Microeconomics 1A .. (TD9). October 16, 2023

(26) 6B.2 in MWG.

Show that if the preference rel. $\succsim$ on $\mathcal{L}$ is represented by a utility function $U(\cdot)$ that has the expected utility form, then $\succsim$ satisfies the independence axiom.

Def. $U: \mathcal{L} \to \mathbb{R}$ has an expected utility form iff $\exists\, (u_1, \dots, u_n)$ s.th. $\forall L = (p_1, \dots, p_n) \in \mathcal{L}$ (lottery with probabilities p_i)

$$U(L) = u_1 p_1 + \dots + u_n p_n.$$

"assignment" to the n outcomes.

Show again that U has the expected utility form iff it is linear ie.

$\forall L_k \in \mathcal{L}$, $k = 1, \dots, K$ and prob. $(\alpha_1, \dots, \alpha_K) \geq 0$ s.th. $\sum_k \alpha_k = 1$.

$$U\left(\sum_k \alpha_k L_k\right) = \sum_k \alpha_k U(L_k).$$

IMPORTANT PROP. to be used.

Proof: ($\Longrightarrow$) Suppose U has the expected utility form, ie $\forall L = (p_1, \dots, p_n) \in \mathcal{L}$ (after).

(before) $\underbrace{\exists\, (u_1, \dots, u_n)}_{\text{UNIQUE}}$ s.th. $\forall$ $U(L) = u_1 p_1 + \dots + u_n p_n$.

Let $(\alpha_1, \dots, \alpha_K) \geq 0$ with $\sum_k \alpha_k = 1$.

Let $L_k \in \mathcal{L}$, $k = 1, \dots, K$. $\Longrightarrow \exists (u_{k1}, \dots, u_{kn})$ s.th. $U(L_k) = \sum_l u_{kl} p_{kl}$ $\forall k$.

and.

$L' = \sum \alpha_k L_k \in \mathcal{L}$, so $\exists (u_1^*, \dots, u_n^*)$.

s.th. $U(L') = \sum_l u_l^* p_l'$.

and, by def of a convex combination of lotteries.

$$\forall l \quad p_l' = \sum_k \alpha_k p_{kl}.$$

$$\longrightarrow U(L') = \sum_{lk} u_l^* \alpha_k p_{kl} = \sum_k \alpha_k \sum_l u_l^* p_{kl}$$

$$= \sum_k \alpha_k U(L_k).$$

how do I know that the assignments are the same?

for a single U, the assignment of utility to all outcomes is unique.

$(\Longleftarrow)$ Suppose U is linear.

i.e. $\forall L_k$. $\sum_k \alpha_k = 1$.

$$U(\sum \alpha_k L_k) = \sum \alpha_k U(L_k).$$

assign utility.

Suppose. $L_k = (0, \dots, \underset{\substack{\uparrow \\ k^{th} \text{ position}}}{1}, \dots, 0)$.

then. $L' = \sum \alpha_k L_k = (\alpha_1, \dots, \alpha_n) \in \mathcal{L}$, lottery of totally general form.

and we have that. $U(L') = \sum_{k=1}^{n} \underset{\substack{\uparrow \\ \text{its probabilities}}}{\alpha_k} \underbrace{U(L_k)} = u_k$ the utilities assigned to each outcome

$\square$.

Def. $\succsim$ satisfies the <u>INDEPENDENCE AXIOM</u> iff $\forall L, L', L'' \in \mathcal{L}$ and $\alpha \in]0,1[$.

$$L \succsim L' \iff \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

*

Assume U has the <u>expected utility form</u>.
i.e. U is <u>linear</u>.

Remember: CONTINUITY AXIOM.
$\Rightarrow \exists\, U : \mathcal{L} \to \mathbb{R}$ representing $\succsim$
i.e. s.th. $L \succsim L' \iff U(L) \geq U(L')$

Let L, L', L'' s.th. $L \succsim L'$ and $\alpha \in]0,1[$.

$$L \succsim L' \iff \underset{\displaystyle\sum_i u_i p_i}{\overset{\Vert}{U(L)}} \geq \underset{\displaystyle\sum_i u_i p_i'}{\overset{\Vert}{U(L')}}$$

and $U(L'') = \sum_i u_i p_i''$

$$\Rightarrow \text{thus} \ldots \quad \alpha \sum_i u_i p_i + (1-\alpha)\sum_i u_i p_i'' \geq \alpha \sum_i u_i p_i' + (1-\alpha)\sum_i u_i p_i''$$

$$\iff \sum_i u_i (\alpha p_i + (1-\alpha)p_i'') \geq \sum_i u_i (\alpha p_i' + (1-\alpha)p_i'').$$

$$\iff \quad U(\alpha L + (1-\alpha)L'') \geq U(\alpha L' + (1-\alpha)L'').$$

$$\iff \quad \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

i.e. $\boxed{\succsim \text{ satisfies the independence axiom}}$ $\square$.

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(27) $\boxed{\begin{array}{l}\text{Show that.}\\ \succsim \text{ representable by } U \text{ of the expected utility form} \implies \succsim \text{ transitive.}\end{array}}$

Proof: We want to show.

$$\Big(\exists U \text{ linear s.th. } \forall L, L'.\quad L \succsim L' \iff U(L) \geq U(L'). \Big)$$

$$\implies \Big(\forall L, L', L'' \quad L \succsim L' \text{ and } L' \succsim L'' \implies L \succsim L''. \Big).$$

So... assume. $L \succsim L'$ and $L' \succsim L''$.

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\Big(U(L) \geq U(L') \qquad U(L') \geq U(L''). \Big).$$

$$\implies \quad U(L) \geq U(L'').$$

$\iff L \succsim L''$. So $\succsim$ TRANSITIVE $\square$.

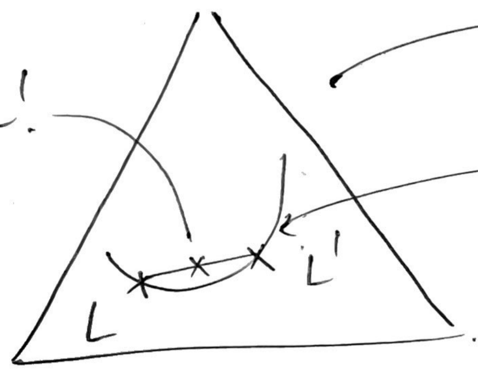
(28) Consider $\mathcal{L} = \{(p_1, p_2, p_3);\ 0 \le p_i \le 1 \text{ and } \sum_i p_i = 1\}$.

Show *graphically* that if a preference relation $\succsim$ on lotteries $\in \mathcal{L}$ satisfies the independence axiom, then the indifference curves are parallel.

(Hint: By contradiction).

Assume successively that indifference curves are *curved lines*, then straight lines but *not* parallel.

$\frac{1}{2}L + \frac{1}{2}L'$



depiction of $\mathcal{L}$, simplex, because $\sum_i p_i = 1$.

assume this is an indifference curve s.th. $L \sim L'$

$L = (p_1, p_2, p_3)$, $L' = (p_1', p_2', p_3')$.

The compound lottery $\frac{1}{2}L + \frac{1}{2}L'$

is $\left(\frac{1}{2}(p_1 + p_1'), \frac{1}{2}(p_2 + p_2'), \frac{1}{2}(p_3 + p_3')\right)$;

i.e. on the straight segment linking L to L' in the simplex,

Assuming the indifference curve "strictly" curved, would imply: $\frac{1}{2}L + \frac{1}{2}L' \not\sim L, L'$.

(i.e. either $\succ$ or $\prec$).

But. the independence axiom would imply:
(IA).

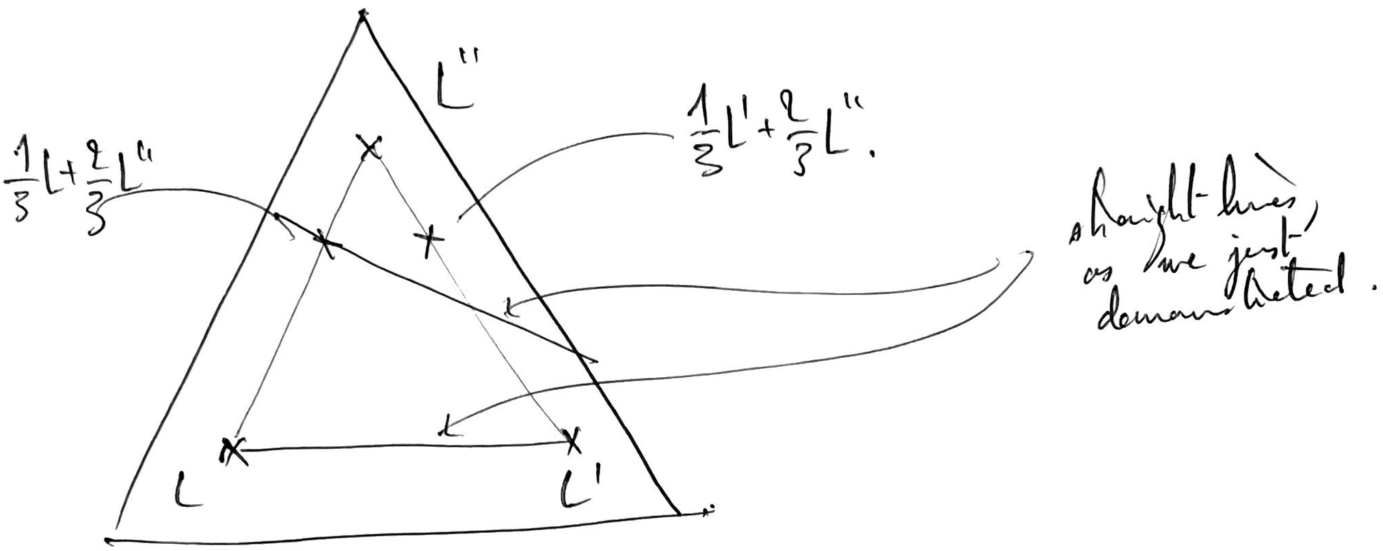
$$L \sim L'$$
$$\Updownarrow$$
$$L \succsim L' \quad \text{and} \quad L \precsim L'$$
$$\Updownarrow (IA) \qquad\qquad \Updownarrow (IA)$$
$$\underbrace{\frac{1}{2}L + \frac{1}{2}L}_{= L} \succsim \frac{1}{2}L' + \frac{1}{2}L \qquad \underbrace{\frac{1}{2}L + \frac{1}{2}L}_{L} \precsim \frac{1}{2}L' + \frac{1}{2}L$$

$$\Longrightarrow \text{so} \quad L \sim \frac{1}{2}L' + \frac{1}{2}L.$$
$$\left(\begin{matrix}\wr \\ L'\end{matrix}\right).$$

$\Longrightarrow$ This actually already proves that indifference curves are straight lines, without the need to show contradiction... $\square$.
$\varphi \in]0,1[$.

This contradicts the previous...

* Assume indifference curves are straight lines, but not parallel.



Let. $L \sim L' \iff (L \succsim L'$ and $L \precsim L')$.

by the independence axiom, we must have.

$$\left(\frac{1}{3}L + \frac{2}{3}L'' \succsim \frac{1}{3}L' + \frac{2}{3}L'' \quad \text{and} \quad \frac{1}{3}L + \frac{2}{3}L'' \precsim \frac{1}{3}L + \frac{2}{3}L''\right)$$

$$\implies \frac{1}{3}L + \frac{2}{3}L'' \sim \frac{1}{3}L' + \frac{2}{3}L''.$$

Contradiction with the drawing if $I(\frac{1}{3}L + \frac{2}{3}L'')$ is not parallel to $I(L)$.

$\implies$ If written more generally for $\alpha \in]0,1[$, this proves directly that indifference curves are parallels.

$\square$

(29) MWG. 6.B.4. $\alpha = 1\%$ prob. that a flood happens.

(A) no evacuation is necessary = no flood, none is performed = no evacuation.
(B) evacuation, no flood.
(C) evacuation, flood.
(D) no evacuation, flood.

Suppose.
$$\begin{cases} B \sim pA + (1-p)D. \\ C \sim qB + (1-q)D. \\ A \succ D. \end{cases}$$

$p, q \in]0,1[$.

a) Construct a utility fct of the expected utility form.

b) criterion 1 : $p(\text{evacuation} \mid \text{flood}) = 0.9$
$p(\text{evacuation} \mid \text{no flood}) = 0.1$.

criterion 2 : resp. 0.95 and 0.05. Which criterion should the agency prefer ??

a) Suppose.
$$\begin{cases} B \sim pA + (1-p)D. \\ C \sim qB + (1-q)D \\ A \succ D. \end{cases}$$

$q(1-p) + (1-q)$
$= \cancel{q} - pq + 1 - \cancel{q}$
$= 1 - pq.$

Suppose. U rep. $\succsim$. linear.

$$\Longleftrightarrow \begin{cases} U(B) = pU(A) + (1-p)U(D) \\ U(C) = qU(B) + (1-q)U(D) \\ \text{and} \\ U(A) > U(D). \end{cases}$$

let. $u_A = U(A)$, $u_D = U(D)$ with $u_A > u_D$.

$\Longrightarrow$ $u_B = pu_A + (1-p)u_D.$

and. $u_C = qu_B + (1-q)u_D. = pq\, u_A + (1-pq) u_D.$

b). Assume. <u>Criterion 1:</u>

$$\star \quad p(\text{evacuation} \mid \text{flood}) = 0.9.$$

$$= \frac{p(\text{evacuation } \underline{\text{AND}} \text{ flood})}{p(\text{flood})} = \boxed{\frac{p(C)}{\alpha} = 0.9}$$

$$\star \quad p(\text{evacuation} \mid \text{no flood}) = 0.1.$$

$$= \frac{p(\text{evacuation } \underline{\text{AND}} \text{ no flood})}{p(\text{no flood})} = \boxed{\frac{p(B)}{1-\alpha} = 0.1.}$$

AND, ~~furthermore~~, $p(\text{no evacuation} \mid \text{flood})$

$$= 1 - p(\text{evacuation} \mid \text{flood}).$$

$$\frac{p(\text{no evac AND flood})}{p(\text{flood})} = \boxed{\frac{p(D)}{\alpha} = 1 - 0.9 = 0.1.}$$

and. $p(\text{no evacuation} \mid \text{no flood}) = 1 - p(\text{evacuation} \mid \text{no flood}).$

$$\frac{p(\text{no evac AND no flood})}{p(\text{no flood})} = \boxed{\frac{p(A)}{1-\alpha} = 1 - 0.1 = 0.9.}$$

$\longrightarrow$ So the criterion gives us the probabilities of each outcome. (A, B, C, D).

$\Longrightarrow$ We have.

$$L_1 = (\underbrace{0.891}_{p_A}, \overbrace{0.099}^{p_B}, \overbrace{0.009}^{p_C}, \overbrace{0.001}^{p_D})$$

Likewise. $L_2 = (0.9405, 0.0495, 0.0095, 0.0005)$

So using. $\forall L$. $U(L) = u_A p_A + u_B p_B + u_C p_C + u_D p_D$.

$$U(L) = u_A p_A + [p u_A + (1-p) u_D] p_B + [pq u_A + (1-pq) u_D] p_C + u_D p_D.$$

with. $u_A > u_D$.

So.

$$U(L_1) = u_A (0.891 + p\, 0.099 + pq\, 0.009) + u_D ((1-p)\, 0.099 + (1-pq)\, 0.009 + 0.001)$$

and.

$$U(L_2) = u_A (0.9405 + p\, 0.0495 + pq\, 0.0095) + u_D ((1-p)\, 0.0495 + (1-pq)\, 0.0095 + 0.0005)$$

So .

$$U(L_2) - U(L_1).$$

$$= u_A \left(0.0495 - p\, 0.0495 + pq\, 0.005 \right).$$

$$+ u_D \left((1-p)\, 0.0495 + (1-pq)\, 0.005 - 0.0005 \right).$$

$$= 0.0495 \overbrace{(u_A - u_D)}^{>0} + \overbrace{0.005 - 0.0005}^{0.0045}$$

$$+ p \left(0.0495 \underbrace{(u_D - u_A)}_{<0.} + q\, 0.005 \underbrace{(u_A - u_D)}_{>0} \right).$$

but $p \in [0,1]$. $\longrightarrow$ $U(L_2) - U(L_1) > 0.$

$$\Updownarrow$$

$$L_2 \succ L_1.$$

so they should prefer criterion 2 ...

(30) MWG. 6B7.

Let. $L: \begin{cases} 200\ \$ \text{ with. } p = 0.7. \\ 0\ \$ \quad — \quad 1-p = 0.3. \end{cases}$

$L': \begin{cases} 1200\ \$ \text{ with } p' = 0.1. \\ 0\ \$ \quad — \quad 1-p' = 0.9. \end{cases}$

let x_L and $x_{L'}$ the sure amount of money that an individual. finds indifferent to L and L'.

Show that if his pref. are TRANSITIVE. and MONOTONE, the individual. must prefer L to L' iff $x_L > x_{L'}$.

Note: a pref $\succsim$ over lotteries is monotone iff. given two real outcomes $C_1 > C_2$, the lottery with "sure outcome C_1 is strictly preferred to the lottery with sure outcome C_2.

ie.

$$L_1 = \begin{cases} C_1 & \text{with prob } 1. \\ 0 & \text{with prob } 0. \end{cases}$$

$$L_2 = \begin{cases} 0 & \text{with prob } 0 \\ C_2 & \text{with prob } 1. \end{cases}$$

$$C_1 > C_2 \Longrightarrow L_1 \succ L_2.$$

Suppose. $\succsim$ is Transitive and Monotone.

Suppose. x_L s.th.

$L_{x_L} = \{ x_L$ with prob 1 $\sim L$.

$L_{x_{L'}} = \{ x_{L'}$ with prob 1. $\sim L'$

Assume. $\succsim$ monotone. $\Longleftrightarrow$.

$$x_L > x_{L'} \Longrightarrow \boxed{L_{x_L} \succ L_{x_{L'}}}.$$

and $\succsim$ transitive. $\Longrightarrow$.

$$L_{x_L} \sim L \Longleftrightarrow (L_{x_L} \succsim L \text{ and } L_{x_L} \precsim L).$$

$$L_{x_{L'}} \sim L' \Longleftrightarrow (L_{x_{L'}} \succsim L' \text{ and } L_{x_{L'}} \precsim L').$$

.... using what we know from WEEK 1. $x \succsim y \succ z \Rightarrow x \succ z$.

$\Rightarrow L \succsim L_{x_l} \succ L_{x_{l'}} \Rightarrow L \succ L_{x_{l'}}$

and. $L \succ L_{x_{l'}} \succsim L' \Rightarrow \boxed{L \succ L'}$. $\square$.

otherwise, complete proof uses again proof by <u>contradiction</u> ...

~~Assume.~~

31

$$A = \{(3500, 2800, 0) ; p = \{0.3, 0.66, 0.04\}\}$$

$$B = \{3500, p = 1\}.$$

and.

$$C(\{A, B\}) = \{B\}.$$

i.e. $\boxed{B \succ^* A.}$

$$A' = \{(3500, 2800, 0) ; p = (0.3, 0, 0.7)\}.$$

$$B' = \{(3500, 2800, 0) ; p = (0, 0.34, 0.66)\}$$

and $C(\{A', B'\}) = \{A'\}$ i.e. $\boxed{A' \succ^* B'.}$

is $\succsim^*$ consistent with the expected utility form ??

Assume. $\succsim^*$, as revealed by $C(\cdot)$.
is representable by U <u>linear</u>.

i.e. $\exists$ an assignment $(u_{3500}, u_{2800}, u_0)$.
to each of the three possible outcomes.

$(3500, 2800, 0)$.

s.th. $\forall L = (p_{3500}, p_{2800}, p_0)$.

$$U(L) = u_{3500}\, p_{3500} + u_{2800}\, p_{2800} + u_0\, p_0 .$$

Now,

$$B \succ^{\alpha} A \iff U(B) > U(A)$$

$$\iff u_{3500} > u_{3500} \times 0.3 + 0.66\, u_{2800} + 0.04\, u_0$$

$$\iff \boxed{0.7\, u_{3500} > 0.66\, u_{2800} + 0.04\, u_0} \quad (1)$$

and

$$A' \succ^{\alpha} B' \iff U(A') > U(B')$$

$$\iff 0.3\, u_{3500} + 0.7\, u_0 > 0.34\, u_{2800} + 0.66\, u_0 .$$

$$\iff \boxed{0.3\, u_{3500} + 0.04\, u_0 > 0.34\, u_{2800}} \quad (2)$$

(1) $\Rightarrow$

$$0.7\, u_{3500} > 0.66\, u_{2800} + 0.04\, u_0 > 0.34\, u_{2800} - 0.3\, u_{3500}$$

as sum up (1) and (2).

$$\boxed{u_{3500} > u_{2800}} \Longleftarrow \geq \text{MONOTONE}.$$

I cannot see any contradiction ...

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* Now assume $B'' = \{(2800)\,;\,1\}$.

and $B'' \succ^* A$ (and still $A' \succ^* B'$).

$\Updownarrow$

$$u_{2800} > u_{3500} \times 0.3 + 0.66\, u_{2800} + 0.04\, u_0$$

$\Longleftrightarrow$

$$u_{2800} > 0.66\, u_{2800} + 0.34\, u_{2800} = u_{2800}$$

still

$$0.3\, u_{3500} + 0.04\, u_0 > 0.34\, u_{2800}$$

$\longrightarrow$ CONTRADICTION!

So in that second case, the decision maker's revealed ~~the~~ preference is <u>NOT</u> consistent with the expected utility form.

$\square$