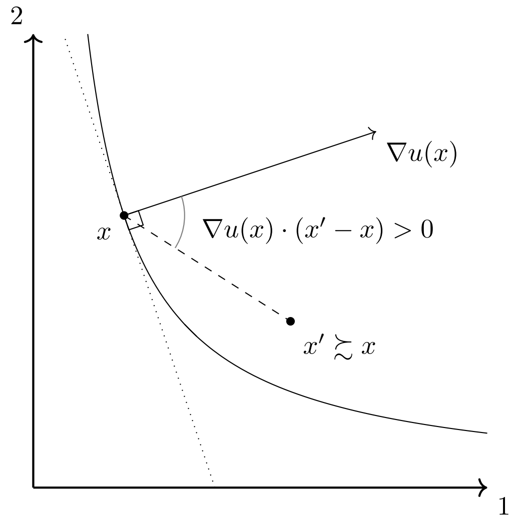



0.1. **C13.** (Cobb-Douglas utility function.) For all $x = (x_1, x_2) \in \mathbb{R}^2_+$, the utility function representing Cobb-Douglas preference relation takes the general form $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, with $\alpha_{i=1,2} > 0$.

*a)* Show that the utility function defined by $v(x) = u(x)^{\frac{1}{\alpha_1+\alpha_2}}$, $\forall x$ represents the same preference as $u$. Show that the utility function defined by $v'(x) = \ln u(x)$, $\forall x$ represents the same preference as $u$. [More generally, show that the preference relation represented by a utility function is preserved under strictly increasing transformation.]

*b)* Determine and draw the indifference set $I(y)$ and upper contour set $U(y)$ for all $y \in \mathbb{R}^2_+$, for the following three cases: i) $\alpha_1 = \alpha_2$; ii) $\alpha_1 > \alpha_2$; and iii) $\alpha_1 < \alpha_2$.

*c)* Show that $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ is: (i) continuous; (ii) differentiable; (iii) strictly increasing.

*d)* Show that $u(x)$ is strictly quasiconcave. [Show that (i) quasiconcavity is preserved under strictly increasing transformations (i.e., it is an ordinal property). Show that (ii) $u(x)$ is (strictly) concave iff its Hessian matrix $Hu(x)$ is negative (semi-)definite $\forall x$, i.e., $z \cdot Hu(x)z \leq 0$ (resp. $< 0$), $\forall z \in \mathbb{R}^l$, $\forall x$.]

*e)* Let $p = (p_1, p_2) >> 0$ be a price system and $w$ the wealth of the consumer with Cobb-Douglas preference. Determine the demand of the consumer. [Show that the utility maximization problem is invariant under strictly increasing transformation of the considered utility function.]

*f)* Provide a graphical representation of the solution to the previous utility maximization problem in the $(x_1, x_2)$-plane. Assume that the price of good 1 changes; draw the associated supply curve. Assume that the wealth changes; draw the associated wealth-consumption curve.

*Proof.* □

0.2. **C12.** Consider a twice continuously differentiable utility function $u : \mathbb{R}^2_+ \to \mathbb{R}$ representing a locally nonstatiated consumer's preference. Let $p >> 0$, and $w > 0$. Prove the following results:

*a)* Convexity of the preference, that is quasiconcavity of the utility function $u$, implies that at any bundle $x = (x_1, x_2)$, the marginal rate of substitution $MRS_{12}(x) = \frac{\partial u(x) \partial x_1}{\partial u(x) \partial x_2}$ is decreasing in $x_1$.

*b)* Show that (i) $u$ is (strictly) concave iff $u(x') \leq u(x) + \nabla u(x) \cdot (x' - x)$, $\forall x, x'$ (resp. $u(x') < u(x) + \nabla u(x) \cdot (x' - x)$, $\forall x \neq x'$); (ii) $u$ is (strictly) quasiconcave iff $u(x') \geq u(x)$ (and $x' \neq x$) implies $\nabla u(x) \cdot (x' - x) \geq 0$ (resp. $> 0$).

*c)* Show that $u$ is (strictly) quasiconcave iff its Hessian matrix $Hu(x)$ is negative (semi-)definite on $\ker \nabla u(x)$, $\forall x$, i.e., $z \cdot Hu(x)z \leq 0$ (resp. $< 0$), $\forall z \in \ker \nabla u(x)$, i.e. $\forall z$ s.th. $z \cdot \nabla u(x) = 0$, $\forall x$.

*d)* Prove that local nonsatiation of the represented preference implies Walras' law, i.e., show that if $\bar{x} \in x(p, w)$ is a solution to the utility maximization problem $\bar{x} = \arg\max\{u(x) \,|\, x \in B_{p,w}\}$, then $p.\bar{x} = w$. (*Hint:* by contradiction.)

*e)* (Necessary conditions.) Prove that if $\bar{x}$ is a local extremum, then $\bar{x}$ satisfies the Kuhn-Tucker conditions, i.e., $\nabla u(\bar{x}) = \lambda p$ and $\lambda(p \cdot x - w) = 0$. Prove that if $u$ is monotonous, then $\lambda > 0$. (Prove that the Kuhn-Tucker conditions are equivalent to $MRS_{ij}(\bar{x}) = \frac{\partial u(\bar{x}) \partial x_i}{\partial u(\bar{x}) \partial x_j} = \frac{p_i}{p_j}$, $\forall i \neq j$.)

*f)* (Sufficient conditions.) Let $\bar{x}$ satisfies the Kuhn-Tucker conditions. Show that: (i) if $u$ is concave, then $\bar{x}$ is a global maximizer; (*Hint:* by contradiction, using *b)* i).) (ii) if $u$ is monotone and quasiconcave, then $\bar{x}$ is a global maximizer. (*Hint:* by contradiction, using *b)* ii).)

*g)* Show that $u$ strictly concave implies $\bar{x}$ is unique. (*Hint:* by contradiction.)

*Proof.* □

(C13). a).

Show that if $u$ rep. $\succsim$ and $f$ strictly increasing fct, then $f \circ u$ rep $\succsim$.

Let $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, $\alpha_{1,2} > 0$. Cobb-Douglas.

Conclude w.r.t. $v(x) = u(x)^{\frac{1}{\alpha_1+\alpha_2}}$ and $v(x) = \ln u(x)$.

Proof: let $u$ rep $\succsim$.

i.e. $x \succsim y \iff u(x) \geq u(y)$.

$\iff$ ($f$ strictly increasing) $f(u(x)) \geq f(u(y))$.

$\iff f \circ u$ rep. $\succsim$. $\square$.

So $v(x) = u(x)^{\frac{1}{\alpha_1+\alpha_2}}$ also rep. the Cobb-Douglas pref.

$\iff$ so we can always consider CONSTANT RETURN TO SCALE.

$\iff$ homogeneity of degree 1.

$\iff \alpha_1 + \alpha_2 = 1$. R.

$\Rightarrow$ So $v(x) = \ln(u(x)) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2$.

also rep. the Cobb-Douglas pref. (1/12)

$\Rightarrow$ so we can use this form which has the advantage of separating our two variables. R

with $\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} u(x) = 0$ ! always!

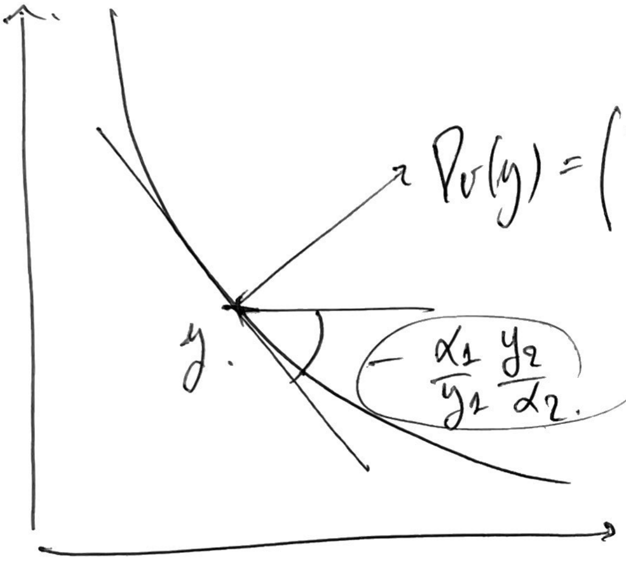
(C13) h) Determine and draw the indifference set and $I(y)$ upper contour set $U(y)$. $\forall y$.

when i) $\alpha_1 = \alpha_2$. ; ii) $\alpha_1 > \alpha_2$

(iii) $\alpha_1 < \alpha_2$.

$u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

Using (C13) a) it is equivalent to draw the indifference curves for $\boxed{v(x) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2}$ (*)




Along an indifference curve.

$v(x) = v^* = cste$.

$\Downarrow$

$\alpha_2 \ln x_2 = -\alpha_1 \ln x_1 + v^*$.

So when. $x_2 \to 0$

$\Longrightarrow x_1 \to +\infty$ at $v^*$ cste.

and when. $x_1 \to 0$,

$\Longrightarrow x_2 \to +\infty$, at $v^*$ cste.

we know that $\{x,\ u(x) = u(y)\}$ passes through $y$.

$\Longrightarrow$ Show that the slope of the tangent is $-\frac{\alpha_1}{y_1}\frac{y_2}{\alpha_2}$:

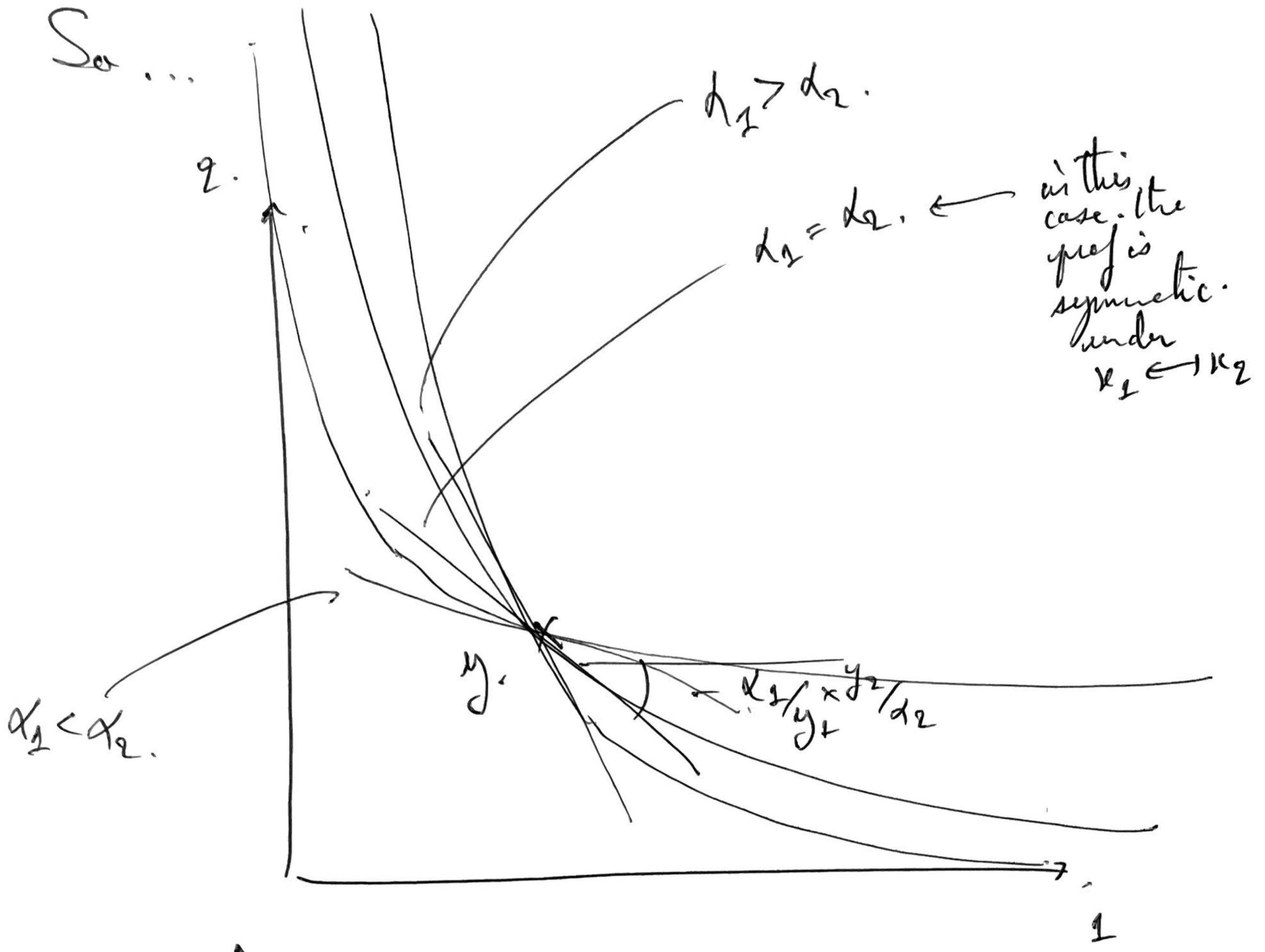
The tangent satisfies $(x-y)\cdot\nabla v(y) = 0$ ie, it is perpendicular to the gradient.

t. passes through y. we have $\nabla v(y) \overset{(*)}{=} \begin{pmatrix} \alpha_1/y_1 \\ \alpha_2/y_2 \end{pmatrix}$.

$(x_1 - y_1)\frac{\alpha_1}{y_1} + (x_2 - y_2)\frac{\alpha_2}{y_2} = 0$.

$\Longleftarrow \quad \| \; x_2 = \left(-\frac{\alpha_1}{y_1}\frac{y_2}{\alpha_2}\right) x_2 + \overbrace{\frac{\alpha_1+\alpha_2}{\alpha_2} y_2}^{cte} \; \|$. $\square$.

So ...




$$\nabla v(y) = \begin{pmatrix} \frac{\partial v}{\partial x_1}(y) \\ \frac{\partial v}{\partial x_2}(y) \end{pmatrix} = \begin{pmatrix} \alpha_1/y_1 \\ \alpha_2/y_2 \end{pmatrix}.$$

MARGINAL UTILITIES.

So, if we are at $y_1 = y_2$.

then, e.g. $\alpha_1 < \alpha_2$ means that to maintain a cste level of utility, a loss $-dx_2$ of commodity 2 needs to be compensated by an increase $dx_1 > |dx_2|$

strictly larger.

... and conversely for $\alpha_1 > \alpha_2$ ...

(C13). c). Show that $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ is
(i) continuous, (ii) differentiable,
(iii) strictly increasing.

Proof: (i). $f$ is continuous iff $\forall y \; \lim_{x \to y} f(x) = f(y)$

$\Updownarrow$ by def. of "being the limit";

$\forall y \; \forall \varepsilon > 0 \; \exists \delta > 0$, s.th.
$\forall x$, $|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$

For simplicity, consider $f: x \longmapsto x^{\alpha}$, $a > 0$.
$\Longrightarrow$ show that $f$ is continuous:

let $y$, $\varepsilon > 0$.
let $x = y + a$. $\Longrightarrow$ $|x - y| = |a|$.

$$f(x) - f(y) = (y + a)^{\alpha} - y^{\alpha}$$
$$= y^{\alpha}\left[\left(1 + \frac{a}{y}\right)^{\alpha} - 1\right].$$

So $|f(x) - f(y)| < \varepsilon$. assuming $f(x) \geq f(y)$

$$\Longleftrightarrow \left(1 + \frac{a}{y}\right)^{\alpha} < \frac{\varepsilon}{y^{\alpha}} + 1.$$

$$\Longleftrightarrow \boxed{a < \underbrace{y\left[\left(\frac{\varepsilon}{y^{\alpha}} + 1\right)^{1/\alpha} - 1\right]}_{=:\delta}.}$$

$\Downarrow$

So any $|a| < \delta$ will do!

$\Downarrow$

So $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ is continuous. $\square$

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(ii) differentiability:

$$\boxed{\begin{array}{l} f: x \longmapsto x^\alpha \text{ is differentiable at } y. \\ \text{iff. } \lim_{h \to 0} \dfrac{f(y+h) - f(y)}{h} = M \text{ finite} \end{array}}$$

$$\frac{f(y+h) - f(y)}{h} = \frac{y^\alpha}{h}\Big[\underbrace{\Big(1 + \frac{h}{y}\Big)^\alpha}_{} - 1\Big].$$

$$\underset{\alpha \to 0}{=} 1 + \alpha \frac{h}{y} + O(h^2).$$

$$\underset{\alpha \to 0}{=} \alpha y^{\alpha - 1} + O(h).$$

$\Longrightarrow$ So $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ is differentiable.

So $\displaystyle\lim_{h \to 0} \frac{f(y+h) - f(y)}{h} = \underbrace{\alpha y^{\alpha - 1}}_{= f'(y)}$ finite at $y \neq 0$.

$\square$.

---

(iii). $\succsim$ strictly increasing iff $x >_{\mathbb{R}^2_+} y \Longrightarrow x \succ_X y$. translated to utility.

i.e. $\forall i$ $x_i \geq y_i$ and $\exists_j$ $x_j > y_j$ $\Longleftarrow$

$$\boxed{\begin{array}{l} u \text{ strictly increasing iff} \\ x >_{\mathbb{R}^2_+} y \Longrightarrow u(x) >_{\mathbb{R}} u(y). \end{array}}$$

So consider $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $x = \begin{pmatrix} y_1 \\ y_2 + \varepsilon \end{pmatrix}$ with $\varepsilon > 0$.

$$\Longrightarrow u(x) = y_1^{\alpha_1}(y_2 + \varepsilon)^{\alpha_2} = \underbrace{y_1^{\alpha_1} y_2^{\alpha_2}}_{= u(y)} \underbrace{\Big(1 + \underbrace{\frac{\varepsilon}{y_2}}_{>0}\Big)^{\overset{>0}{\alpha_2}}}_{>1}$$

so $\underline{u(x) > u(y)}$ $\square$.

(C13) d).

Show $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, $\alpha_{1,2} > 0$.
is <u>strictly quasiconcave.</u>

<u>Proof:</u> Not possible to show it directly with the "standard" definition. (Try it).

$u$ strictly quasiconcave.
iff. $\forall \lambda \in ]0,1[$, $x \neq x'$.
$$u(\lambda x + (1-\lambda) x') > \min \{u(x), u(x')\}.$$

$\longrightarrow$ We will use a series of IMPORTANT. lemmas.

(C13) d) (i).

Show that (strict.) quasiconcavity is <u>preserved</u> under. strictly increasing transformations. (i.e (strict.) quasiconcavity is an ordinal property).

let $u$ (str.) quasiconcave. $\Longleftrightarrow \forall y \; U_u(y) = \{x;\ u(x) \geq u(y)\}$ (str.) CONVEX.

$\forall f$ strict increasing $U_{f\circ u}(y) = \{x;\ f(u(x)) \geq f(u(y))\}$.
$= U_u(y)$ (str.) CONVEX.

$\Longleftrightarrow$ $f \circ u$ is <u>(str.) quasiconcave</u>! $\square$.

(15) d)

(c) $\boxed{\text{Show that } f \text{ (strict) concave iff } f(x+z) \underset{(<) \text{ for } z\neq 0}{\leq} f(x) + \nabla f(x)\cdot z \quad \forall z.}$

Proof:

$f$ (strict) concave

$\Longrightarrow$ ..

$$f(\underbrace{\alpha x + (1-\alpha)y}_{y + \alpha \underbrace{(x-y)}_{=: z}}) = f(y+\alpha z) \underset{(>)_{z\neq 0}}{\geq} \underbrace{\alpha f(x) + (1-\alpha) f(y)}_{f(y) + \alpha[\underbrace{f(x)}_{=f(y+z)} - f(y)]}.$$

$\Updownarrow$

$$f(y+\alpha z) \geq f(y) + \alpha[f(y+z) - f(y)].$$

$$\Longleftrightarrow \quad \underbrace{\frac{f(y+\alpha z) - f(y)}{\alpha}}_{\xrightarrow{\alpha \to 0} \nabla f(y)\cdot z} \underset{(>)_{z\neq 0}}{\geq} f(y+z) - f(y).$$

$$\Longrightarrow \boxed{f(y+z) \leq f(y) + \nabla f(y)\cdot z.}$$

7/12 H.

(C13) d) ~~(iii)~~

(iii) Show that $f$ (strict) concave
iff $Hf(x)$ is negative (semi) definite. $\forall z, \forall x$.
i.e. $z \cdot Hf(x) z \underset{(\text{resp} \underset{z\neq 0}{<})}{\leq} 0$

Proof: $\cdot$ $f(x+\alpha z) \underset{\alpha\to 0}{=} f(x) + \alpha \nabla f(x) \cdot z$

LIMITED EXPANSION order 2. $+ z \cdot Hf(x) z \frac{\alpha^2}{2} + O(\alpha^3)$.

By the previous lemma.

$$f(x+\alpha z) - f(x) - \alpha \nabla f(x) \cdot z \underset{(\text{resp} <)}{\leq} 0$$

So. $$z \cdot Hf(x) z + O(\alpha) \underset{(\text{resp} >)}{\geq} 0.$$

$$\Longrightarrow z Hf(x) z \underset{(>)}{\geq} 0 \quad \forall z \ \forall x.$$

$\square$.

(C13) d) ~~(iii)~~ Now use ~~(ii)~~ and (iii) to show (and (i)) that
$u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$ is strict quasiconcave.

Proof: We start by proving that.

$v(x) = \ln u(x) = \alpha_1 \ln x_1 + \alpha_2 \ln x_2$.

is strictly concave. :

$$Hv(x) = \begin{pmatrix} \frac{\partial^2 v}{\partial x_1^2}(x) & \frac{\partial^2 v}{\partial x_1 \partial x_2}(x) \\ '' & \frac{\partial^2 v}{\partial x_2^2}(x) \end{pmatrix} = \begin{pmatrix} -\alpha_1/x_1^2 & 0 \\ 0 & -\alpha_2/x_2^2 \end{pmatrix}.$$

this is the great advantage of considering the log of $u(x)$.

→ $Hv(x)$ is diagonal, we immediately see that its eigenvalues are. $< 0$.

so. $Hv(x)$ is NEGATIVE DEFINITE.

⟹ so $v$ is strictly concave. (using (iii)).

⟹ so. $v$ is strictly quasiconcave.

⟹ so $u = \exp(v)$. is strictly quasiconcave. (using (i))

↑ strictly increasing transformation. □.

remembering that:

$$\begin{array}{ccc} \text{strictly concave} & \Longrightarrow & \text{concave} \\ \Downarrow & & \Downarrow \\ \text{strictly quasiconcave} & \Longrightarrow & \text{quasiconcave.} \end{array}$$

(13) e). Let $p = (p_1, p_2) \gg 0$ and $w > 0$.

(ii) Determine the demand of the consumer.

(i) Show that the UMP is invariant under strictly increasing trf of the considered utility fct.

Proof: (i) $\text{UMP}_u$: $\begin{cases} \max u(x) \\ p \cdot x \leq w \end{cases}$.

$\Downarrow$

$\bar{x} \in x(p,w)$ i.e. $\bar{x}$ solution to the utility maximization problem iff.

KT conditions: $\begin{cases} \nabla u(\bar{x}) = \lambda p \\ \lambda(p \cdot \bar{x} - w) = 0. \end{cases}$

Consider the $\text{UMP}_{f \circ u}$ $\begin{cases} \max f(u(x)) \\ p \cdot x \leq w. \end{cases}$ with $f$ strictly increasing fct.

$\Downarrow$ let $\bar{x}$ solve $\text{UMP}_u$

we have: $\nabla(f \circ u)(\bar{x}) \underset{\text{chain rule}}{=} \underbrace{f'(u(\bar{x}))}_{> 0 \text{ by hypothesis}} \nabla u(\bar{x})$.

$= f'(u(\bar{x})) \underbrace{\lambda p}_{\text{since } \bar{x} \text{ solves } \text{UMP}_u}$. $\Longrightarrow$ $\nabla(f \circ u)(\bar{x}) = \lambda' p$. with $\lambda' = f'(u(\bar{x}))\lambda$.

and. $\lambda'(p \cdot \bar{x} - w) = f'(u(x))\lambda \underbrace{(p \cdot \bar{x} - w)}_{= 0}$ since $\bar{x}$ solves $\text{UMP}_u$.

$\longrightarrow$ So $\bar{x}$ also solves $\text{UMP}_{f \circ u}$!

$\square$.

→ Note: it was actually obvious since any tf strict increasing *preserves* the represented preference $\succsim$.

and the utility maximization problem can be redefined in terms of $\succsim$:

$$\boxed{\bar{x} \in x(p,w) \text{ iff. } \underbrace{\bar{x} \succsim x}_{\text{preferred to all } x} \quad \forall x \in \underbrace{B_{p,w}}_{\text{i.e. } x \text{ satisfying the budget constraint}}}$$

(ii) So, instead of $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$, we can solve the UMP for $v(x) = \ln u(x)$
$= \alpha_1 \ln x_1 + \alpha_2 \ln x_2$.

$$\Longrightarrow \nabla v(x) = \begin{pmatrix} \alpha_1 / x_1 \\ \alpha_2 / x_2 \end{pmatrix} \underset{\text{KT conds.}}{=} \lambda p = \lambda \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

and Walras' law (from local non-satiation), (otherwise KT just says $\lambda(p \cdot x - w) = 0$)

$p \cdot x = p_1 x_1 + p_2 x_2 = w$.

remembering

$$\boxed{\begin{array}{c} \text{strict monotone (increasing)} \\ \Downarrow \\ \text{monotone (increasing)} \\ \Downarrow \\ \text{locally nonsatiated} \end{array}}$$

$$\Longrightarrow \left.\begin{array}{l} \dfrac{\alpha_1}{x_1} = \lambda p_1 \\ \dfrac{\alpha_2}{x_2} = \lambda p_2 \end{array}\right\} \Rightarrow \lambda(\underbrace{p_1 x_1 + p_2 x_2}_{=w}) = \alpha_1 + \alpha_2$$

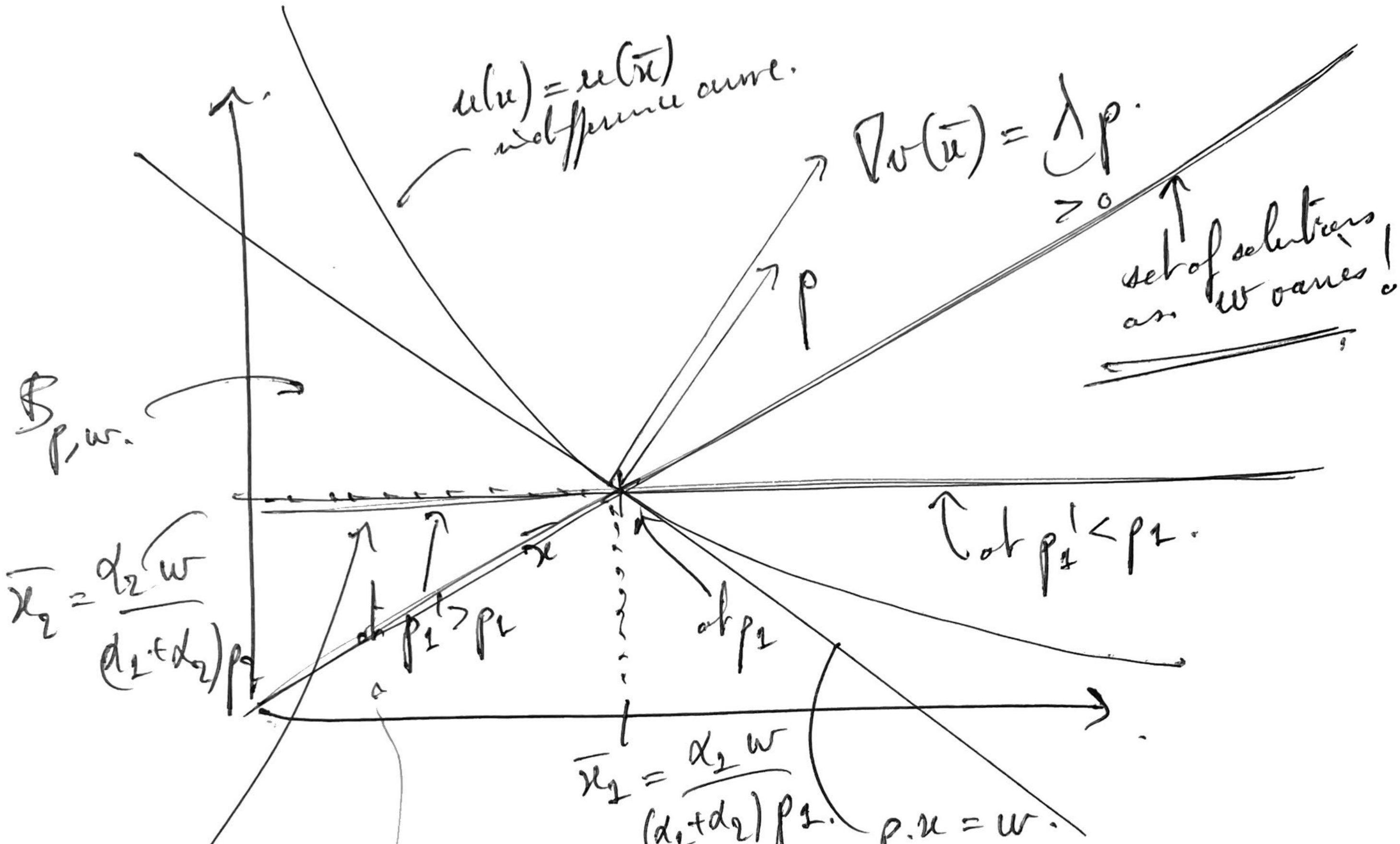
$$\Longrightarrow \lambda = \frac{\alpha_1 + \alpha_2}{w}$$

$$\Longrightarrow \boxed{x_i = \frac{\alpha_i w}{(\alpha_1 + \alpha_2) p_i} \quad i = 1, 2}$$

□.

f) Provide a graphical rep. of the solution to the UMP.

* Assume that the price of good 1 changes $\Longrightarrow$ draw the associated supply curve.
* Assume that the wealth changes $\Longrightarrow$ draw the associated wealth-consumption curve.




"supply" curve = "set of solutions as $p_1$ varies".

* As $p_1 \to p_1'$ ; $\bar{x}_1^*$ is changed to $= \frac{\alpha_1 w}{(\alpha_1+\alpha_2)p_1'}$

BUT $\bar{x}_2^*$ is not changed.

$\Longrightarrow$ "supply curve" $= \{x ;\ x_2 = \bar{x}_2\}$ $\leftarrow$ horizontal.

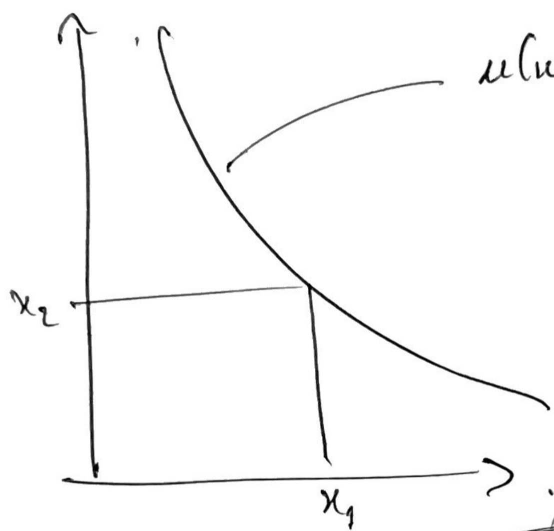
* wealth-consumption curve $= \left\{x ;\ x_1 = \frac{\alpha_1 w}{(\alpha_1+\alpha_2)p_2},\ x_2 = \frac{\alpha_2 w}{(\alpha_1+\alpha_2)p_2} ;\right.$ with $\left. w \in ]0,+\infty[ \right\}$. □

linear.

(C12) a) Convex pref, i.e. quasiconcave $u$.

$$\Longrightarrow \boxed{MRS_{12}(x) = \frac{\partial u / \partial x_1 (x)}{\partial u / \partial x_2} \text{ decreasing in } x_1.}$$

Proof:




$u(x) = u^* = cste$ along an indifference curve.

$$\Downarrow$$

$$du^* = 0 = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2.$$

$$\Downarrow$$

$$\boxed{\frac{\partial u / \partial x_1}{\partial u / \partial x_2} = - \frac{dx_2}{dx_1}.} \quad \text{at } u^* = cste$$

$u(x_1, x_2) = u^*$. IMPLICIT SET.

$$\Downarrow$$

$$\boxed{\exists f.\ x_2 = f(x_1, u^*)}$$

$$\hookrightarrow \frac{dx_2}{dx_1} = \frac{\partial f}{\partial x_1}(x_1, u^*) \text{ at } u^* = cste. = -MRS_{12}(x_1, u^*)$$

$\longrightarrow$ So $\dfrac{\partial MRS_{12}(x)}{\partial x_1} \leq 0 \iff MRS_{12}$ decreasing in $x_1$.

$$\Updownarrow$$

$$-\frac{\partial^2}{\partial x_1^2} f(x_1, u^*) \leq 0. \iff \frac{\partial^2}{\partial x_1^2} f \geq 0 \text{ i.e. } f \text{ CONVEX}$$

$$\Updownarrow$$

$\{x;\ x_2 \geq f(x_1, u^*)\}$ CONVEX SET.

$$\Updownarrow$$

$u$ quasi-CONCAVE! $\iff$ $\{x;\ u(x_1, x_2) \geq u^*\}$ CONVEX SET □

(12) b)

(ii). show $u$ (strictly) quasiconcave

iff $u(x') \geq u(x)$ (and $x' \neq x$).

$\Longrightarrow \nabla u(x) \cdot (x'-x) \geq 0$ (resp. $> 0$).

Proof:

$u$ quasiconcave

iff $\forall \alpha \in [0, 1]$ (and $x \neq x'$).
( ] [ ).

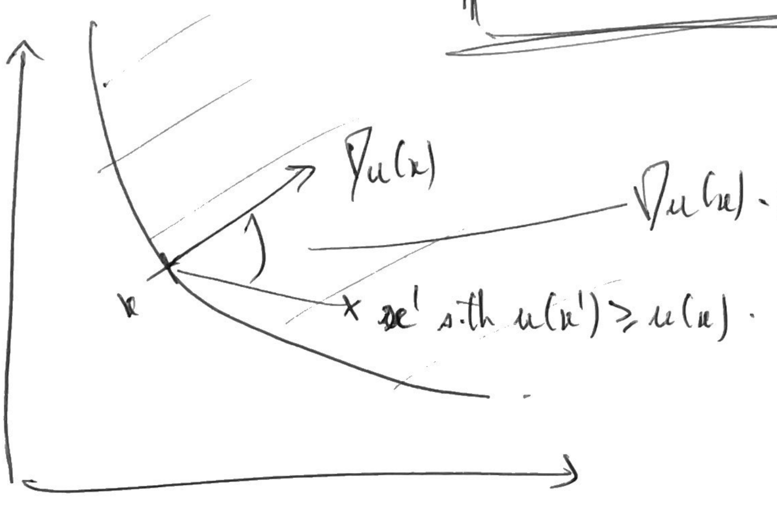
$$u(\underbrace{\alpha x' + (1-\alpha)x}_{= x + \alpha(x'-x) \text{ if } \alpha \to 0}) \underset{(>)}{\geq} \underbrace{\min\{u(x), u(x')\}}_{= u(x)}$$

(since we assume $u(x') \geq u(x)$)

LIMITED. EXPANSION.

$$\cancel{u(x)} + \underbrace{\alpha}_{\geq 0} \nabla u(x) \cdot (x'-x) \geq \cancel{u(x)}.$$

$$\Longleftrightarrow \boxed{\nabla u(x) \cdot (x'-x) \geq 0}$$




$\nabla u(x) \cdot (x'-x) \geq 0$.

$\Longleftrightarrow$ ACUTE ANGLE !!

(C12) c) Show that $u$ (strict) quasiconcave.

iff. $Hu(x)$ is negative (semi) definite.

on $\operatorname{Ker} \nabla u(x)$. $\forall x$.

i.e. $z \cdot Hu(x) z \leq 0$.

(resp. $< 0$).

$z \neq 0$.

$\forall z \in \operatorname{Ker} \nabla u(x) \iff \forall z$ s.th. $z \cdot \nabla u(x) = 0$.

$\forall x$.

$u$ quasiconcave.

iff $\forall \alpha \in [0,1]$.

$$\underbrace{u(\alpha x' + (1-\alpha)x)}_{\shortparallel} \geq \min\{u(x), \underbrace{u(x')}_{\shortparallel}\}.$$

$x' = x + z$.

$$u(x + \alpha \underbrace{(x' - x)}_{= z})$$

$\shortparallel$

$$u(x + \alpha z) \underset{\alpha \to 0}{=} u(x) + \alpha \nabla u(x) \cdot z + \frac{\alpha^2}{2} z \cdot Hu(x) z + O(\alpha^3).$$

$\uparrow$

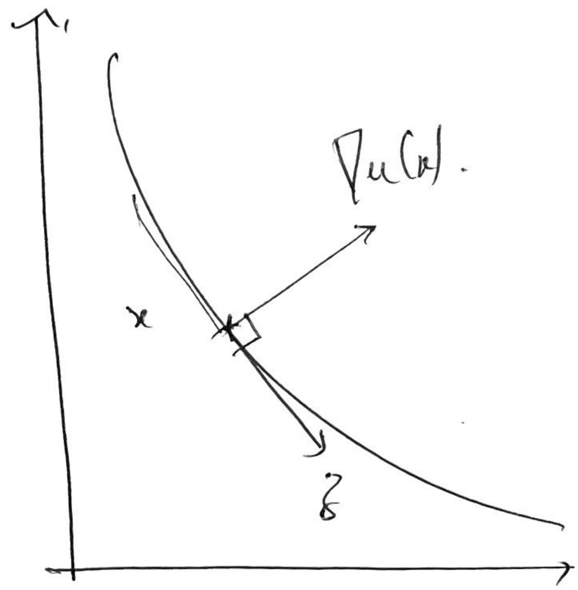
LIMITED expansion at the 2nd order.

$\geq$ ~~$u(x+z)$~~ $\min\{\underline{u(x), u(x+z)}\}$. □

~~because~~

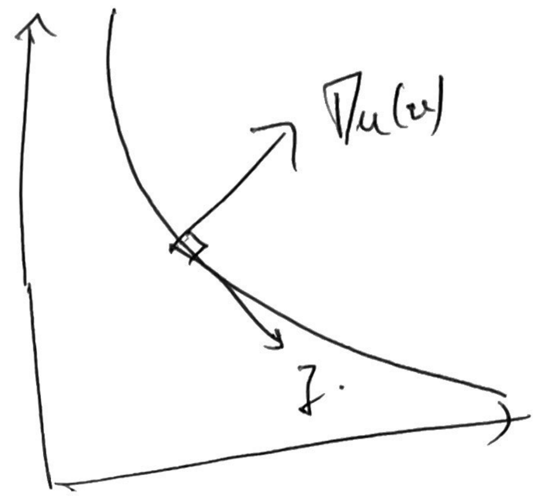
But at $\nabla u(x) \cdot z = 0$.

$\implies \underline{x + z \in L(x)} \implies u(x+z) \leq u(x)$



$$u(x+\alpha z) \underset{\alpha \to 0}{=} u(x) + \alpha \nabla u(x)\cdot z + \frac{\alpha^2}{2} z \cdot Hu(x) z + O(\alpha^3).$$

LIMITED
EXPANSION.




at $\nabla u(x)\cdot z = 0$.

$\Rightarrow x+\alpha z \in L(x)$.

$\Rightarrow u(x+\alpha z) \le u(x)$.

This is where.

where have I used quasi-concavity ??

corollary of (ii)

So at $\nabla u(x)\cdot z = 0$.

$u(x+\alpha z) - u(x)$

$= \frac{\alpha^2}{2} z \cdot Hu(x) z + O(\alpha^3)$

$\le 0$.

$$\boxed{z Hu(x) z + O(\alpha) \le 0}$$

$\alpha \to 0$

0

and $u(x') = u(x)$ $\Rightarrow \nabla u(x)\cdot(x'-x) = 0$

shows that

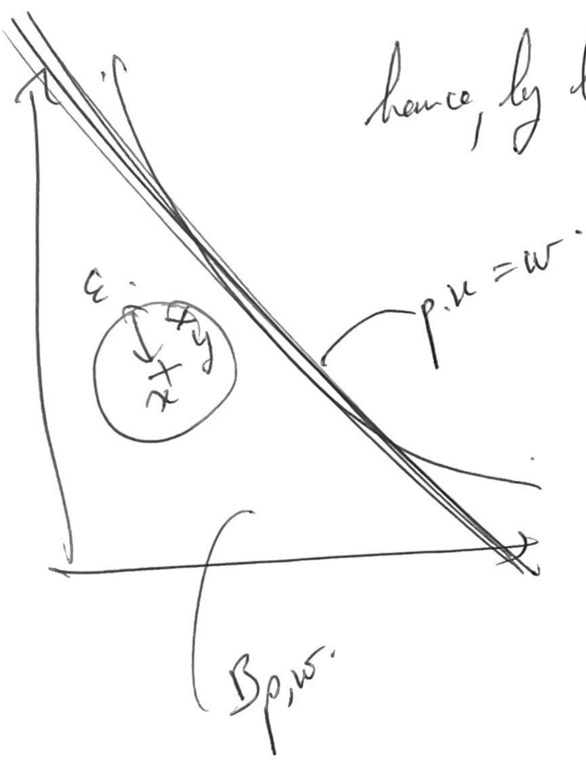
because we've shown if $u(x') \geq u(x)$ $\Rightarrow \nabla u(x)\cdot(x'-x) \geq 0$

So if $\alpha \nabla u(x)\cdot z = 0$.

$\Rightarrow$ we must have $u(\alpha z + x) \le u(x)$

(C12) d) $\boxed{\text{Prove that local nonsatiation} \implies \text{Walras' law.}}$

**Proof:** Assume $\bar{x}$ is a global maximizer (w.r.t. UMP) s.th. $p \cdot \bar{x} < w$. (ie it does not satisfy Walras' law.




hence, by local nonsatiation, $\exists y$ s.th. $p \cdot y < w$ (since it can be as close to $x$ as I want, in an "open ball").

s.th. $y \succ \bar{x}$.

ie $u(y) > u(\bar{x})$.

which contradicts that $\bar{x}$ was a maximizer!

hence $x \in x(p,w)$. ie Walrasian demand correspondence $\equiv$ sol to UMP

$\Downarrow$

$p \cdot x = w$.

C12 e). (~~Necessary conditions.~~)

(i) Prove that if $\bar{x}$ is a local extremum, then it satisfies the KT conditions. (LAGRANGE)

$$\nabla u(\bar{x}) = \lambda p \quad \text{and} \quad \lambda(p\cdot x - w) = 0.$$

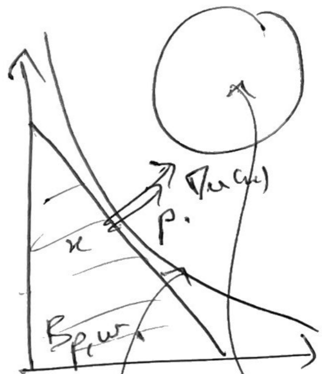
(ii) Prove that if $u$ monotonous $\implies$ $\lambda > 0$. ??

(iii) Prove. KT cond. $\iff$ $MRS_{ij}^{(\bar{x})} = \frac{\partial u/\partial x_i}{\partial u/\partial x_j}(\bar{x}) = \frac{p_i}{p_j}$ when $\lambda \neq 0$.

(ii). $u$ monotonous iff.

$$\boxed{x > y \implies x \succ y}$$

ie. $\forall i \; x_i \geq y_i$, $\exists j \; x_j > y_j$.




convex ≠ NOT monotone.

quasi concave and monotone !!

let $y$. So let $\varepsilon > 0$. $x = \begin{pmatrix} 0 \\ \vdots \\ \varepsilon \\ \vdots \\ 0 \end{pmatrix} + y > y$. on the $i^{th}$ position.

$$\implies u(x) > u(y)$$

$$\implies \underbrace{u(x) - u(y)}_{} > 0$$

$$= u(y + \varepsilon e_i) - u(y)$$

$$\underset{\varepsilon \to 0}{=} \cancel{u(y)} + \varepsilon \frac{\partial u}{\partial x_i}(y) + O(\varepsilon^2) - \cancel{u(y)}$$

$$\implies \boxed{\frac{\partial u}{\partial x_i}(y) + \underbrace{O(\varepsilon)}_{\to 0} > 0} \quad \forall i \quad \blacksquare.$$

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remember $\lambda$ = MARGINAL UTILITY OF WEALTH.

$\implies$ we want it to be $> 0$!

$\frac{\partial u(\bar{x})}{\partial w} = \frac{\partial \bar{x}}{\partial w} \cdot \nabla u(\bar{x})$ (chain rule)

$\frac{1}{p} \cdot \lambda p$ (KT)

$= \lambda$ !! ∎.

$$\begin{cases} \max u(x) \\ \text{subject to} \\ p.x \leq w. \end{cases}$$ ; as we've seen, for $u$ locally ~~non satiated~~.

$\bar{x} \in x(p,w)$.

$\Longrightarrow p.x = w$.

$\Longrightarrow$ So the UMP is equivalent to.

$$\begin{cases} \max u(x) \\ \text{subject to.} \\ p.x = w. \end{cases}$$

We can now use Lagrange method. Define the Lagrangian.

$$L(x,\lambda) = u(x) - \lambda(p.x - w).$$

$\longrightarrow$ the condition $p.x = w$. is reflected by imposing $\frac{\partial L}{\partial \lambda} = 0$.

~~Note~~ $\oplus$ note that under the condition $p.x = w$, we have.

$L(x,\lambda) = u(x)$. s.th.

maximizing $L(x,\lambda)$ ~~w.r.t~~ w.r.t $x$. is the same as maximizing $u(x)$. ~~$L(x$~~

So, the maximum of $u(x)$ subject to the constraint $p.x = w$. ($\Longleftarrow \frac{\partial L}{\partial \lambda} = 0$.)

is obtain when.

$$\nabla L(x,\lambda) = 0.$$

i.e. $\nabla u(x) - \lambda p = 0$.

i.e $\boxed{\nabla u(x) = \lambda p}$

(12) f). (SUFFICIENT CONDITIONS).

Let $\bar{x}$ satisfies the KT conds.
Show that (ii).
if $u$ monotone and quasiconcave.
$\Longrightarrow$ $\bar{x}$ is a global maximizer.

i.e $\bar{x}$ is NOT a global max.

Suppose, $\exists x$. $u(x) > u(\bar{x})$.
s.th. $p.x \leq w$. $\leftarrow$ i.e $x \in B_{p,w}$. is feasible.

$u$ monotone $\Longrightarrow$ ~~u locally nonsatiated~~.
$\Longrightarrow$ . $\lambda > 0$.
so . $\bar{x}$ satisfies KT conds. $\Longrightarrow$ $\overset{>0}{\lambda}(p.\bar{x} - w) = 0$
$\Downarrow$
$\underline{\underline{p.\bar{x} = w}}$.

and. $u$ quasiconcave,
$\Longrightarrow$ $\left( u(x) > u(\bar{x}) \Longrightarrow \nabla u(\bar{x}).(x - \bar{x}) > 0 \right)$.

same remark that before...

$\nabla u(\bar{x}) = \lambda p$ if $\bar{x}$ satisfies KT conds.

$\Longrightarrow$ $\lambda p.(x - \bar{x}) > 0$.

$\underbrace{\lambda p.(x - \bar{x}) = \lambda(\underset{\leq w}{p.x} - \underset{= w}{p.\bar{x}})}_{\text{because } x \in B_{p,w}}$ $> 0$. $\leq 0$

CONTRADICTION !!

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(C12) g) Show. u strictly concave.

$\Longrightarrow \bar{x}$ is unique.

suppose. $x$ and $\bar{x} \in x(p,w)$. with $x \neq \bar{x}$.

by strict concavity.

unique value of the maximum

$u(x) = u(\bar{x})$.

and $x \neq \bar{x}$.

$$\longrightarrow \nabla u(\bar{x}) \cdot (x - \bar{x}) > 0$$

KT.

$\lambda p$.

$>0$

$\gg 0$

but in both cases.

Walras' law.

$\Longrightarrow px = w$ and $p\bar{x} = w$.

$\Downarrow$

$0 > 0$. CONTRADICTION.

$\Longrightarrow$ hence. $\bar{x}$ is UNIQUE.

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