MIDTERM 2019.

Exercice 1: $\succsim$. $u: \mathbb{R}_+^2 \longrightarrow \mathbb{R}$.

$$u(x_1, x_2) = \sqrt{x_1} + \sqrt{x_2}.$$

1. Show. $\succsim$ continuous, monotone, and convex

* $\succsim$ continuous iff. it is preserved under taking limits

i.e.

$$\forall \{x_n\} \{y_n\}. \quad x_n \succsim y_n \quad \forall n. \qquad \begin{matrix} x_n \to x \\ y_n \to y \end{matrix} \Longrightarrow x \succsim y.$$

equivalently. iff. ~~$\forall y, U(y)$ and $L(y)$ are both closed.~~

~~$U(y) = \{x \; u(x) \geq u(y)\}$ is closed because~~ ~~u is continuous~~

$$x_n \succsim y_n \iff u(x_n) \geq u(y_n).$$

$\Downarrow$ u is continuous. $x_n \to x$, $y_n \to y$.

$$x \succsim y \iff u(x) \geq u(y). \qquad \square.$$

* $\succsim$ monotone

iff. $\underbrace{x \gg y}$ $\Longrightarrow$ $x \succ y$.

i.e.

$\forall i \; x_i > y_i$

IN FACT IT IS EVEN STRONGLY MONOTONE.

Assume $x \gg y$, i.e. $\forall i \; x_i > y_i$

$\Longrightarrow$ $u(x) = \underbrace{\sqrt{x_1}}_{> \sqrt{y_1}} + \underbrace{\sqrt{x_2}}_{> \sqrt{y_2}}$.

because $x \mapsto \sqrt{x}$ is strictly increasing.

so $u(x) > u(y)$.

$\Longleftrightarrow$ $x \succ y$ $\square$.

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* $\succsim$ convex. $\succsim$ convex iff.

$$\boxed{\begin{array}{l} x \succsim y \text{ and } z \succsim y \\ \Longrightarrow \forall \alpha \in [0,1] \; \alpha x + (1-\alpha) z \succsim y \end{array}}$$

| it is equivalent to saying that $\forall y$ $U(y)$ is a convex set. |

which is equivalent to saying that
u is quasi concave.

let $x \succsim y$ and $z \succsim y$.

i.e. $u(x) \geq u(y)$ and $u(z) \geq u(y)$

$\Updownarrow$ equiv.

$z \geq y$

since $\succsim$, (u) is monotone.

let $\alpha x + (1-\alpha) z$.

$$u(\quad) = \sqrt{\alpha x_1 + (1-\alpha) z_1} + \sqrt{\alpha x_2 + (1-\alpha) z_2}$$

$$\geq$$

$\Longrightarrow \begin{cases} z_1 \geq y_1 \\ z_2 \geq y_2 \end{cases}$

$\Longleftrightarrow \begin{cases} x_1 \geq y_1 \\ \text{and } x_2 \geq y_2 \end{cases}$

so $\alpha x_1 + (1-\alpha) z_1 \geq \alpha y_1 + (1-\alpha) y_1 = y_1$.

same for z.

and $\sqrt{\cdot}$ is increasing, so.

$$= \sqrt{\cdot} + \sqrt{\cdot} \geq \sqrt{\cdot} + \sqrt{\cdot}$$

$$u(\alpha x + (1-\alpha) z) \geq u(y)$$

A.

* alternatively, we could have shown u quasiconcave.

$$Hu(x) = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} & \frac{\partial^2 u}{\partial x_1 \partial x_2} \\ \frac{\partial^2 u}{\partial x_2 \partial x_2} & \frac{\partial^2 u}{\partial x_2^2} \end{pmatrix}.$$

$$= \begin{pmatrix} \frac{\partial}{\partial x_1}\left(\frac{1}{2\sqrt{x_1}}\right) & 0 \\ 0 & \frac{\partial}{\partial x_2} \frac{1}{2} x_2^{-\frac{1}{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} x_1^{-3/2} & 0 \\ 0 & -\frac{1}{4} x_2^{-3/2} \end{pmatrix}.$$

$Hu(x)$ has strictly negative eigenvalues $\forall x \gg 0$.

$\Longrightarrow$ $Hu(x)$ is negative definite.

or. u. is strictly concave.

$\Longrightarrow$ u is strictly quasiconcave,

$\Longleftrightarrow$ $\succsim$ is strictly convex.

$\Longrightarrow$ $\succsim$. is <u>convex</u> $\square$

2) Let $p \gg 0$, $w > 0$.

u is continuous. $\Longrightarrow$

u is quasi concave and monotone.

$\Longrightarrow$ The Kuhn-Tucker conditions are sufficient (not just necessary but...).

u is continuously twice-differentiable.

~~[illegible]~~ UMP $\begin{cases} \max\limits_{x \in X} u(x) \\ p \cdot x \leq w. \end{cases}$

~~[illegible]~~

$\bar{x} \in x(p,w) \equiv$ Walrasian demand corresp.

iff. $\boxed{\begin{aligned} &Du(\bar{x}) = \lambda p. \\ &\text{and. } \lambda(p \cdot x - w) = 0. \end{aligned}}$

$$\begin{pmatrix} \frac{\partial u}{\partial x_1}(\bar{x}) \\ \frac{\partial u}{\partial x_2}(\bar{x}) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\sqrt{x_1}} \\ \frac{1}{2\sqrt{x_2}} \end{pmatrix} = \lambda \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}$$

we could also argue u strictly monotone $\Longrightarrow \lambda > 0$

u monotone $\Longrightarrow$ u locally non satiated.

$\Longrightarrow$ Walras' law. i.e.

$$\underline{p \cdot \bar{x} = w.}$$

So ...

$$\lambda (p \cdot \bar{x}) = \lambda w.$$

$$= \frac{1}{2}\left(\sqrt{\bar{x}_1} + \sqrt{\bar{x}_2}\right) = \frac{1}{2} u(\bar{x}).$$

$$\Longrightarrow \boxed{\lambda = \frac{u(\bar{x})}{2w}} \longrightarrow \boxed{\frac{1}{2\sqrt{x_i}} = \frac{u(x)}{2w} p_i}$$

$$\frac{1}{\cancel{2}\sqrt{x_i}} = \frac{u(x)}{\cancel{2}w}\,p_i$$

$$\Longrightarrow \quad \boxed{\sqrt{x_i} = \frac{w}{p_i\, u(x)}}$$

$$u(x) = \frac{w}{u(x)}\left(\frac{1}{p_1} + \frac{1}{p_2}\right).$$

$$u(x)^2 = w\,\frac{p_1 + p_2}{p_1 p_2}.$$

$$\Longrightarrow \quad \cancel{\sqrt{x_i} = \frac{w}{p_i \sqrt{w}}\sqrt{\frac{p_1+p_2}{p_1 p_2}}}$$

OK!

$$\Longrightarrow \quad x_i = \frac{w^{\cancel{2}}}{p_i^2}\,\frac{1}{\cancel{w}}\,\frac{p_1 p_2}{(p_1+p_2)}. \qquad \square$$

$$\boxed{x_1 = \frac{w}{p_1+p_2}\cdot\frac{p_2}{p_1} \quad \text{and} \quad x_2 = \frac{w}{p_1+p_2}\cdot\frac{p_1}{p_2}.}$$

Exercice 2 :

$$x_1 = \frac{w}{p_2} \qquad x_2 = \frac{w(p_2-p_1)}{p_2^2}$$

1) . Homo. of deg 0. $\forall t$.

$$x(tp, tw) = \begin{pmatrix} \dfrac{tw}{tp_2} \\ \dfrac{t^2 w(p_2-p_1)}{t^2 p_2^2} \end{pmatrix} = \begin{pmatrix} \dfrac{w}{p_2} \\ \dfrac{w(p_2-p_1)}{p_2} \end{pmatrix} = x(p,w)$$

so YES. $\square$

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2) Walras' law.

$$p \cdot x = p_1 \frac{w}{p_2} + \cancel{p_2} \frac{w(p_2-p_1)}{p_2^{\cancel{2}}}$$

$$= w.$$

So YES it holds $\square$

3). WARP in the framework of the demand.

~~[illegible]~~ satisfies WARP iff.

$x(p,w)$.

$$\boxed{\begin{aligned} & p'.x(p,w) - w' \leq 0. \\ & \text{and. } x(p',w') \neq x(p,w) \\ & \Longrightarrow p.x(p',w') - w > 0. \end{aligned}}$$

4) Set $p_2 = 1$.

$$\boxed{\begin{aligned} x_1 &= w. \\ x_2 &= w(1-p_1). \end{aligned}}$$

Suppose $p'.x(p,w) \leq w'$.

i.e. $p_1' w + 1\, w.(1-p_1) \leq w'$.

$$\boxed{w(1+p_1'-p_1) \leq w'.}$$

~~then I cannot have.~~

$\cancel{p_2.x(p',w') \leq (p_1-p_2'+1)\,w'}$

The question is, can I find $x \neq x'$ s.th.

$p \cdot x' \leq w$ and $p' \cdot x \leq w'$?

assume $\begin{cases} p \cdot x' \leq w \iff w'(1 + p_1 - p_1') \leq w \\ \text{and} \\ p' \cdot x \leq w' \iff w(1 + p_1' - p_2) \leq w' \end{cases}$

$\Updownarrow$

note: $p_2 < 1$. $\Downarrow$

$w' - w \leq (p_1' - p_1) w'$
AND
$w' - w \geq (p_2' - p_1) w$

let $w = w' \longrightarrow p_1' = p_2$.
does not work.

suppose $w' = w + \underbrace{\Delta w}_{> 0}$.

$0 \leq (p_2' - p_2) \underbrace{\Delta w}_{> 0}$

≥ 0.

$\Delta w \leq (p_1' - p_1)(w + \Delta w)$. (1)

AND $\Delta w \geq (p_1' - p_2) w$. (2)

$- \quad \leq \quad - \quad \hat{(2)}$

~~$(p_1' + p_2) w \not\geq \Delta w + \Delta w (p_1' - p_1)$~~

So here for instance.

$$p_1' = p_2 + \underbrace{\Delta p_1}_{\geq 0} \cancel{\text{th}}$$

$$\Delta w \leq \Delta p_1 (w + \Delta w)$$

$$\Delta w \geq \Delta p_1 w.$$

$$\Updownarrow$$

So long as. $\Delta p_2 \leq \dfrac{\Delta w}{w}.$

$\Downarrow$

then we can find counter example to

WARP! $\square$.

Exercise 3:

1) $\succsim$ satisfies the independence axiom on $\mathcal{L}$

iff $\forall L, L', L'' \in \mathcal{L}$,

$\forall \alpha \in (0,1]$. convex.

$$L \sim L' \Longrightarrow \underbrace{\alpha L + (1-\alpha) L''}_{\text{this is a "compound lottery"}} \sim \alpha L' + (1-\alpha) L''$$

2) $U: \mathcal{L} \to \mathbb{R}$ has the expected utility form

iff $\exists (u_1, \dots, u_n)$ assignment to each outcomes $(c_1, \dots, c_n)$

s.th. $\forall L = (p_1, \dots, p_n) \in \mathcal{L}$

$$\boxed{U(L) = \sum_{i=1}^{n} u_i p_i}$$

2) b) Show that $\succsim$ satisfies the independence axiom

a). first,

U represents $\succsim$ on $\mathcal{L}$ iff $\forall L, L'$

$L \succsim L' \iff U(L) \geq U(L')$

let L, L', L''

(p_i) (p_i') (p_i'').

$\Rightarrow$ let $\alpha \in [0, 1]$.

~~then~~ let $L \sim L'$

$$\Updownarrow$$

$$U(L) = U(L')$$

$U(\alpha L + (1-\alpha) L'')$

$= \sum_i u_i \times (\alpha p_i + (1-\alpha) p_i'')$

the ~~compound~~ lottery is also representable.

$= \alpha \sum_i u_i p_i + (1-\alpha) \sum_i p_i'' u_i$

$= \alpha U(L)$

$= \alpha U(L')$

$= \alpha \sum_i u_i p_i'$

$= \sum_i u_i (\alpha p_i' + (1-\alpha) p_i'') = U(\alpha L' + (1-\alpha) L'')$

INDEPENDENCE AXIOM.