Graduate Studies Mathematical Models in Economics and Finance
Master 1 Modèles Mathématiques en Economie et en Finance
Introductory Finance - Tutorial - Isabelle Nagot

Exercise 1.
On a a saving account, an interest is paid m times per year and compounded.

a. Compute the annualised interest rate on each period, r_m, knowing that after 1 year the total interest has to correspond to the annual rate r_d.
Numerical application when an interest is paid monthly, and compounded: compute the monthly rate corresponding to 10% after 1 year.

b. Prove that for $m \geq 2$ and $r_d > 0$, $(1 + \frac{r_d}{m})^m > 1 + r_d$.

c. Prove that, after each interest payment, the value of the investment belongs to an exponential curve that is independent of m.

d. Compute $\lim_{m \to +\infty} r_m$.

e. Let $m \in \mathbb{N}^*$. An investor A invests M in a product that pays a discrete interest at each date $\frac{k}{m}$, for $k = 1, ..., m$. The interest rate is the same on each sub-period.

An investor B invests M in a product that pays a continuous interest.

Both investors have the same wealth after 1 year, equal to $M(1 + r_d)$.

What is the maximum difference of wealth between them during the year?

Exercise 2.
Let's assume that the interest rate for maturity 1 year (resp: 2, 3) is worth 4% (resp: 4.2%, 4.5%).
What is the Present Value of the series of cash-flows:

100M€ in 1 year, 200M€ in 2 years, 300M€ in 3 years?

Year	1	2	3
Rate	4	4.2	4.5
CF	100	200	300

Exercise 3. Annuities
An *annuity* is an investment where one receives or pays cash-flows every year. The annuity is called *ordinary* if all cash-flows are the same.
We assume that the discount rate is r for any maturity (ie the risk-free yield curve is flat), with $r > 0$.

Reminder: $\forall T \in \mathbb{N}^*$, $\forall x \neq 1$, $\sum_{t=1}^{T} x^t = x\frac{1 - x^T}{1 - x}$ (sum of a finite geometric series). Prove it.

a. Finite annuities.
How much is the PV of an annuity of A€ every year during T years?
Compute the result with $A = 100$k€, $r = 5\%$ and $T = 10$ years.

b. Perpetual ordinary annuities: the same cash-flow A is paid forever.
What is now the present value? Draw it as a function of r.
Compute the result with $A = 100$€ and $r = 5\%$.

Exercise 4. We consider a standard bond having annual coupons with nominal N, coupon rate c and maturity T. Today is a coupon payment date and the coupon has just been paid.

a. We denote by $\frac{P}{N}$ its clean price now. Write it as a function of the yield-to-maturity ρ : $\frac{P}{N} = f(\rho)$.

b. Compute $f(c)$ and prove that $\rho < c$ means the bond is trading "at a premium".

c. Ex: the Treasury bond "T 5.5% 15 Aug 2028" (issue date 17/08/1998, semi-annual) is trading at 126.65 on 15 Aug 2020. Its yield-to-maturity is 3.232%, we consider it is computed assuming annual coupons (it is an approximation).

Compute its current yield and compare it to the coupon rate and the yield-to-maturity.

d. We assume that the bond trades at a premium as in the example. We want to prove that: yield-to-maturity $<$ current yield $<$ coupon rate.

d.1 Prove that: current yield $<$ coupon rate.

d.2 Compare the investment of P in the bond at current conditions to the investment of P that would pay the same coupons as the bond, but P instead of N at time T (compare their returns).

Deduce that we have: yield-to-maturity $<$ current yield.

e. Prove that:
yield-to-maturity $>$ current yield $>$ coupon rate when the bond trades at a discount
yield-to-maturity $=$ current yield $=$ coupon rate when the bond trades at par.

Exercise 5. Compute the yield-to-maturity:

a. at issue date, for the German 10Y bond (Bund) issued in July 2019 - Bond issued at negative yield (ISIN: DE0001102473, coupon rate $= 0\%$, price at issue date $= 102.64$).

b. on 2 Oct 23, for the Austrian 100Y bond issued in June 2020 - Century bond
(ISIN: AT0000A2HLC4, coupon rate $= 0.85\%$, price on 2 Oct 23 $= 33.94$).
Prove that the yield-to-maturity is $\rho = 2,858\%$.

Exercise 6. Duration

a. Compute the duration of a standard annual bond such that:
yield-to-maturity is $\rho = 5\%$, coupon rate is $c = 3\%$, and maturity is $T = 5$ years.

b. Compute the duration of the investment of Exercise 3.b (take the first derivative in the reminder).
How is this duration changing with r?
N.A. with $A = 100$€ and $r = 5\%$.

Exercise 7. The **par yield** (or **par rate**) denotes the coupon rate that a bond with a given maturity would have to pay in order to sell at par today.

a. Prove that the par yield curve graphs the yield-to-maturity of coupon-paying bonds trading at par (e.g. issued today).

b. At time 0, compute the par rate $c(0,T)$ for the maturity T, given the 0-coupon yield curve $t \mapsto r(0,t)$.

c. We assume that the yield curve has a normal shape. Prove that the par yield curve is below the 0-coupon yield curve.

Exercise 8. a. Extend the definition of the yield-to-maturity to a bond having m coupon payments per year (e.g. $m = 2$ for semi-annual coupons), the coupon rate being denoted by c.

b. Compute the duration of the bond.

c. Compute the rate of change in the price, $\frac{\Delta P}{P}$, implied by a small variation $\Delta\rho = \Delta r$.

Exercise 9. Floating Rate Bond

1. A standard bond pays fixed coupons (yearly or semi-yearly) to its owner. Explain why the holder of a standard bond is exposed to an interest rate risk.

2. A Floating Rate Bond (FRB) pays a series of variable coupons:
at time $t \in \{1, ..., T\}$, it pays
$r(t-1,t) \times$ nominal where $r(t-1,t)$ is the rate prevailing at time $t-1$ for maturity t,
and the nominal is reimbursed at T.

Because the coupon is indexed on an interest rate, a FRB should protect the investor against a rate increase. We check this property below.

a. Compute the value of the FRB at $T-1$, then at $T-2$.
Deduce its price at time 0.

b. Compute the price between 2 coupons dates $t-1$ and t: at $s \in]t-1, t[$.

Exercise 10. We consider a general bond.

1. What is the link between the slope of the curve $\rho \mapsto P(\rho)$ and the sensitivity of the bond.

2. Prove that the 1st order approximation of the rate of change:
overvalues (in absolute value) the effect of a rate increase.
undervalues the effect of a rate drop.

Exercise 11. Consider a bond with no coupon, maturing at $T = 10$ years.
Paid in 10Y: $N = 100M$€ (ie 100M units of 0-coupon).
The yield-to-maturity is assumed to be: $\rho = 10\%$.

Compute the price variation if the rate goes from 10% to 9% for 3 methods: approximation to the 1st order, to the 2nd order, and exact computation.

Exercise 12. Perpetual annuity. Coupon = A, YTM = $\rho = 10\%$.
Compare the relative variation of price implied by a rate increase of 1% and a rate drop of 1%.

Exercise 13. In this exercise, we consider Government bonds only (in a given country), assumed to be default-free. Today is time 0. For any t between 1 and 50 years, r_t denotes the risk-free interest rate for maturity t.

We consider a standard bond maturing at time T (≥ 2 years), we denote by c its coupon rate and by ρ its yield-to-maturity at time 0.

1. Writing in two different ways the price of this bond at 0, derive an equation linking $r_1, r_2, ..., r_T$ and c, ρ.

2. What is the yield-to-maturity at time 0 for a 0-coupon bond maturing at T?

We assume $0 < r_1 < r_2 < ... < r_T$ below.

3. Let $f(\rho) = c \sum_{t=1}^{T} \frac{1}{(1+\rho)^t} + \frac{1}{(1+\rho)^T}$. Prove that $f(r_1) > f(\rho) > f(r_T)$. Deduce: $\rho \in]r_1, r_T[$.

4. Prove that an annuity having the same maturity date T has a lower yield-to-maturity than this standard bond (qualitative and quantitative arguments).

Exercise 14. We want to check how the duration of a standard bond depends on its coupon rate. We consider a standard bond of a given issuer. We denote by T its maturity, N its nominal, c its coupon rate, and P its price at time 0.

The dependence on c of the yield-to-maturity is complex, so we write the price and the duration without involving it.

For $t = 0, 1, ..., T$, $B(0, t)$ denotes the price at 0 of the 0-coupon bond maturing at time t for this issuer (not the usual risk-free discount factor).

1. Prove that $\frac{P}{N}$ can be written $cA_1 + B$, with A_1 and B constants proper to the issuer (you can express them in terms of the $B(0, t)$, $t = 1, ..., T$), i.e. independent of the particular bond that is considered with this given time-to-maturity.

a. Explain that formula and compare it to the usual one.

b. Prove that D can be written $\dfrac{cA_2 + TB}{cA_1 + B}(*)$, with A_2 a constant proper to the issuer.

3. We consider a second bond with same issuer, same nominal and same timing ($t = 1, 2, ..., T$) of annual coupons and nominal payments.

a. Explain why this second bond can have a different coupon rate c'.

b. Let P' and D' be the price and duration of the second bond at time 0.
Compare D' to D when $c' > c$, intuitively, then through computation, using formula (*).

Exercise 15. Prove that (by contradiction: if not, then there exists an arbitrage opportunity):
If two portfolios have same value at a future time T (equality of random variables),
they must have same value at any earlier time, t.

Exercise 16. Call-put parity relationship:
Consider European options (call and put) with the same strike price K and the same maturity T, on a non-dividend paying stock. At a time $t < T$, with S_t being the stock price, C_t the call price and P_t the put price, prove that: $C_t + KB(t, T) = P_t + S_t$.

Build an arbitrage opportunity if this equality is not satisfied.

Exercise 17. A Floating Rate Bond (FRB) pays a series of variable coupons (see exercise 9):
at time $t \in \{1, ..., T\}$, it pays
$r(t-1, t) \times$ nominal where $r(t-1, t)$ is the rate prevailing at time $t-1$ for maturity t,
and the nominal is reimbursed at T (see also exercise 8).
We assume that 0-coupon bonds are available for any maturity.

1. Replicate, at time 0 (just after a coupon date), the variable coupon of time t with a portfolio of 0-coupon bonds. Deduce the value at time 0 of this coupon.

2. Compute the price of the FRB at time 0.

3. Compute its duration at time 0.

Exercise 18. We consider forward contracts on a risky asset, all maturing at T. The risk free continuous interest rate is assumed to be constant.

We denote by:

$F(t, T)$ the delivery price in the contract agreed at time t, for $t \leq T$.
$V_{t,T}(t')$ the value at time t' of the long forward contract agreed at time t, for $t \leq t' \leq T$.

1. What are $V_{t,T}(t)$ and $V_{t,T}(T)$?

2. What is the relationship between $F(t, T)$, $F(t', T)$, and $V_{t,T}(t')$, $\forall t \leq t' \leq T$?

3. What if there are dividends or storage costs on the underlying asset?

Exercise 19. Forward exchange rate

A \$-based investor is considering at time t a forward contract on the exchange rate €/\$ with a maturity in 1 year. We denote by:

$r_{\$}$ the discrete interest rate in dollars between t and $t + 1$,
$r_{€}$ the discrete interest rate in euros between t and $t + 1$,
S_t the exchange rate EUR/USD at time t: 1€ is worth S_t \$ at time t.

1. Compute $F(t, T)$ the forward exchange rate at t for the maturity $T = t + 1$.

Numerical value in two different market conditions :

on 27 Jan 2012, the money market rates and spot rate are as follows:
Spot rate = 1.3142 EUR/USD, USD 1 year LIBOR = 1.102%, EUR 1 year Euribor = 1.768%.

on 15 Nov 2013: $S_t = 1.3443$, $r_{\$} = 0.588\%$, $r_{€} = 0.496\%$.

2. Exhibit an arbitrage opportunity if this equality is wrong.

Exercise 20. Future on commodity

Compute the forward price on a commodity that has a storage cost.
You will assume that the maturity date is in 1 year, and that the storage cost for 1 year, C\$, has to be paid in 6 months.

Exercise 21. Interest rate future (and future contract on a 0-coupon bond).

A forward interest rate is an interest rate which is specified now for a loan that will occur at a specified future date.

1. Compute $r^F(t; T, T')$, the forward interest rate contracted at date t, for a loan done at date T and to be reimbursed at date T' (with $t < T < T'$). Compute it for discrete rates and for continuous rates.

2. For continuous rates, compute $\lim_{\Delta T \to 0} r^F(t; T, T + \Delta T)$.

Exercise 22. We consider a given sovereign issuer, with an upward sloping zero-coupon yield curve, at time t.

All the rates being continuously compounded, plot on a same graph (i.e. order the 3 curves):

- the zero-coupon yield curve $T \mapsto r(t,T)$,

- the coupon-bearing yield curve $T \mapsto \rho(t,T)$, where $\rho(t,T)$ is the continuous yield-to-maturity for a standard coupon-bearing bond issued at t and maturing at T
(i.e. $\rho(t,T) = ln(1 + YTM(t,T))$ if $YTM(t,T)$ is the corresponding discrete yield-to-maturity, usually defined), and
- the forward rate curve $T \mapsto r^F(t,T_0;T)$, i.e. $r^F(t,T_0;T)$ being the rate for the maturity T delivered at a given future time $T_0 \in]t,T[$.

Exercise 23. Prove that when the underlying asset is an equity paying no dividend, an American call is worth the same as the European call with the same characteristics (assuming that the risk-free interest rate is positive).

Exercise 24. Analogue of the call-put parity relationship for the American options. When the underlying asset is an equity paying no dividend, prove that:

$S_0 - K \leq C_0^A - P_0^A \leq S_0 - \frac{K}{(1+r)^T}$.

Exercise 25. Options combination: we combine some European calls and puts on a same underlying asset (an equity paying no dividend) in order to obtain different payoff shapes.
All options have same maturity T.
Draw the payoff of the following portfolios: **1.** {-1 call, + 1 equity }

{ +1 call with strike K_1, -1 call with strike K_2 }, where **2.** $K_2 > K_1$ or **3.** $K_2 < K_1$

4. { +1 put with strike K_1, +1 put with strike $K_2 > K_1$, -2 puts with strike $\frac{K_1+K_2}{2}$ }

5. { +1 call, +1 put with same strike }

6. { +1 put with strike K_1, +1 call with strike $K_2 > K_1$ }

Each time, describe the use of such a payoff, and if the portfolio has a positive value at the initial date.

Exercise 26. We consider a financial market in discrete-time, with two dates $t = 0$ and $t = 1$. The risk-free asset is worth 1 at $t = 0$, and $1 + r$ for any state of the world at $t = 1$,
the risky asset is an equity paying no dividend, being worth S at $t = 0$ and S^d or S^u at $t = 1$, with $S^d < S(1 + r) < S^u$.

Prove that $S^d < S(1 + r) < S^u$ is mandatory to ensure that there is no arbitrage opportunity between the 2 basis assets.