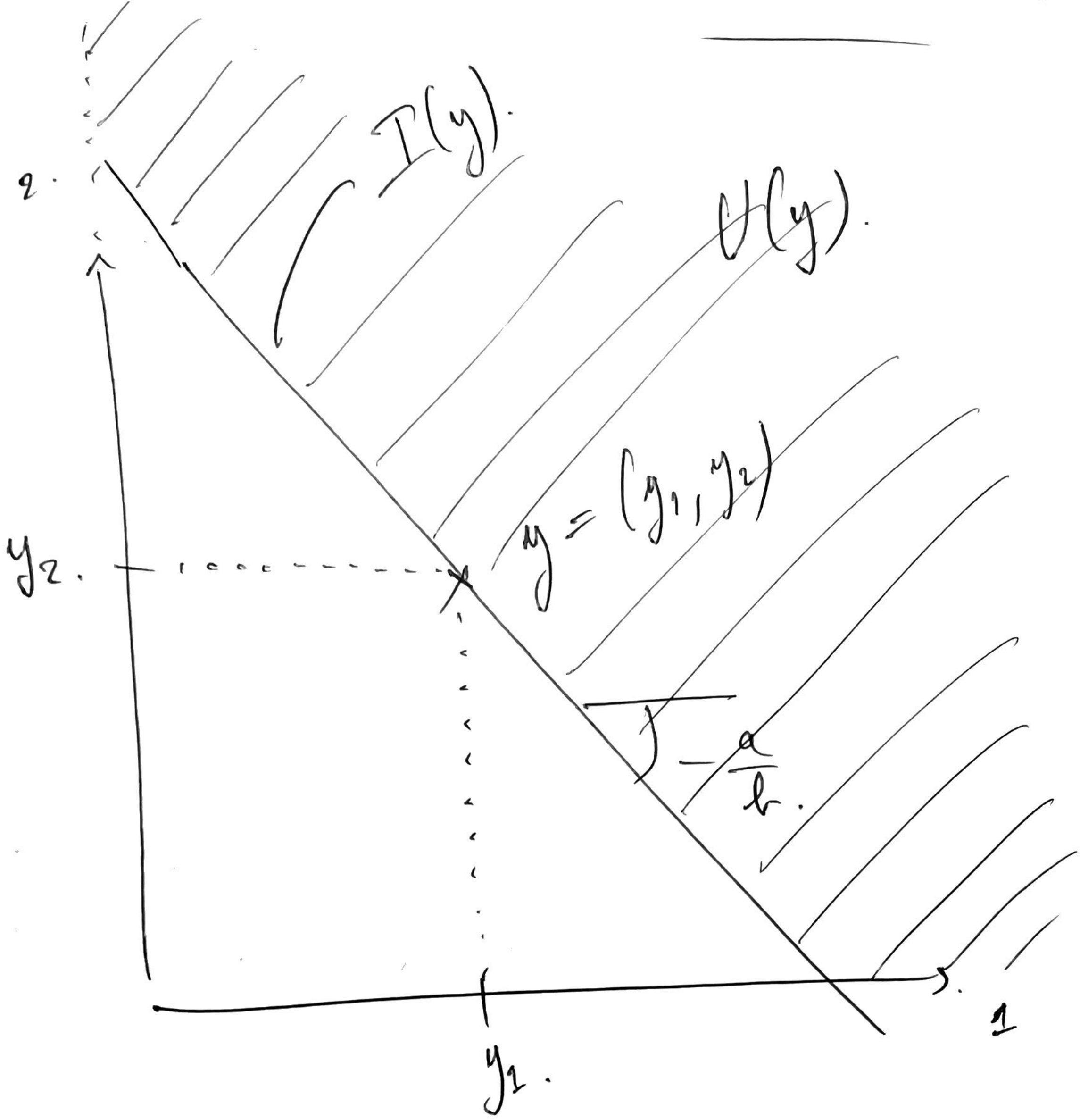
$= \{x ; x \succsim y$ and $y \succsim x\}$.

$$= \{x = (x_1, x_2) ; a x_1 + b x_2 = \underbrace{a y_1 + b y_2}_{=: c}\}.$$

$$= \{x ; x_2 = -\frac{a}{b} x_1 + \frac{c}{b}\}.$$

and we know, $y \in I(y)$ because $\sim$ reflexive.

" $U(y)$ " $\succsim$ "



$U(y) = \{x,\ x \succsim y\}$

$= \{x=(x_1,x_2)\ ;\ ax_1 + bx_2 \geq ay_1 + by_2\}$

is everything on the "upper right" of $I(y)$...

1). Show $\succsim$ is CONTINUOUS (i), CONVEX (ii),
STR. MONOTONE (iii)
but NOT Str. convex (iv).

Recall: $\succsim$ is CONTINUOUS iff. $U(y)$ and $L(y)$ are closed. $\forall y$.

(i) Df: $\Updownarrow$

iff $\forall \{x_n\}, \{y_n\}$ $x_n \succsim y_n$.

$$\left.\begin{array}{l} x_n \longrightarrow x \\ y_n \longrightarrow y \end{array}\right\} \Longrightarrow x \succsim y.$$

(i.e. the preference rel. is preserved under limits.

Suppose $\{x_n\}, \{y_n\}$. s.th. $x_n \succsim y_n$ $\forall n$,

and. $x_n \longrightarrow x$.

$y_n \longrightarrow y$.

we have. $x_n \succsim y_n$.

$\Longleftrightarrow$ $ax_{n1} + bx_{n2} \geq ay_{n1} + by_{n2}$.

$\lim_{n\to\infty}$

$\Longrightarrow$. $ax_1 + bx_2 \geq ay_1 + by_2$.

using that.

$f: (x_1, x_2) \longmapsto ax_1 + bx_2$.

is continuous! so. $\succsim$ is continuous

$\Longleftrightarrow$ $x \succsim y$. $\square$.

ii) **Def**: $\succsim$ CONVEX iff $\forall y$. $U(y)$ is CONVEX.

$\iff (x \succsim z \text{ and } y \succsim z) \implies \alpha x + (1-\alpha)y \succsim z.$ $\forall \alpha \in [0,1]$.

Suppose. $x \succsim z$ and $y \succsim z$.

$$\iff \quad a x_1 + b x_2 \geq a z_1 + b z_2.$$
$$\underline{\text{and}} \quad a y_1 + b y_2 \geq a z_2 + b z_2$$

$\implies \forall \alpha \in [0,1]$.

$$a(\alpha x_1 + (1-\alpha) y_1) + b(\alpha x_2 + (1-\alpha)(y_2))$$
$$\geq a z_2 + b z_2.$$

$$\iff \alpha x + (1-\alpha) y \succsim z \quad \forall \alpha \in [0,1] \quad \square.$$

(iii) We want to show that.

$$x > y \implies x \succ y.$$

ie. $x \succsim y$ and. $y \not\succsim x$

ie. $a x_2 + b x_2 \geq a y_1 + b y_2$
and. $a x_1 + b x_2 \not\leq a y_1 + b y_2.$

$$\iff \quad a x_1 + b x_2 > a y_1 + b y_2$$

so now we just write.

$$x > y \iff \text{either} \begin{cases} x_1 \geq y_1 \\ \text{and} \\ x_2 > y_2 \end{cases} \text{or} \begin{cases} x_1 > y_1 \\ \text{and} \\ x_2 \geq y_2 . \end{cases}$$

$\Longrightarrow$ in both cases, that implies $a x_1 + b x_2 > a y_1 + b y_2$.

$$\iff x \succ y . \qquad \square .$$

(iv).

$$\boxed{\begin{array}{l} \underline{\text{Def}}: \succsim \text{ str. convex iff. } x \succsim z \text{ and } y \succsim z. \\ \quad \text{and } x \neq y. \\ \Longrightarrow \forall \alpha \in]0,1[, \ \alpha x + (1-\alpha) y \succ z. \end{array}}$$

to show that $\succsim$ is not str. convex, we need to find a counter example!

take. $x \sim y \sim z$ with e.g. $x \neq y$ and. $y = z$.

$$x \sim y \iff a x_1 + b x_2 = a y_1 + b y_2$$

$x \neq y$.

we have, $\forall \alpha \in]0,1[$,

$$\cancel{\alpha(ax_1 + bx_2)} + \cancel{(1-\alpha)}(ay_2 + by_2)$$

$$= ay_2 + by_2.$$

$\Longrightarrow$

using $x \sim y$. Thus.

$$\alpha x + (1-\alpha) y \sim y \sim z$$

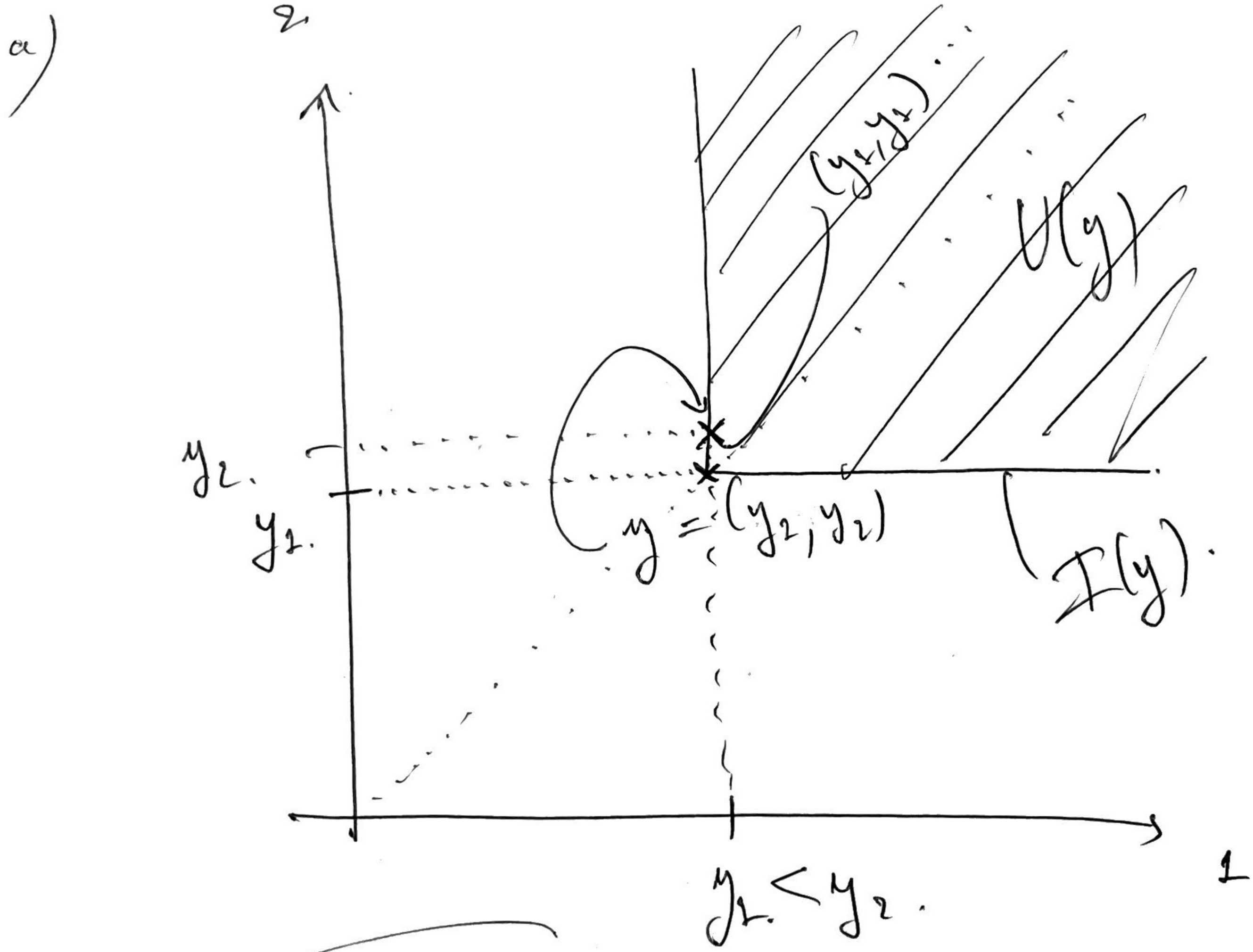
$$\forall \alpha \in]0,1[.$$

thus. $\alpha x + (1-\alpha) y \not\succ z$ $\square$.

(C8) LEONTIEF PREF. :

$$x \succsim y \iff \min\{x_1, x_2\} \geq \min\{y_1, y_2\}.$$

a)



→ here let's assume $\min\{y_1, y_2\} = y_1$.

we seek all the $x = (x_1, x_2)$ s.th.

$$\min\{x_1, x_2\} \geq y_1.$$

1) Show. $\succsim$ is continuous (i), convex, monotone (iii), but ~~not~~ str. convex (ii). and not str. mono.

(i) likewise, using that min is a continuous fct.

Suppose $\{x_n\}, \{y_n\}$. $x_n \succsim y_n$.

$x_n \to x$

$y_n \to y$

$\iff \min\{x_n\} \geq \min\{y_n\}$.

$\Longrightarrow \min\{x\} \geq \min\{y\}$.

$\lim_{n \to \infty}$ $\iff$ $x \succsim y$.

□.

(ii)

~~Suppose $x \succsim z$ and $y \succsim z$.~~

~~$\Updownarrow$ $\Updownarrow$~~

~~$\min\{x\} \geq \min\{z\}$ $\min\{y\} \geq \min\{z\}$.~~

let $\alpha \in [0,1]$.

~~$\min\{\alpha x + (1-\alpha) y\}$.~~ = ~~either $\alpha x_1 + (1-\alpha) y_1$.~~

~~or $\alpha x_2 + (1-\alpha) y_2$.~~

(ii)

$\succsim$ convex $\iff$ $U(y)$ convex $\forall y$.

$\iff \forall x, z \in U(y)$.
$\forall \alpha \in [0,1]$ $\alpha x + (1-\alpha) z \in U(y)$

let $x, z \in U(y)$.
$\iff \min\{x\} \geq c = \min\{y\}$.
and $\min\{z\} \geq c$.

case 1: $\min\{x\} = x_1$, $\min\{z\} = z_1$.
case 2: $\min\{x\} = x_1$, $\min\{z\} = z_2$
the two other possibilities are symmetrical...

case 1: in this case it is obvious that

$$\min(\alpha x + (1-\alpha) z) = \alpha x_1 + (1-\alpha) z_1 \geq \alpha c + (1-\alpha) c = c.$$

case 2:

$$\min(\alpha x+(1-\alpha)z)$$

because $\min z = z_2$.

$$=\begin{cases}\text{either } \alpha x_1+(1-\alpha)z_1 \geq \alpha x_1+(1-\alpha)z_2 \\ \text{or } \alpha x_2+(1-\alpha)z_2 \geq \alpha x_1+(1-\alpha)z_2\end{cases}$$

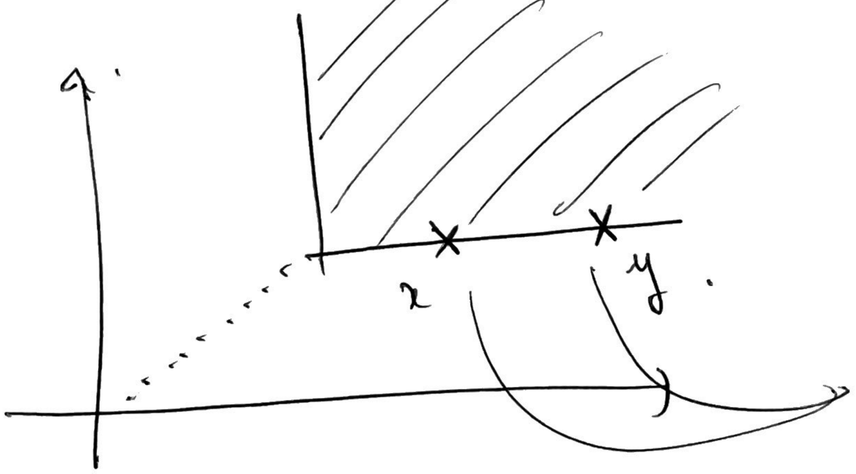
because $\min x = x_1$.

$$\geq \alpha x_1+(1-\alpha)z_2$$

$$\geq \alpha c+(1-\alpha)c = c.$$

$\Longrightarrow$ so $\alpha x+(1-\alpha)z \in U(y)$ $\square$.

to show that it is not strictly convex, we again consider two consumption bundle on the same indifference set,



$x \sim y$.

...

(iii) Show $\succsim$ is monotone, but not strongly.

i.e show. $x \gg y \Longrightarrow x \succ y$.

$x \gg y \Longleftrightarrow \forall i = 1,2 , x_i > y_i$

$\Longrightarrow \min\{x\} > \min\{y\}$.

$\Longleftrightarrow x \succ y$.

$\square$.

if we only have. $x > y$.

i.e. $\forall i = 1,2.\ x_i \geq y_i$

and $\exists j \quad x_j > y_j$.

if we are in the situation. when. $\min\{x\} = x_i$

and $\min\{y\} = y_i$

for some i

and. $x_i = y_i$

then we don't have. $x \succ y$.

$\square$.

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