Microeconomics 1. $\boxed{\text{MIDTERM 2023.}}$

October 25 2023.

$$X = \{ x = (x_1, x_2) \in \mathbb{R}^2_+ \; ; \; x \geq c \}.$$

with $c = (c_1, c_2) \in \mathbb{R}^2_{++}$

$\succsim$ on X is represented by.

$$u(x) = (x_1 - c_1)^{\alpha_1} (x_2 - c_2)^{\alpha_2}$$

with $\alpha_{1,2} > 0$.

I. $\alpha_{1,2} = 1/2$ and $c_{1,2} = 1$.

1). (i) utility level at $(3, \frac{3}{2})$?

$$u\left(3, \frac{3}{2}\right) = (3-1)^{1/2} \left(\frac{3}{2} - 1\right)^{1/2}.$$

$$= 2^{1/2} / 2^{1/2} = 1.$$

(ii) ~~Do~~ we have that $\left(\frac{3}{2}, 3\right) \sim \left(3, \frac{3}{2}\right)$?

Yes, obviously, since $u(x)$ is symmetric in $1 \leftrightarrow 2$: for $\alpha_1 = \alpha_2$ and $c_1 = c_2$.

2) Trace the indifference curves corresponding to $u^* = 1$ (i) and $u^* = 0$ (ii).

(i)

We note that $u(1,1) = 1$.

So $(1,1)$, $(\frac{3}{2}, 3)$ and $(3, \frac{3}{2}) \in I(u^* = 1)$.

$$I(u^*=1) = \{ x = (x_1, x_2) ; \\ (x_1 - 1)^{1/2} (x_2 - 1)^{1/2} = 1 \}.$$

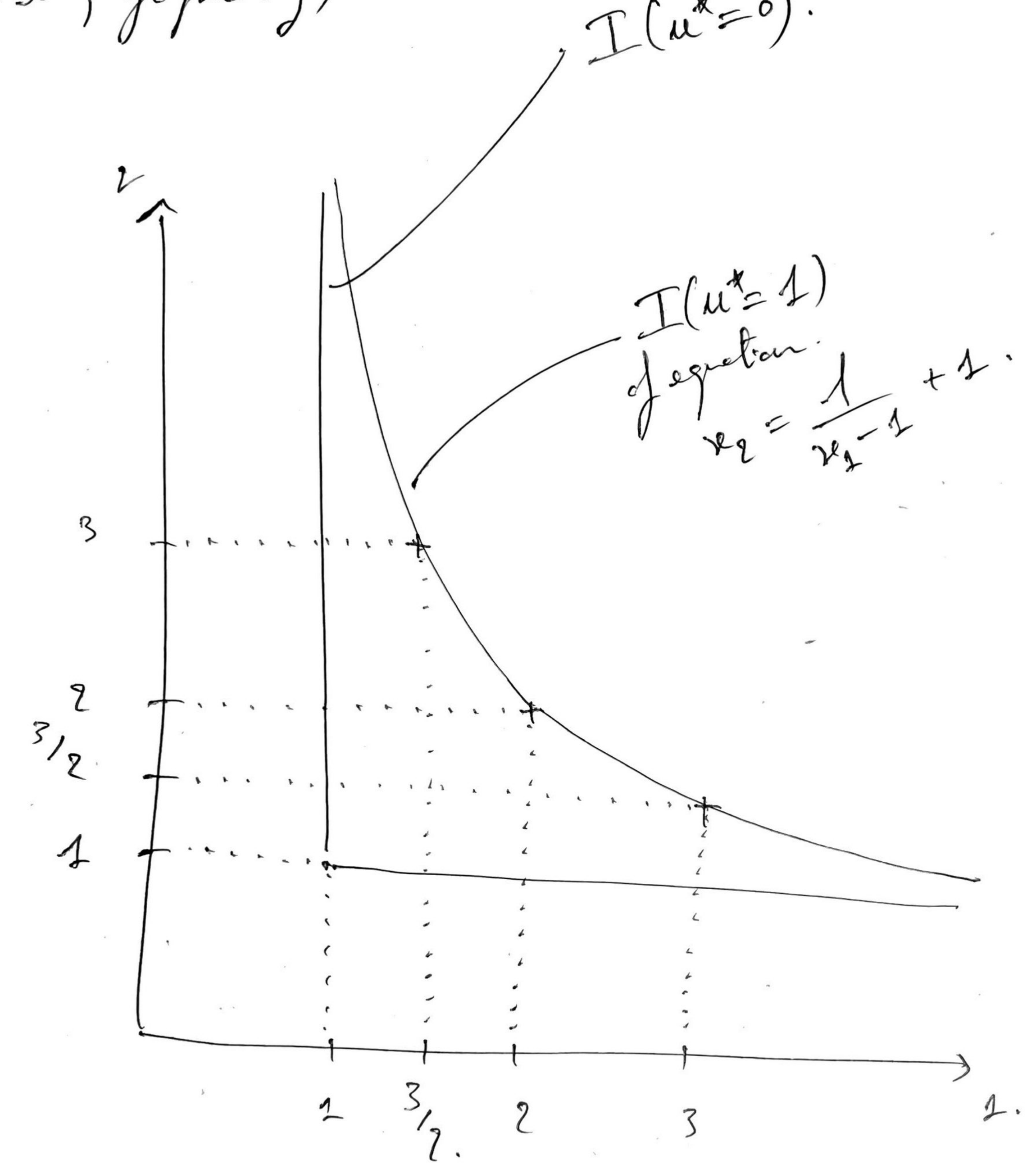
$$= \{ x \in X ; \ x_2 = 1 + \frac{1}{x_1 - 1} \}.$$

(ii) $I(u^* = 0) = \{ x ; (x_1 - 1)^{1/2} (x_2 - 1)^{1/2} = 0 \}$.

$= \{ x ;$ either $x_1 = 1$ or $x_2 = 1$ (or both) $\}$.

and $x_1, x_2 \geq 1$ of course ...

...So, graphically,



3). Compute. $\nabla u\left(3, \frac{3}{2}\right)$.

$$\nabla u_{(x)} = \begin{pmatrix} \frac{\partial u}{\partial x_1}(x) \\ \frac{\partial u}{\partial x_2}(x) \end{pmatrix} = \begin{pmatrix} \alpha_1 (x_1 - c_1)^{\alpha_1 - 1} (x_2 - c_2)^{\alpha_2} \\ \alpha_2 (x_1 - c_1)^{\alpha_1} (x_2 - c_2)^{\alpha_2 - 1} \end{pmatrix} = \dots$$

$$\dots = \begin{pmatrix} \dfrac{\alpha_1}{x_1 - c_1} u(x) \\ \dfrac{\alpha_2}{x_2 - c_2} u(x) \end{pmatrix}.$$

So, evaluating at $\left(3, \frac{3}{2}\right)$ and for $\alpha_{1,2} = \frac{1}{2}$
and $c_{1,2} = 1$.

$$\nabla u\left(3, \frac{3}{2}\right) = \begin{pmatrix} \dfrac{1/2}{3-1} \, 1 \\ \dfrac{1/2}{3/2 - 1} \, 1 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/4 \\ 1 \end{pmatrix}}}.$$

4). Let $p = \left(\frac{1}{4}, 1\right)$ and $w = \frac{9}{4}$.
determine the demand of the consumer graphically.

We first need to determine and represent the Walrasian budget set.

$$B_{p,w} = \{x ;\ \underbrace{p \cdot x}_{p_1 x_1 + p_2 x_2} \leq w\}.$$

$$= \{x^t ;\ x_2 \leq \frac{w}{p_2} - \frac{p_1}{p_2} x_1\}.$$

so far. $p = \left(\frac{1}{4}, 2\right)$ and $w = 9/4$.

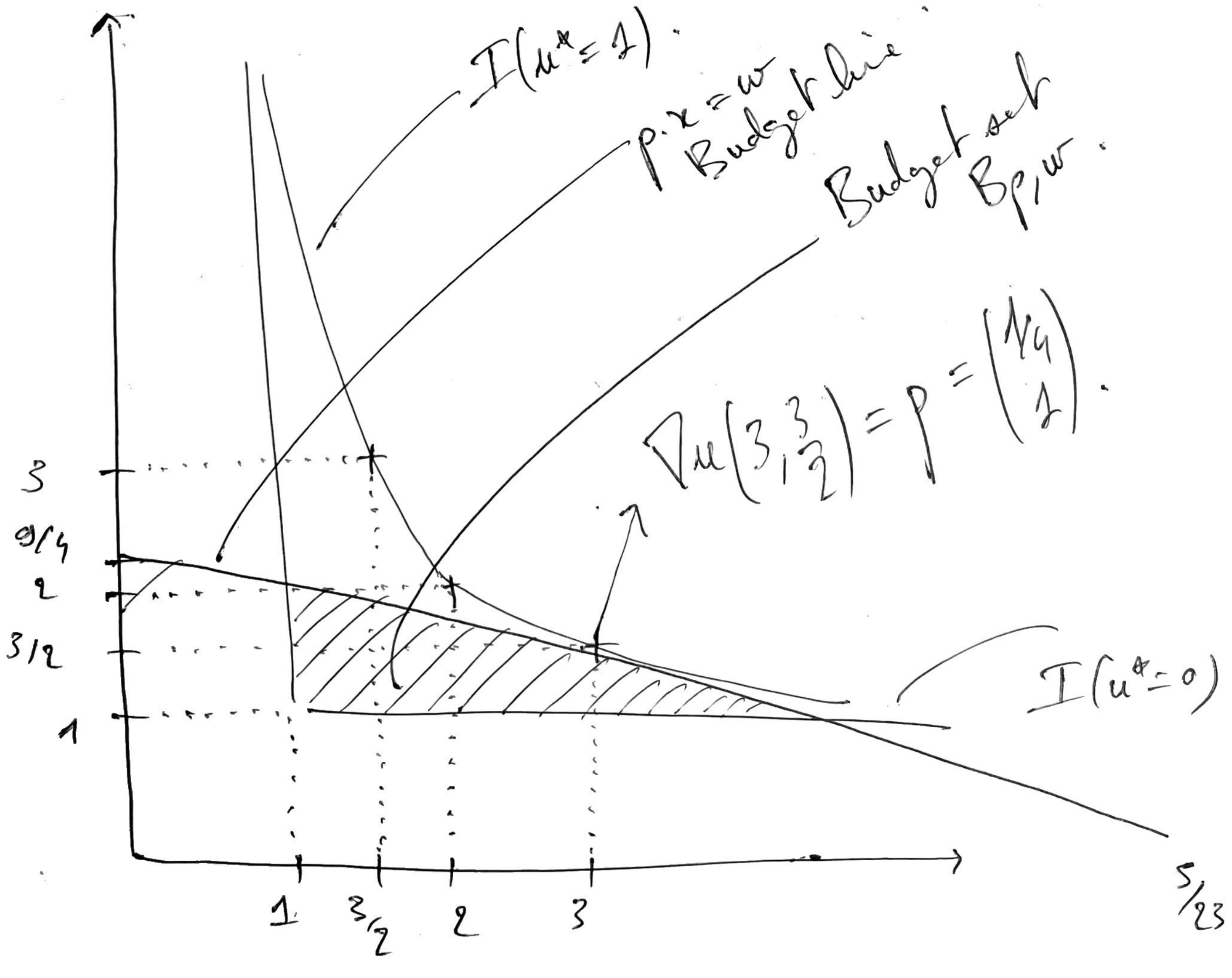
The budget line has equation.

$$x_2 = \frac{9}{4} - \frac{1}{4}x_1 .$$

We can note that $\left(3, \frac{3}{2}\right)$ is on the budget line.

since.

$$\frac{9}{4} - \frac{1}{4}3 = \frac{6}{4} = \frac{3}{2} .$$



We remark that the indifference curve $I(u^* = 1)$ is tangent to the budget line at $\left(3, \frac{3}{2}\right)$. $\iff$ $\nabla u\left(3, \frac{3}{2}\right) \propto p = \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$.

and $\left(3, \frac{3}{2}\right) \in$ Budget line.

Hence

$\left(3, \frac{3}{2}\right)$ is the demand of the consumer at $p = \begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$ and $w = 9/4$

II. We revert to the general case: $\alpha_{1,2} > 0$ and $c \gg 0$.

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1) (i) A fct u is said to represent a preference relation $\succsim$ on X iff:

$$\forall x, y \in X. \quad x \succsim y \iff u(x) \geq u(y).$$

(ii) Show that if u represents $\succsim$ and f strictly increasing, then $f \circ u$ also rep. $\succsim$.

Let u rep. $\succsim$ i.e.

$$\forall x, y \in X \qquad x \succsim y \iff u(x) \geq u(y).$$

Let $f : \mathbb{R} \to \mathbb{R}$ strictly increasing,
then $u(x) \geq u(y) \iff f(u(x)) \geq f(u(y))$.
$\forall x, y \in X$.

Hence we have that

$$\forall x, y \quad x \succsim y \iff f \circ u(x) \geq f \circ u(y).$$

i.e. $f \circ u$ represents $\succsim$ on X.

(iii) The fct $x \longmapsto x^{\frac{1}{\alpha_1 + \alpha_2}}$ is strictly increasing for all $\alpha_1, \alpha_2 > 0$.

...

... hence if $u(x)$ rep. $\succsim$,

so does $v(x) = \left(u(x)\right)^{\frac{1}{\alpha_1+\alpha_2}}$.

s.th. we can set $\alpha_1 + \alpha_2 = 1$ in the LES.
without changing the preference relation being represented.

2) Show that $\succsim$ is (i) continuous,
(ii) monotone, (iii) convex.

(i). <u>Def</u>. $\succsim$ continuous iff.
$\forall \{x_n\}, \{y_n\}$ s.th. $x_n \succsim y_n \ \forall n$.

$$\begin{matrix} x_n \longrightarrow x \\ y_n \longrightarrow y \end{matrix} \implies x \succsim y.$$

so ... let. $\{x_n\}, \{y_n\}$ s.th. $\forall n \ x_n \succsim y_n$
and. s.th. $x_n \longrightarrow x$.
$y_n \longrightarrow y$.

$\succsim$ is represented by u, so.

$$\forall n \quad x_n \succsim y_n.$$

$$\iff \forall n. \quad u(x_n) \geq u(y_n).$$

and $u(x) = (x_1 - c_1)^{\alpha_1}(x_2 - c_2)^{\alpha_2}.$
is continuous as the product of two continuous functions

($x \longmapsto x^\alpha$, $\alpha > 0$ being continuous).

So.

$\forall n \quad u(x_n) \geq u(y_n)$
and $x_n \longrightarrow x$ $\Longrightarrow$ $u(x) \geq u(y)$.
and $y_n \longrightarrow y$.

$$\Updownarrow$$

$$x \succsim y.$$

(We used the definition of the continuity of a fct.

$$\lim_{x_n \to x} f(x_n) = f(x).)$$

$\square$.

(ii) $\succsim$ monotone :

Def . $\succsim$ monotone . iff .

$$\boxed{\underbrace{x \gg y}_{\text{i.e. } x_i > y_i \ \forall i=1,2} \Longrightarrow x \succ y}$$

Let . $x = y + \varepsilon \cancel{\begin{pmatrix}1\\1\end{pmatrix}}$ with $\varepsilon \gg 0$.

i.e. $\varepsilon = (\underbrace{\varepsilon_1}_{>0} , \underbrace{\varepsilon_2}_{>0})$.

we have that $x \gg y$.

we compute .

$$u(x) = (y_1 + \varepsilon_1 - c_1)^{\alpha_1} (y_2 + \varepsilon_2 - c_2)^{\alpha_2} .$$

$$= \underbrace{(y_1 - c_1)^{\alpha_1} (y_2 - c_2)^{\alpha_2}}_{\overset{\shortparallel}{u(y)}} \underbrace{\left(1 + \frac{\varepsilon_1}{y_1 - c_1}\right)^{\alpha_1}}_{>1} \underbrace{\left(1 + \frac{\varepsilon_2}{y_2 - c_2}\right)^{\alpha_2}}_{>1} .$$

with in both cases .

$\left(1 + \frac{\varepsilon_i}{y_i - c_i}\right)^{\alpha_i} > 1$. given that $y_i \geq c_i$, $\varepsilon_i > 0$ and $\alpha_i > 0$.

... so. $u(x) > u(y)$.

$$\Longleftrightarrow x \succ y \qquad \square.$$

(In fact, $\succsim$ is strongly monotone — ie it is enough that one of the coordinates of x be strictly bigger than y).

(iii) $\succsim$ convex.

Recall. $\boxed{\succsim \text{ convex } \Longleftrightarrow u \text{ representing } \succsim \text{ is quasiconcave.}}$

The easiest proof consists in using the fact that Quasiconcavity is an ordinal property; ie it is preserved under strictly increasing tf.

Using 1)(ii), we know that

$v(x) = \ln u(x)$ also represent $\succsim$ given that $x \longmapsto \ln x$ is strictly increasing.

...

so let's study $v(x) = \alpha_1 \ln(x_1 - c_1)$
$+ \alpha_2 \ln(x_2 - c_2)$

We have that. $x \in X$.

$$H_{v}(x) = \begin{pmatrix} -\alpha_1 \dfrac{1}{(x_1-c_1)^2} & 0 \\ 0 & -\alpha_2 \dfrac{1}{(x_2-c_2)^2} \end{pmatrix}$$

so, given $\alpha_{1,2} > 0$,
~~[illegible]~~

$H_{v}(x)$ is <u>negative definite</u> $\forall x \gg c$.
or $\geq$ even.
(since both its eigen values are strictly negative.

Hence v is strictly concave.
$\Longrightarrow$ v is strictly quasiconcave.
$\Longrightarrow$ $u = e^{v}$ is strictly quasiconcave.
(since $x \longmapsto e^{x}$ is strictly increasing and strict quasiconcavity is an ordinal property.)

$\longrightarrow$ hence u is quasiconcave.

hence $\succsim$ is convex. $\square$

(iv). $\succsim$ monotone $\Longrightarrow$ $\succsim$ locally nonsatiated.

$\Downarrow$

the demand $x(p,w)$ satisfies Walras' law i.e. $p.x(p,w) = w$.

(v) u is strictly quasiconcave hence we know that the solution to the utility maximization problem is unique.

(and we knew that a solution existed because u quasiconcave and monotone).

3) (i). The utility maximization problem

$$UMP_u = \begin{cases} \max\limits_{x \in X} u(x). \\ p.x \leq w. \end{cases}$$

can be rewritten in term of the preference relation $\succsim$ as

$\bar{x} \in x(p,w)$ iff.

$$\bar{x} \succsim x \quad \forall x \in B_{p,w}.$$

Hence, using 1)(ii), if. u rep. $\succsim$ and $f \circ u$ also rep $\succsim$ when f strictly increasing,

we have that. a solution to UMP_u is also a solution to $UMP_{f \circ u}$

$\forall f$ strictly increasing.

$x \longmapsto \ln x$ is strictly increasing, hence we can solve the UMP for $v(x) = \ln u(x)$ instead of $u(x)$.

(ii) The Kuhn-Tucker conditions are <u>necessary</u> conditions,

but under the circumstance where u is quasiconcave and monotone (hence also $v = \ln u$), they are also <u>sufficient</u>;

The Kuhn-Tucker conditions write

$\bar{x} \in x(p, w)$ iff

$$\begin{cases} \nabla v(\bar{x}) = \lambda p & (1) \\ \text{and } \lambda(p \cdot \bar{x} - w) = 0. & (2) \end{cases}$$

We compute:

$$\nabla v(x) = \begin{pmatrix} \dfrac{\alpha_1}{x_1 - e_1} \\ \dfrac{\alpha_2}{x_2 - e_2} \end{pmatrix} \gg 0 \quad \forall x \in X,$$

hence, from $p \gg 0$, we have that $\lambda > 0$.

hence (2) implies $p.\bar{x} = w$

i.e Walras' law.
(that we know would be verified from 2) (iv))

Thus we have

I am using the "hint".

$$\nabla v(\bar{x}).(\bar{x} - c)$$

$$= \frac{\alpha_1}{x_1 - c_1}(x_1 - c_1) + \frac{\alpha_2}{x_2 - c_2}(x_2 - c_2)$$

$$= \alpha_1 + \alpha_2 = 1.$$

from 1)(iii)

$$\underset{\substack{(2)\\ \text{Kuhn-Tucker}}}{=} \lambda p.(\bar{x} - c) \underset{\text{Walras' law}}{=} \lambda(w - p.c).$$

$$\Longrightarrow \boxed{\lambda = 1/(w - p.c)}$$

$$\Longrightarrow \quad \nabla v(\overline{x}) = \begin{pmatrix} \dfrac{\alpha_1}{\overline{x_1} - c_1} \\[2ex] \dfrac{\alpha_2}{\overline{x_2} - c_2} \end{pmatrix} = \frac{1}{w - p.c} \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}.$$

$$\Longrightarrow \quad \boxed{\begin{aligned} \overline{x_1} &= c_1 + \alpha_1 \frac{w - p.c}{p_1} \\ \text{and} \quad \overline{x_2} &= c_2 + \alpha_2 \frac{w - p.c}{p_2}. \end{aligned}}$$

(iii) $$MRS_{12}(x) = \frac{\partial v / \partial x_1 \, (x)}{\partial v / \partial x_2 \, (x)}$$

$$\boxed{= \frac{\alpha_1}{x_1 - c_1} \, \frac{x_2 - c_2}{\alpha_2}.}$$

at $\overline{x} \in x(p, w)$, if we substitute the values found previously, we get:

$$MRS_{12}(\overline{x}) = \frac{\cancel{\alpha_1}}{\cancel{\alpha_1} \frac{\cancel{(w - p.c)}}{p_1}} \; \frac{\cancel{\alpha_2} \frac{\cancel{(w - p.c)}}{p_2}}{\cancel{\alpha_2}} \boxed{= \frac{p_1}{p_2}}.$$

The result $MRS_{12}(\bar{u}) = \frac{p_1}{p_2}$ was to be expected as a general result immediately implied by the Kuhn-Tucker conditions, since:

$$MRS_{12}(\bar{x}) = \frac{\frac{\partial u}{\partial x_1}(\bar{u})}{\frac{\partial u}{\partial x_2}} = \frac{\cancel{\lambda} p_1}{\cancel{\lambda} p_2} = \frac{p_1}{p_2}.$$

4) (i) We can normalize $p_2 = 1$ without loss of generality because $x(p, w)$ is <u>homogeneous of degree 0</u>,

i.e. $\forall t > 0 \quad x(tp, tw) = x(p, w)$

(ii). The Weak Axiom of Revealed Preference writes, in the framework of the demand, as

$$\boxed{\begin{array}{l} p \cdot x(p', w') \leq w \text{ and} \\ x(p, w) \neq x(p', w') \implies p' \cdot x(p, w) > w' \end{array}}$$

(iii). Verify that the LES satisfies the WARP in the simpler case. $c_1 = c_2 = 0$ (i.e. Cobb-Douglas) and $\alpha_1 = \alpha_2 = 1/2$.

So we have.

$$x(p, w) = \begin{pmatrix} \frac{\alpha_1}{p_1} w \\ \\ \frac{\alpha_2}{p_2} w \end{pmatrix}.$$

and we apply 4)(i) to normalize $p_2 = 1$ without loss of generality.

$$x(p, w) = \begin{pmatrix} w/2p_1 \\ w/2 \end{pmatrix}.$$

Assume. $p \cdot x(p', w') \leq w$

$$p_1 \frac{w'}{2p_1'} + \frac{w'}{2} \leq w. \qquad \times \frac{1}{2}\left(\frac{p_1'}{p_1} + 1\right).$$

$$\Longleftrightarrow \frac{w'}{2}\left[\frac{p_1}{p_1'} + 1\right] \leq w.$$

$$\Longrightarrow \frac{w'}{2}\left(\frac{p_1}{p_1'} + 1\right)\left(\frac{p_1'}{p_1} + 1\right)\frac{1}{2} \overset{(*)}{\leq} \frac{w}{2}\left(\frac{p_1'}{p_1} + 1\right).$$

$$\Longrightarrow \quad \frac{w'}{4}\underbrace{\left[2+\frac{p_1}{p_1'}+\frac{p_1'}{p_1}\right]}_{\substack{\geq 4 \text{ for all } p_1, p_1' > 0 \\ \text{with equality iff } p_1 = p_1'}} \leq p' \cdot x(p,w)$$

(**)

(Using the fact that

$$\frac{\partial}{\partial x}\left(x+\frac{1}{x}\right) = 1-\frac{1}{x^2} = 0 \quad \text{iff } x = 1.$$

and $\frac{\partial^2}{\partial x^2}\left(x+\frac{1}{x}\right) = \frac{2}{x^3} > 0 \quad \forall x > 0$.

so the minimum is attained at $x = 1$. . .).

Assume $x(p,w) \neq x(p',w')$.

* if $w' = w$ $\Longrightarrow$ we must have $p_1 \neq p_1'$.
  in which case $\frac{w'}{4}\underbrace{\left[2+\frac{p_1}{p_1'}+\frac{p_1'}{p_1}\right]}_{>4} > w'$.

hence $p' \cdot x(p,w) > w'$

* if $p_1 = p_1'$ $\Longrightarrow$ we must have $w' \neq w$
  and it is the left inequality (*) that is strict.
* if both $p_1 \neq p_1'$ and $w \neq w'$, both inequalities (*) and (**) are strict. . .

Hence we have verified:

$$p \cdot x(p', w') \leq w \quad \text{and} \quad x(p,w) \neq x(p',w')$$
$$\Longrightarrow \quad p' \cdot x(p,w) > w'.$$

i.e. $x(p,w)$, the Walrasian demand, satisfies the WARP. $\square$

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5) (bonus)

(i) The demand is unaffected by an equal percentage change in all prices and the wealth because it is homogeneous of degree 0. (as in 4.(i)).

(ii). We have

$$\begin{cases} x_1(p,w) = c_1 + \dfrac{\alpha_1}{p_1}(w - p.c) \\[2ex] x_2(p,w) = c_2 + \dfrac{\alpha_2}{p_2}(w - p.c) \end{cases}$$

We compute:

$$\frac{\partial x_1}{\partial p_1} = 0 - \frac{\alpha_1}{p_1^2}(w - p.c) + \frac{\alpha_1}{p_1}(-c_1)$$
$$= -\frac{\alpha_1}{p_1^2}(w - p_2 c_2) \ \underline{\underline{\leq 0}}.$$

(iii)

$$\frac{\partial x_1}{\partial p_2} = -\frac{\alpha_1}{p_1} c_2 \leq 0$$

and. $$\frac{\partial x_1}{\partial w} = \frac{\alpha_2}{p_1} \geq 0.$$

and. $\frac{\partial x_2}{\partial p_1}$, $\frac{\partial x_2}{\partial p_2}$, $\frac{\partial x_2}{\partial w}$ has symmetric expressions w.r.t. $1 \leftrightarrow 2$.

(iii). You should know from class that the indirect utility fct. is expected to be homogeneous of degree 0, hence unaffected by a percentage change in all prices and the wealth.

It is obvious since the indirect utility fct, is

$$i(p, w) = v(x(p, w)).$$

and. $x(p, w)$ is homogeneous of deg. 0 from 4.(i) and 5.(i).

(iv). We compute the fct.

$$\begin{aligned} i(p, w) &= v(x(p, w)) \\ &= \alpha_1 \ln(x_1(p, w) - c_1) + \alpha_2 \ln(x_2(p, w) - c_2). \\ &= \alpha_1 \ln \frac{\alpha_1}{p_2}(w - p.c) + \alpha_2 \ln(w - p.c)\frac{\alpha_2}{p_2}. \end{aligned}$$

$$\Longrightarrow \quad \frac{\partial i}{\partial w}(p,w) = \frac{\alpha_1+\alpha_2}{w} = \frac{1}{w} \quad (=\lambda) \quad \underline{\underline{>0}}.$$

(as you should have expected.)

$$\frac{\partial i}{\partial p_1}(p,w) = \frac{-\alpha_1 c_1}{w-p\cdot c} - \frac{\alpha_2 c_1}{w-p\cdot c} - \frac{\alpha_1}{p_1}.$$

$$= -\frac{\alpha_1}{p_1} - \frac{c_1}{w-p\cdot c} \quad \underline{\underline{<0}}.$$

likewise $\frac{\partial i}{\partial p_2}(p,w)$ has symmetric expression w.r.t. $1 \leftrightarrow 2$. $\square$.

$$\underline{\underline{<0}}.$$

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Microeconomics 1 – MMMMEF/ QEM

Midterm Exam (90 mins)

No mobile phone or calculator. One sheet containing personal notes authorised.

We consider throughout a consumer with consumption set $X = \{x = (x_1, x_2) \in \mathbb{R}^2_+ \mid x \geq c\}$ where $c = (c_1, c_2) \in \mathbb{R}^2_+$ is a subsistence consumption bundle. Furthermore, the consumer has preferences $\succsim$ on X represented by an utility function of the form $u(x) = (x_1 - c_1)^{\alpha_1}(x_2 - c_2)^{\alpha_2}$, with $\alpha_1, \alpha_2 > 0$. The resulting demand is known as the *linear expenditure system* (LES) and is due to Stone (1954).

Part 1, Graphical study. In this part we consider $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $c_1 = c_2 = 1$.

1. (i) What is the utility level at $(3, \frac{3}{2})$? (ii) Do we have that $(\frac{3}{2}, 3) \sim (3, \frac{3}{2})$?
2. Trace the indifference curves corresponding to utility levels (i) $u^* = 1$, (ii) $u^* = 0$.
3. Compute $\nabla u(3, \frac{3}{2})$.
4. Let $p = (\frac{1}{4}, 1)$ and $w = \frac{9}{4}$, determine the demand of the consumer graphically.

Part 2. Analytical study. In this part we revert to the general case where $\alpha_1, \alpha_2 > 0$ and $c_1, c_2 \geq 0$.

1. (i) When is a function u said to represent a preference relation $\succsim$? (ii) Show that if u represents $\succsim$ and f is a strictly increasing function, then $f \circ u$ also represents $\succsim$. (iii) Conclude that we can set $\alpha_1 + \alpha_2 = 1$ in the LES.
2. Show that $\succsim$ is (i) continuous, (ii) monotone, and (iii) convex on X. Let $p \gg 0$ and $w > 0$. (iv) Why should we expect the demand of the consumer to satisfy Walras' law? (v) Why should we expect the demand of the consumer to be unique (i.e, a singleton)?
3. (i) Using 1.(ii), explain why we can solve the utility maximization problem for $v(x) = \ln u(x)$ instead of $u(x)$. (ii) Determine the demand $x(p, w)$ of the consumer using the Kuhn-Tucker conditions. (*Computational hint:* Use that $p \cdot \bar{x} = w \implies p \cdot (\bar{x} - c) = w - p \cdot c$.) (iii) Compute the marginal rate of substitution $MRS_{12}(x) = -\frac{\partial v / \partial x_1}{\partial v / \partial x_2}(x)$. (iv) Evaluate MRS_{12} at $\bar{x} \in x(p, w)$. Why was this result expected?

4. (i) Why can we normalize $p_2 = 1$ without losing generality? (ii) State the WARP in the framework of the demand. (iii) Verify that the LES satisfies the WARP in the simpler case when $\alpha_1 = \alpha_2 = \frac{1}{2}$ and $c_1 = c_2 = 0$. (*Hints:* Use the fact that $x + \frac{1}{x} \geq 2$ for all $x > 0$, with equality iff $x = 1$.)

5. (bonus) How is the demand affected by (i) an equal percentage change in all prices and the wealth, (ii) separate changes in the wealth and each price? How is the indirect utility function (i.e., the utility level at the demand) affected by (iii) an equal percentage change in all prices and the wealth, (iv) separate changes in the wealth and each price?