Microeconomics 1A. October 3, 2023.

Exercice 1: C10. $p \gg 0$ $w > 0$.

Determine graphically the demand of the consumer for:

a) lexicographic pref.

b) linear pref.

c) Leontief pref.

demand of the consumer.

Def: We call $x(p,w)$ s.th: $x \in x(p,w)$ iff x solution to the utility maximization problem. $\begin{cases} \max u(x) \\ p \cdot x \leq w \end{cases}$

a) The lexicographic preference is defined by (on $\mathbb{R}_+^2$).

$$x \succsim y \Longleftrightarrow \begin{cases} x_1 > y_1 \\ \text{or} \\ (x_1 = y_1 \text{ and } x_2 \geq y_2) \end{cases}$$

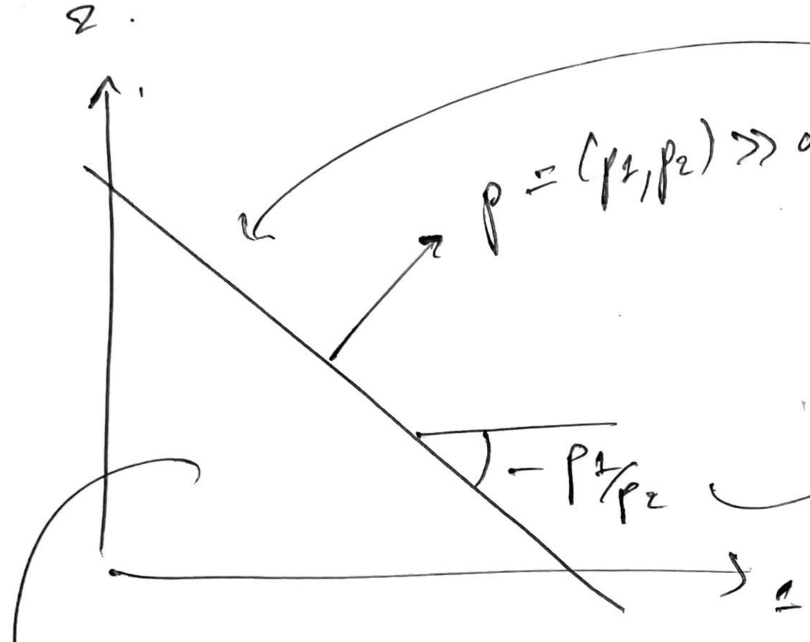
As we've seen in a previous tutorial, the lexicographic preference is not representable by any utility function.

However, we can still define the "maximization problem" of the consumer as:

$$\boxed{\underbrace{x \in x(p,w)}_{\text{demand of the consumer}} \text{ iff } \forall y \in B_{p,w}.\ x \succsim y.}$$

i.e. in their budget set.

The budget set is represented in $\mathbb{R}^2_+$ as:



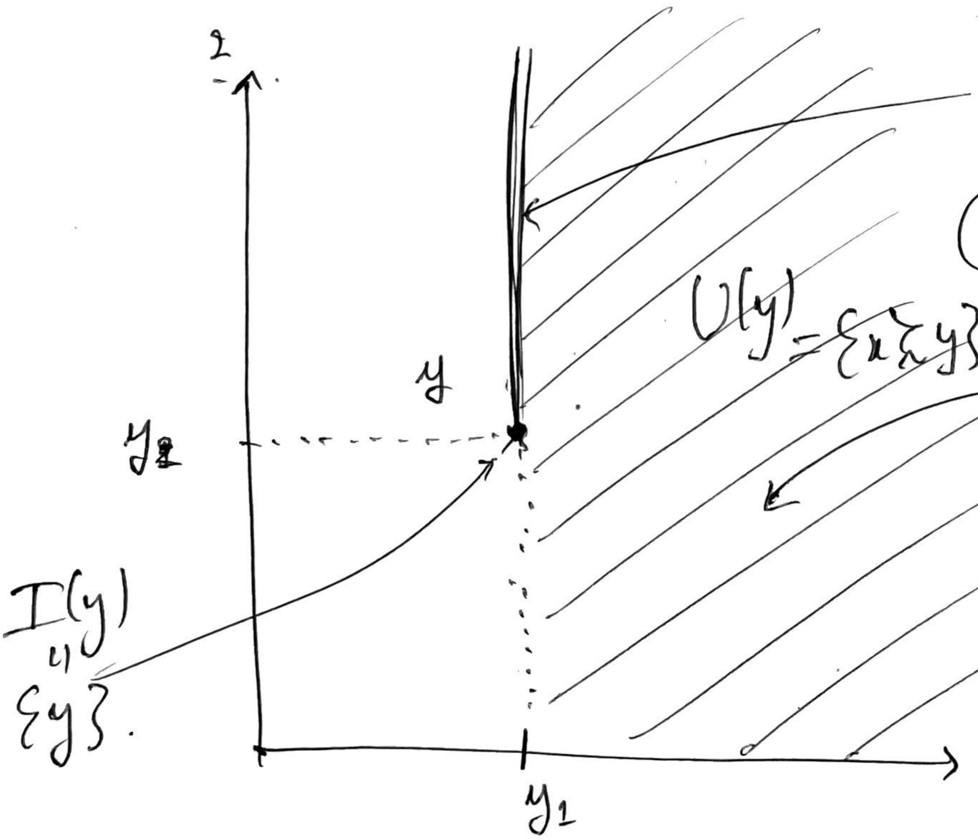
budget line
$p \cdot x = w$
$\Updownarrow$
$x_2 = -\frac{p_1}{p_2} x_1 + \frac{w}{p_2}$

slope.

$B_{p,w} = \{x;\ p \cdot x \leq w\}$.

It will be the same for b) and c).

As we've seen in a previous tutorial, the lexicographic pref. is represented in $\mathbb{R}^2_+$ as:
(drawing its indifference curve $I(y)$ and upper contour set $U(y)$ for a "randomly" picked y.)



the line above y including y correspond to the x s.th.
$(x_1 = y_1 \text{ and } x_2 \geq y_2)$.

the dashed area on the right of y excluding the line passing through y corresponds to the x s.th. $(x_1 > y_1)$.

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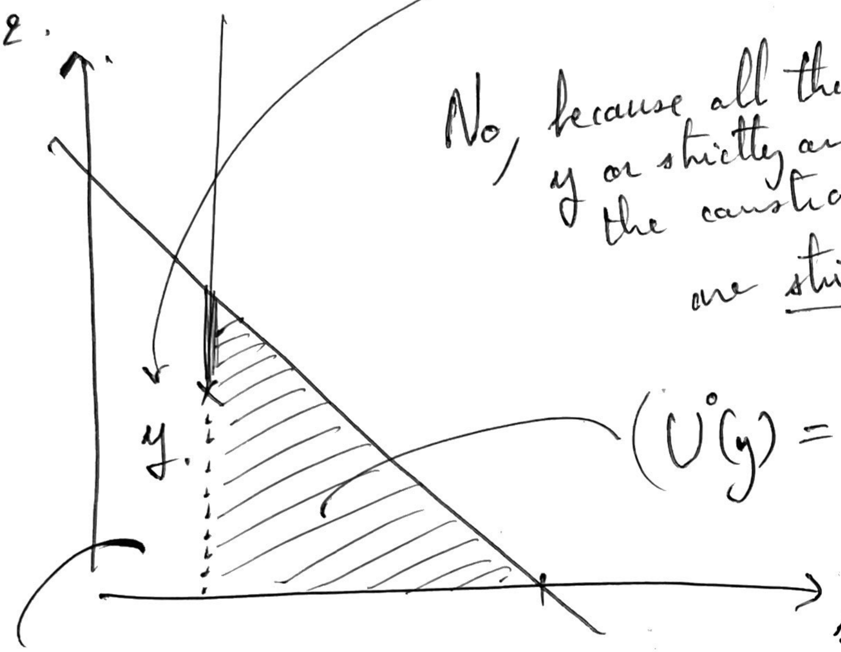
Now suppose you pick a point $y \in B_{p,w}$,

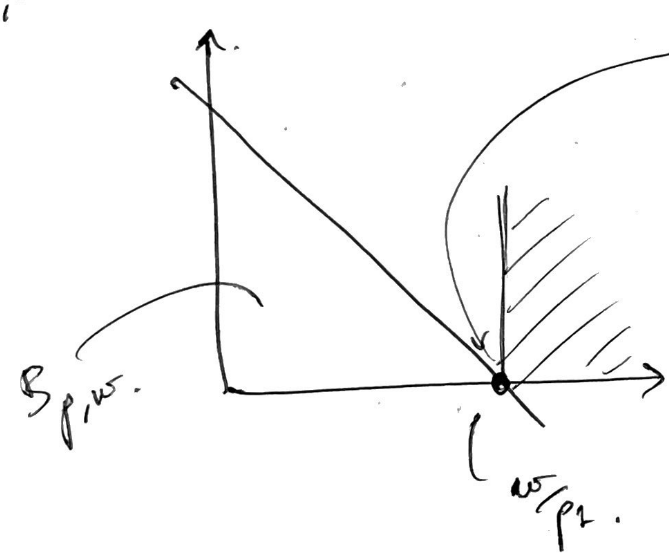
is it maximized under our constraint?



No, because all the point strictly above y or strictly on its right, yet satisfying the constraint $p \cdot x \leq w$, are strictly prefered to it.

$$\left(U^\circ(y) = \{x \succ y\}\right) \cap B_{p,w} \neq \emptyset.$$

$B_{p,w}$.



the bundle $\left(\frac{w}{p_1}, 0\right)$ is the only point in $B_{p,w}$ s.th. it has nothing strictly over it or strictly on its right.

$$x_{p,w} = \left\{\left(\frac{w}{p_1}, 0\right)\right\}.$$

In general, here, we are looking for $y \in B_{p,w}$ s.th. $\{x \succ y\} \cap B_{p,w} = \emptyset$.

$\Longrightarrow$ y is a maximum under the constraint.

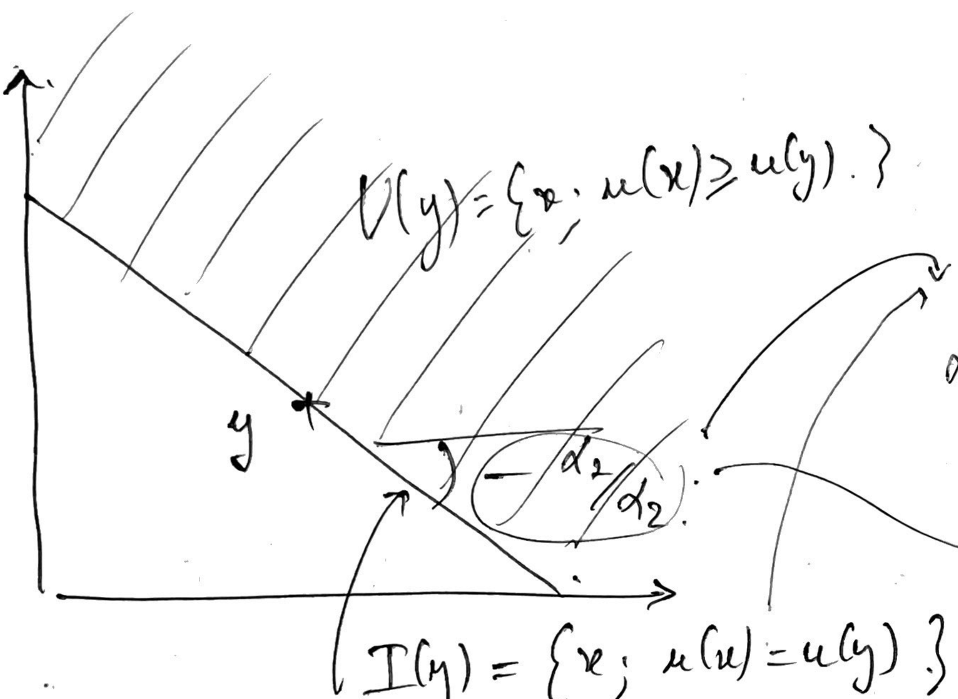
$\square$.

b) Linear preference $\Leftrightarrow$ $x \succsim y$ iff $\alpha_1 x_1 + \alpha_2 x_2 \geq \alpha_1 y_1 + \alpha_2 y_2$.

$\alpha_{1,2} > 0$

The same is representable by the utility function $u(x) = \alpha_1 x_1 + \alpha_2 x_2$

Its indifference sets and upper contour sets look like:



from the same computation:

$$\alpha_1 x_1 + \alpha_2 x_2 = u(y) = \text{cste}$$

$$\Downarrow$$

$$x_2 = -\frac{\alpha_1}{\alpha_2} x_1 + \frac{\text{cste}}{\alpha_2}$$

slope

So we need to distinguish three cases,

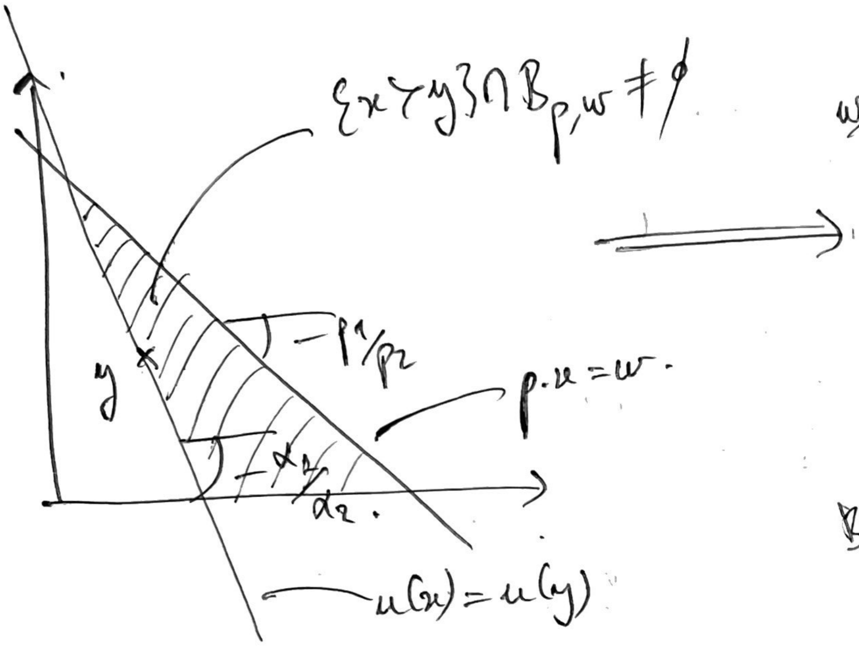
either $\frac{\alpha_1}{\alpha_2} = \frac{p_1}{p_2}$, i.e. the budget line is parallel to the indifference curves

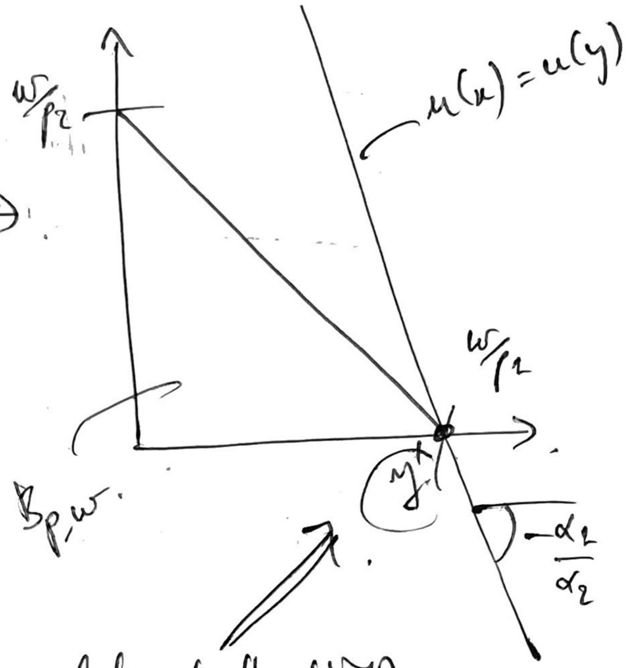
or $-\frac{\alpha_1}{\alpha_2} < -\frac{p_1}{p_2}$ i.e. the indifference curves are steeper.

or " $>$ " ——— less steep.

$\Longrightarrow$

$-\frac{\alpha_1}{\alpha_2} < -\frac{p_1}{p_2}$:

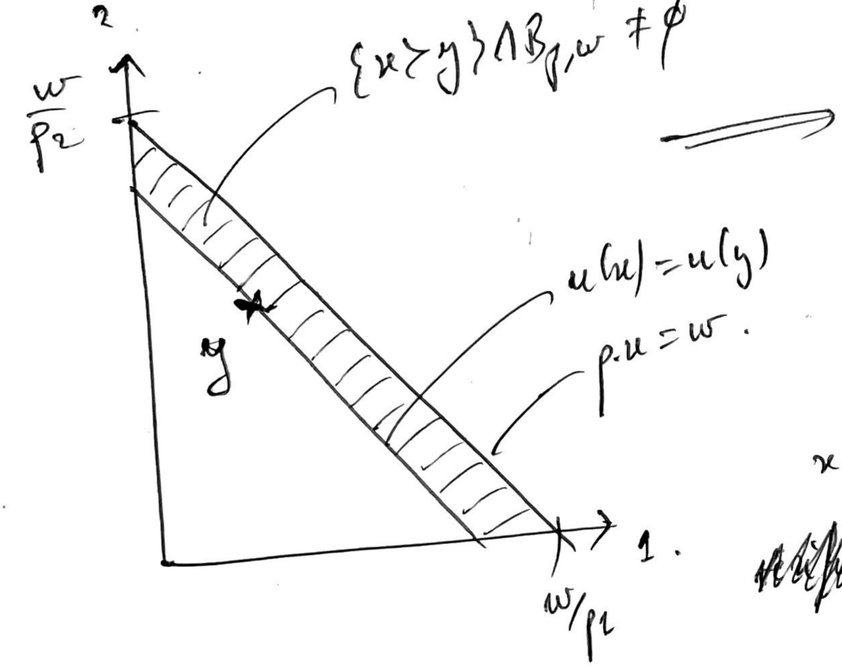


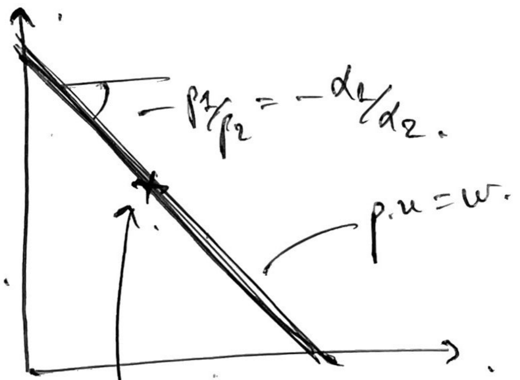


$y' = \left(\frac{w}{p_1}, 0\right)$ is the unique solution to the UMP.
i.e. unique y s.th. $\{x \succ y\} \cap B_{p,w} = \emptyset$.

<u>Symmetrically</u>, the case $-\frac{\alpha_1}{\alpha_2} > -\frac{p_1}{p_2}$ will yield $\left(0, \frac{w}{p_2}\right)$ as the unique solution.

Finally, $-\frac{\alpha_1}{\alpha_2} = -\frac{p_1}{p_2}$:





the solution set is the whole line $p.x = w$.

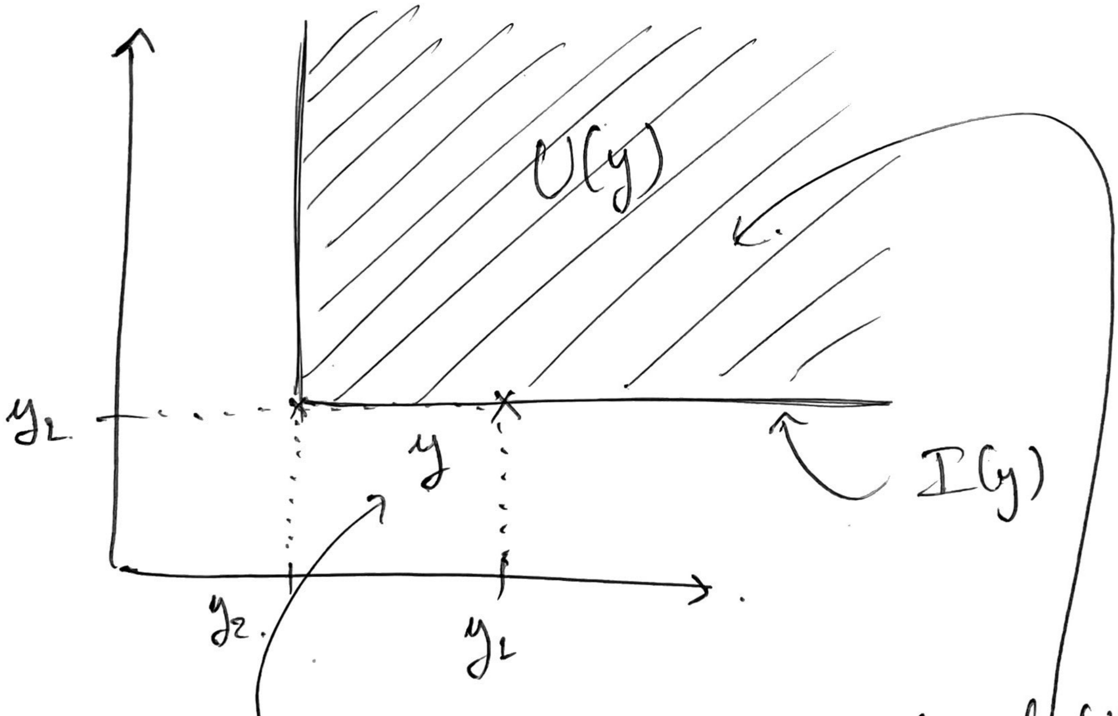
$x(p,w) = \{x;\ p.x = w\}$. □

Note : Note that the solution set to the maximization problem can at most be 1-dimensional. (i.e. reduced to a curve).
This comes from local nonsatiation
$\Longrightarrow$ which ruled out the possibility to have "thick" indifference curves.

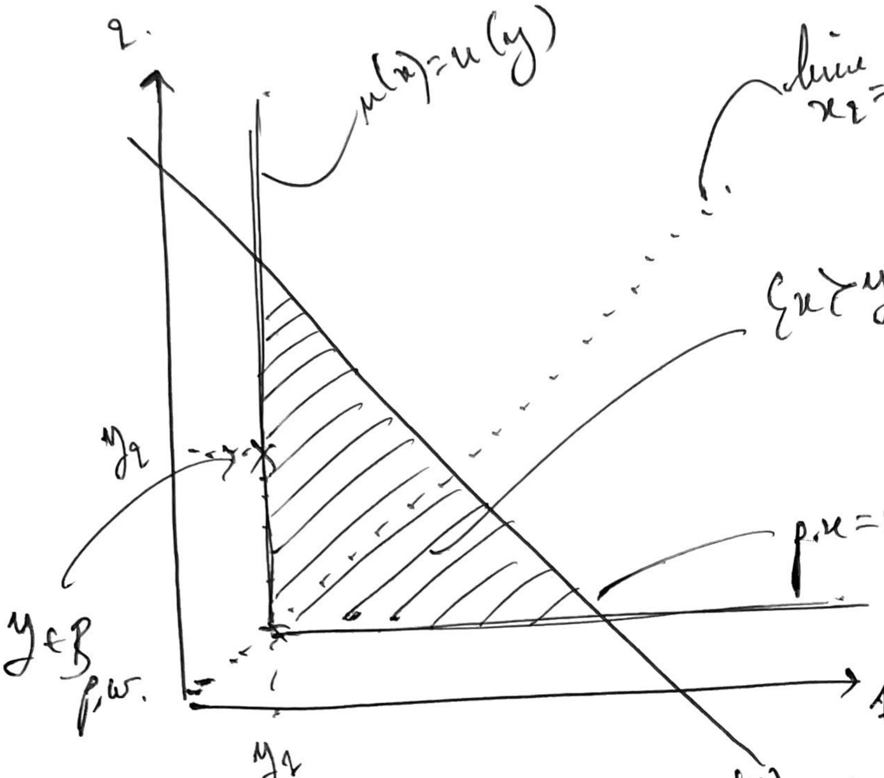
c) Leontief pref. $x \succsim y$ iff. $\min\{x_1, x_2\} \geq \min\{y_1, y_2\}$.

L, it is representable by the utility function $u(x) = \min\{x_1, x_2\}$.

Remember, its upper contours sets and indifference curves look like:

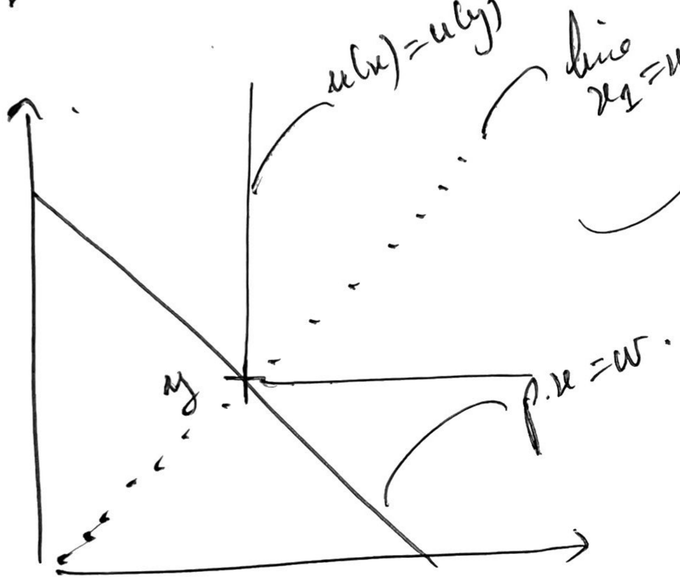


in this case, $\min\{y\} = y_2$; so the set U(y) of x s.th. $\min\{x\} \geq \min\{y\} = y_2$. is the shaded area.



$\{x \succ y\} \cap B_{p,w} \neq \emptyset$ so y is not a solution.

$\Longrightarrow$



in this case

$\{x \succ y\} \cap B_{p,w} = \emptyset$

$\Downarrow$

y is a solution (unique), and it verifies

$y_1 = y_2$

and $p.y = w$

$\Downarrow$

$$y = \left(\frac{w}{p_1+p_2}, \frac{w}{p_1+p_2}\right) \quad \square$$

Note: in all three case a), b) and c), we "landed" on the budget line, i.e.: every time

$x \in x(p,w) \Longrightarrow p.x = w$.

which is <u>Walras' law</u>.

Exercice 2:

Solve the utility maximization problem. (UMP).
for. CES, Cobb-Douglas,
and Leontief.
(using the Kuhn-Tucker conds).

Reminder. Consider the UMP.

$$\max u(x) \quad \left\{ p.x \leq w. \right. \quad p \gg 0, \; w > 0.$$

$\Rightarrow$ $\bar{x} \geq 0$ is a solution to the UMP.
iff. $\nabla u(x) = \lambda p$.
and. $\lambda(p.x - w) = 0$.

= KUHN-TUCKER NECESSARY CONDITIONS.

* for the CES pref, represented by.

$$u(x) = \left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right)^{1/\rho}.$$

$$\alpha_{1,2} > 0$$

For simplifying the calculus, we can consider $u'(x) = \ln u(x)$, using the fact that preferences (and hence the UMP) are preserved under strictly increasing tsf.)

$$u'(x) = \frac{1}{\rho} \ln\left(\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}\right)$$

So, the KT conditions write:

$$\begin{cases} \dfrac{\partial u'}{\partial x_1}(x) = \dfrac{1}{\cancel{\rho}} \dfrac{\cancel{\rho}\, \alpha_1 x_1^{\rho-1}}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}} = \lambda \overbrace{p_1}^{>0}. \\[2ex] \dfrac{\partial u'}{\partial x_2}(x) = \dfrac{\alpha_2 x_2^{\rho-1}}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}} = \lambda \underbrace{p_2}_{>0}. \end{cases}$$

i.e. $\lambda p_i x_i = \dfrac{\alpha_i x_i^{\rho}}{\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho}}$, $i = 1, 2$.

the other KT conds is

$$\lambda(p \cdot x - w) = 0.$$

from these two, we see that for an interior solution. $\bar{x} \gg 0$, and $p \gg 0$ $\Longrightarrow \lambda \neq 0$.

$$\Downarrow$$

So. $p \cdot x = w$.
i.e Walras' law.

So. $p_1 x_1 + p_2 x_2 = \boxed{w = \dfrac{1}{\lambda}}$

so. $\alpha_1 x_1^{\rho} + \alpha_2 x_2^{\rho} = \alpha_i x_i^{\rho-1} \dfrac{w}{p_i}$

indep of i $\quad$ $i = 1, 2$.

substituting $i=2$ into $i=1$,

$$\frac{p_1}{\cancel{w}}\,\alpha_2\,\frac{x_2^{\ell-1}}{p_2}\,\cancel{w} - \alpha_1 x_1^{\ell-1} = 0.$$

$$\Updownarrow$$

$$\boxed{\frac{\alpha_2 x_2^{\ell-1}}{p_2} = \frac{\alpha_1 x_1^{\ell-1}}{p_1}.} \quad (*).$$

using again $p_1 x_1 + p_2 x_2 = w \implies x_2 = -\frac{p_1}{p_2}x_1 + \frac{w}{p_2}$.

in $(*)$:

$$\left(\frac{\alpha_2}{p_2}\right)\left(-\frac{p_1}{p_2}x_1 + \frac{w}{p_2}\right)^{\ell-1} = \frac{\alpha_1 x_1^{\ell-1}}{p_1}.$$

$$\implies \left(\frac{\alpha_2}{p_2}\right)^{\frac{1}{\ell-1}}\left[-\frac{p_1}{p_2}x_1 + \frac{w}{p_2}\right] = \left(\frac{\alpha_1}{p_1}\right)^{\frac{1}{\ell-1}} x_1.$$

$$\implies x_1 = \frac{\frac{w}{p_2}\left(\frac{\alpha_2}{p_2}\right)^{1/\ell-1}}{\left(\frac{\alpha_1}{p_1}\right)^{1/\ell-1} + \frac{p_1}{p_2}\left(\frac{\alpha_2}{p_2}\right)^{\frac{1}{\ell-1}}}.$$

$$\implies \boxed{x_1 = \frac{w}{p_1 + p_2\left(\frac{\alpha_1}{p_1}\frac{p_2}{\alpha_2}\right)^{1/\ell-1}}.}$$

By symmetric of $u(x)$ in $x_1 \longleftrightarrow x_2$, we can immediately guess that the solution for x_2 is the same with $1 \leftrightarrow 2$. $\square$

* <u>Cobb-Douglas</u> : $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$

$\alpha_{1,2} > 0$.

we can consider instead.

$u'(x) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2$.

$\Longrightarrow$ KT: $\dfrac{\partial u'}{\partial x_1} = \dfrac{\alpha_1}{x_1} = \lambda p_1$

$\dfrac{\partial u'}{\partial x_2} = \dfrac{\alpha_2}{x_2} = \lambda p_2$

$\Longrightarrow \lambda \underbrace{(p_1 x_1 + p_2 x_2)}_{= w} = \alpha_1 + \alpha_2$.

Walras' law. or KT. $\Longrightarrow \boxed{\lambda = \dfrac{\alpha_1 + \alpha_2}{w}}$

$\Longrightarrow \boxed{x_i = \dfrac{\alpha_i}{p_i} w}$ for $i = 1,2$.

$\square$.

* Leontief : $u(x) = \min\{x_1, x_2\}$.

The Leontief function is continuous, BUT is not continuously differentiable.

In particular, it is not differentiable at $x_1 = x_2 \longrightarrow$ its derivative at that point is not defined.

So we cannot use the KT conditions!

However, from the exact symmetry of the utility fct under $x_1 \longleftrightarrow x_2$, we can guess that the solution to the UMP must satisfy $\bar{x}_1 = \bar{x}_2$ and Walras' law (from local nonsatiation) $p \cdot \bar{x} = w$.

s.th $\Longrightarrow$ $\boxed{x_i = \dfrac{w}{p_1 + p_2}}$ $i = 1, 2$.

□