The variance as a risk measure

A brief introduction to Markowitz's portfolio theory.

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Introduction

Introduction

- **Description of the market:** We consider a financial market with two dates $t = 0$ and $t = 1$. The financial market is composed of:
  - 1 non-risky asset. Its price at time t is denoted by S_t^0 (Bond price).
  - d risky assets. Their prices at time t are denoted by $(S_t^1, \cdots, S_t^d)$ (Stock prices).

You invest your initial wealth V_0 at time 0 and hold position up to time 1. Here, $(S_1^0, \cdots, S_1^d)$ are random variables, leaving on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, which satisfy some assumptions that will be specified later on.

- **Aim:** Starting from an initial wealth V_0, we want to invest in the non-risky and risky assets in such a way that optimizes our terminal wealth V_1.

- **Questions:**
  - How would you invest? Maximize expected return of the portfolio but limit/control risk.
  - How to measure risk? As originally proposed by H. Markowitz, we can measure the risk of a strategy using its variance/standard deviation.

Mathematical framework

- We will not directly work with prices:
  - Returns:

$$\text{(interest rate)} \quad \frac{S_1^0 - S_0^0}{S_0^0} = r, \quad \text{(return of asset } i) \quad R^i = \frac{S_1^i - S_0^i}{S_0^i}.$$

  - Investment strategy/portfolio is determined by
    - an initial wealth V_0
    - a vector of weights $(w_0, \cdots, w_d)$ where w_i stands for the proportion of initial wealth invested in asset i:

$$\text{(Budget constraint)} \quad \sum_{i=0}^{d} w_i = 1.$$

Note that $w_i \leq 0$ means short position while $w_i \geq 0$ means long position in asset i. The proportion invested in the non-risky asset is entirely determined by $w = (w_1, \cdots, w_d)$ since

$$w_0 = 1 - \mathbf{1}^t w, \quad \mathbf{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^d.$$

Mathematical framework

Mathematical framework

- The terminal wealth is given by

$$V_1 = \sum_{i=0}^{d} \delta^i S_1^i = V_0 \sum_{i=0}^{d} w_i \frac{S_1^i}{S_0^i} \quad \text{using that } \delta^i = \frac{w_i V_0}{S_0^i}$$

so that

$$\frac{V_1}{V_0} = \frac{S_1^0}{S_0^0} + \sum_{i=1}^{d} w_i \Big(\frac{S_1^i}{S_0^i} - \frac{S_1^0}{S_0^0}\Big) = 1 + r + \sum_{i=1}^{d} w_i (R^i - r)$$

which eventually gives

$$\frac{V_1 - V_0}{V_0} - r = \sum_{i=1}^{d} w_i (R^i - r).$$

⇝ Markowitz problem: Given an initial wealth V_0, find a strategy w such that

$$\text{(Maximize expected excess return)} \quad \max_{w \in \mathbb{R}^d} \mathbb{E}\Big[\frac{V_1 - V_0}{V_0} - r\Big]$$

while controlling the risk measured by the variance $var(V_1/V_0) := \sigma^2(w)$.

◦ We solve the following optimization problem:

$$\max_{w\in\mathbb{R}^d}\left\{R(w):=\mathbb{E}\Big[\frac{V_1-V_0}{V_0}-r\Big]\right\} \quad \text{such that}$$

such that

$$\begin{cases}\sigma^2(w)=var(\frac{V_1-V_0}{V_0}-r)\leq\sigma_\star^2,\\ w\in B,\end{cases}$$

where $\sigma_\star^2>0$ is a given risk level that we do not want to exceed and B is a set of constraints.

◦ Assumption on $R=(R^1,\cdots,R^d)$: First two moments exist

- $\mu:=\mathbb{E}[R-r\mathbf{1}]$ is the expected excess return of the risky assets
- $\Sigma:=Cov((R-r\mathbf{1})^i,(R-r\mathbf{1})^j)_{1\leq i,j\leq d}\in\mathbb{S}_+^d$, where $\mathbb{S}_+^d$ is the set of $d\times d$ symmetric semi-definite non-negative matrix.

◦ We now compute the (excess) mean return and variance of the portfolio:

$$\mathbb{E}\Big[\frac{V_1 - V_0}{V_0} - r\Big] = \sum_{i=1}^{d} w_i \mathbb{E}[R^i - r] = w^t \mu$$

and

$$\begin{aligned}
\sigma^2(w) &:= var(\sum_{i=1}^{d} w_i (R^i - r)) \\
&= var(w^t (R - r\mathbf{1})) \\
&= \mathbb{E}\Big[(w^t (R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}]))(w^t (R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}]))\Big] \\
&= \mathbb{E}\Big[(w^t (R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}]))((R - r\mathbf{1} - \mathbb{E}[R - r\mathbf{1}])^t w)\Big] \\
&= w^t \Sigma w.
\end{aligned}$$

Hence, the Markowitz optimization problem writes

$$\max_{w \in \mathbb{R}^d} w^t \mu, \quad s.t. \quad w^t \Sigma w \leq \sigma_\star^2, \quad \text{and} \quad w \in B.$$

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(iv) ℓ^p-constraints: $\|w\|_{\ell^p} := (\sum_{i=1}^{d} |w_i|^p)^{1/p} \leq \alpha$ for e.g. $p = 1$ or $p = 2$.

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(iv) ℓ^p-constraints: $\|w\|_{\ell^p} := (\sum_{i=1}^{d} |w_i|^p)^{1/p} \leq \alpha$ for e.g. $p = 1$ or $p = 2$.

We will solve the Markowitz optimization problem under (i) or (ii).

A brief economical derivation of Markowitz's problem

Economical derivation of Markowitz's problem

◦ In Markowitz's approach, we implicitely assumed that the law of returns of the portfolio are entirely determined by the first two moments.
$\rightsquigarrow$ implicit Gaussian assumption.
◦ Assume that $R - r\mathbf{1} \sim \mathcal{N}(\mu, \Sigma)$ and that the agent preference ordering over different investment strategies is based on the utility function

$$U(x) = 1 - \exp(-\lambda x), \text{ for some parameter } \lambda > 0.$$

Note that $\lambda = -U''/U'$ is called the risk aversion parameter.
◦ The optimization problem of the agent will be to maximize his satisfaction in the future, namely

$$\max_{w \in \mathbb{R}^d} \mathbb{E}\Big[U\Big(\frac{V_1}{V_0}\Big)\Big], \quad \text{where } \frac{V_1}{V_0} = 1 + r + w^t(R - r\mathbf{1}) \sim \mathcal{N}(1 + r + w^t\mu, w^t\Sigma w)$$

and one has

$$\mathbb{E}\Big[U\Big(\frac{V_1}{V_0}\Big)\Big] = 1 - \mathbb{E}\Big[e^{-\lambda\frac{V_1}{V_0}}\Big] = 1 - e^{-\lambda(1+r) - \lambda w^t\mu + \frac{\lambda^2}{2} w^t\Sigma w}.$$

◦ The previous optimization problem is equivalent to

$$\max_{w\in\mathbb{R}^d}\left\{\mathcal{L}(w,\lambda) := w^t\mu - \frac{\lambda}{2}w^t\Sigma w\right\}$$

which is equivalent to the Lagrangian formulation of Markowitz's optimization problem.

◦ The Lagrange multiplier λ corresponds to the risk aversion parameter of the agent and is directly linked to the level of maximal risk $\sigma_\star^2$ the agent is willing to take.

Markowitz with no-risk free asset

○ We fix the risk aversion parameter $\lambda > 0$. Corresponding to the constrained optimization problem

$$\max_{w:\mathbf{1}^t w=1} w^t\mu - \frac{\lambda}{2}(w^t\Sigma w - \sigma_\star^2),$$

we associate the Lagrangian

$$\mathcal{L}(w,\beta) := w^t\mu - \frac{\lambda}{2}(w^t\Sigma w - \sigma_\star^2) - \beta(w^t\mathbf{1} - 1).$$

○ Assumption: Σ is invertible, i.e. no colinearity between expected returns.

○ Karush Kuhn Tucker optimality conditions:

$$\begin{cases}\nabla_w\mathcal{L}(w,\beta) = 0\\ \partial_\beta\mathcal{L}(w,\beta) = 0\end{cases} \Leftrightarrow \begin{cases}\mu - \lambda\Sigma w - \beta\mathbf{1} = 0\\ w^t\mathbf{1} = 1\end{cases} \Leftrightarrow \begin{cases}w = \frac{1}{\lambda}\Sigma^{-1}(\mu - \beta\mathbf{1})\\ w^t\mathbf{1} = 1\end{cases}$$

$$\Leftrightarrow \begin{cases}w = \frac{1}{\lambda}\Sigma^{-1}(\mu - \beta\mathbf{1})\\ \beta = \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} - \frac{\lambda}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\end{cases} \Leftrightarrow \left\{w_\star = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{1}{\lambda}\left(\Sigma^{-1}\mu - \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\Sigma^{-1}\mathbf{1}\right)\right.$$

◦ Examples:

- Maximal risk aversion: $\lambda \uparrow \infty$ then

$$w_\star = w_{MV} = \frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}$$

which formally corresponds to the solution of the unconstrained optimization problem $\min_w w^t\Sigma w$ s.t. $\mathbf{1}^t w = 1$.

- Market portfolio: For $\lambda = \mathbf{1}^t\Sigma^{-1}\mu$, we obtain

$$w_\star = w_{MK} = \frac{\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mu}$$

which corresponds to the portfolio fully invested in risky assets (in the presence of a non-risky asset).

◦ Going back to the original $w_\star$, we observe that

$$w_\star = (1 - \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\lambda})\frac{\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\lambda}\frac{\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mu} = (1-\alpha)w_{MV} + \alpha w_{MK}$$

with $\alpha = \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\lambda}$. This result is known as the two fund theorem.

Two fund theorem with only risky assets

Theorem

Fix a risk aversion parameter $\lambda > 0$, the solution $w_\star$ to the Markowitz optimization problem with only risky-assets can be represented as a linear combination of only two portfolios: the minimum variance portfolio and the market portfolio.

$$w_\star = (1-\alpha) w_{MV} + \alpha w_{MK}, \quad \text{with} \quad \alpha = \frac{\mathbf{1}^t \Sigma^{-1} \mu}{\lambda}.$$

◦ Remarks:

- As $\lambda \uparrow \infty$, $\alpha \to 0$, the portfolio becomes the MV portfolio.
- If $\lambda = \mathbf{1}^t \Sigma^{-1} \mu$ then $\alpha = 1$, the portfolio becomes the MK portfolio.

Efficient frontier

We now compute the mean and the variance of the optimal portfolio $w_\star$ using the first formula.

$$R^\star = \mu^t w_\star = \frac{\mu^t\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{1}{\lambda}\Big(\mu^t\Sigma^{-1}\mu - \frac{(\mathbf{1}^t\Sigma^{-1}\mu)^2}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\Big)$$

and

$$\begin{aligned}
\sigma_\star^2 &= (w_\star)^t\Sigma w_\star \\
&= \Big(\frac{\mathbf{1}^t\Sigma^{-1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{\mu^t\Sigma^{-1}}{\lambda} - \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\lambda\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\mathbf{1}^t\Sigma^{-1}\Big)\Sigma\Sigma^{-1}\Big(\frac{\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{\mu}{\lambda} - \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\lambda\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\mathbf{1}\Big) \\
&= \frac{1}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{1}{\lambda}\frac{\mathbf{1}^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} - \frac{1}{\lambda}\frac{(\mathbf{1}^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mathbf{1})}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2} + \frac{1}{\lambda}\frac{\mathbf{1}^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} \\
&+ \frac{1}{\lambda^2}\mu^t\Sigma^{-1}\mu - \frac{1}{\lambda^2}\frac{\mathbf{1}^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\mathbf{1}^t\Sigma^{-1}\mu - \frac{1}{\lambda}\frac{(\mathbf{1}^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mathbf{1})}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2} - \frac{1}{\lambda^2}\frac{(\mathbf{1}^t\Sigma^{-1}\mu)(\mu^t\Sigma^{-1}\mathbf{1})}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} \\
&+ \frac{1}{\lambda^2}\frac{(\mathbf{1}^t\Sigma^{-1}\mu)^2(\mathbf{1}^t\Sigma^{-1}\mathbf{1})}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2} \\
&= \frac{1}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{1}{\lambda^2}\Big(\mu^t\Sigma^{-1}\mu - \frac{(\mathbf{1}^t\Sigma^{-1}\mu)^2}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\Big)
\end{aligned}$$

Efficient frontier

◦ For the MV portfolio ($\lambda \uparrow \infty$), one has

$$R_{MV} = \frac{\mu^t \Sigma^{-1} \mathbf{1}}{\mathbf{1}^t \Sigma^{-1} \mathbf{1}}, \quad \text{and} \quad \sigma_{MV}^2 = \frac{1}{\mathbf{1}^t \Sigma^{-1} \mathbf{1}}.$$

◦ Observe that the Cauchy-Schwarz ineq. applied to the scalar product $(x, y) \mapsto x^t \Sigma^{-1} y$ gives that

$$R_\star \geq R_{MV} \quad \text{and} \quad \sigma_\star^2 \geq \sigma_{MV}^2.$$

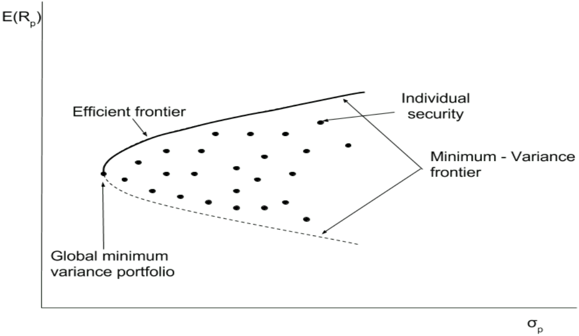
◦ Hence, we get

$$\begin{aligned} R_\star - R_{MV} &= \sqrt{\mu^t \Sigma^{-1} \mu - \frac{(\mathbf{1}^t \Sigma^{-1} \mu)^2}{\mathbf{1}^t \Sigma^{-1} \mathbf{1}}} \times \sqrt{\frac{1}{\lambda^2} \Big(\mu^t \Sigma^{-1} \mu - \frac{(\mathbf{1}^t \Sigma^{-1} \mu)^2}{\mathbf{1}^t \Sigma^{-1} \mathbf{1}}\Big)} \\ &= \sqrt{\mu^t \Sigma^{-1} \mu - \frac{(\mathbf{1}^t \Sigma^{-1} \mu)^2}{\mathbf{1}^t \Sigma^{-1} \mathbf{1}}} \sqrt{\sigma_\star^2 - \sigma_{MV}^2} \end{aligned}$$

Efficient frontier

◦ We thus get the key relation

$$\frac{\sigma_\star^2}{\sigma_{MV}^2} - \frac{(R_\star - R_{MV})^2}{c^2\sigma_{MV}^2} = 1, \quad c := \sqrt{\mu^t\Sigma^{-1}\mu - \frac{(\mathbf{1}^t\Sigma^{-1}\mu)^2}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}}.$$



The above equation says that the efficient frontier is a Hyperbola (half of Hyperbola since $\sigma_\star^2 > \sigma_{MV}^2$) in the plan (σ, R).

Comments on the efficient frontier

◦ Concave relation between $R_\star$ and $\sigma_\star$ reflects the idea of expected utility concavely increasing with level of risk.

◦ For a given level of risk, the efficient portfolio has the highest expected return $R_\star$ in all the investment universe (i.e. exclusively invested in stocks). Said differently, for a given level of return R, the efficient portfolio has the lowest level of risk.

Markowitz with a risk-free asset

Markowitz with a risk-free asset

We now include a risk-free asset. Fix a risk aversion parameter λ. The optimization problem writes

$$\max_{w\in\mathbb{R}^d}\left\{\mathcal{L}(w) := w^t\mu - \frac{\lambda}{2}w^t\Sigma w^t\right\}$$

Recall from the budget constraint that $w_0 = 1 - w^t\mathbf{1}$ is the proportion invested in the risk-free asset.

◦ Assumption: Σ is invertible.

The KKT optimality condition gives

$$\nabla_w\mathcal{L}(w^\star) = 0 \Leftrightarrow \Sigma w^\star = \frac{1}{\lambda}\mu \Leftrightarrow w^\star = \frac{1}{\lambda}\Sigma^{-1}\mu.$$

◦ The portfolio is exclusively invested in risky assets if and only if

$$1 = \mathbf{1}^t w^\star \Leftrightarrow \lambda = \mathbf{1}^t\Sigma^{-1}\mu \Rightarrow w_{MK} = \frac{\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mu} \quad \text{(Market portfolio)}$$

Two fund theorem with a risk-free asset

Theorem

The unique solution $w^\star$ to the Markowitz optimization problem, where investment in the non-risky asset is allowed, can be represented as a combination of two portfolios: the market portfolio and the non-risky asset

$$w^\star = \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\lambda}\frac{\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mu} = \alpha_M \frac{\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mu} = \alpha_M w_{MK}, \quad \textit{with} \quad \alpha_M = \frac{\mathbf{1}^t\Sigma^{-1}\mu}{\lambda}$$

and

$$w_0 = 1 - \mathbf{1}^t w^\star = 1 - \alpha_M.$$

◦ Remarks: Particular cases are:

- As $\lambda \uparrow \infty$, $\alpha_M \to 0$ (fully invested in non-risky asset)
- For $\lambda = \mathbf{1}^t\Sigma^{-1}\mu$, $\alpha_M = 1$ (fully invested in the market portfolio).

Capital Market Line

The mean and variance of optimal portfolios are

$$R^\star = \mu^t w^\star = \frac{1}{\lambda}\mu^t \Sigma^{-1}\mu, \quad \text{and} \quad (\sigma^\star)^2 = \frac{1}{\lambda^2}\mu^t\Sigma^{-1}\mu.$$

Hence, we obtain

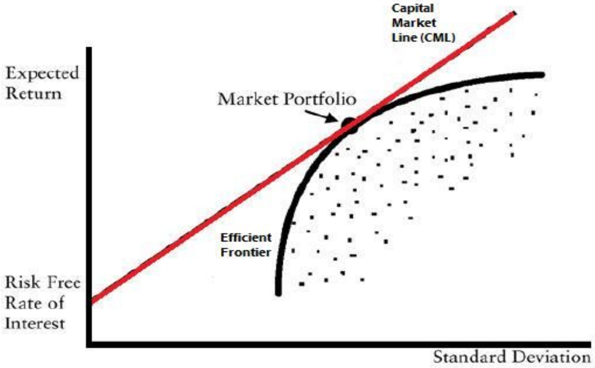
$$R^\star = \sqrt{\mu^t\Sigma^{-1}\mu} \times \sigma^\star.$$

This equation represents a line in the (σ, R) plane, known as the Capital Market Line, which passes through the following two points/portfolios:

- $\sigma^\star = 0$ and $R^\star = 0$. This corresponds to the return of the risk-free asset.
- $\sigma^\star = \sigma_{MK}$ and $R^\star = R_{MK}$, which corresponds to the Market portfolio that lies on the efficient frontier.

From our previous analysis, the CML touches the efficient frontier in only one point, the market portfolio also called tangency portfolio.

Capital Market Line



Sharpe ratio

Based on the two moments, one can define a performance measure known as the Sharpe ratio of the portfolio.

$$\mathcal{S}_p = \frac{R_p}{\sigma_p}.$$

◦ Interpretation: it represents the excess return above the risk-free rate per unit of risk, where the risk is measured by the standard deviation. It represents the additional amount of return an investor receives per unit of increase in risk. When comparing two assets, the one with a higher Sharpe ratio appears to provide a better return for the same risk, which is usually attractive for investors.

◦ Sharpe ratio of portfolios on the CML: Recalling that $R^\star = \sqrt{\mu^t \Sigma^{-1} \mu} \times \sigma^\star$, one has

$$\mathcal{S}_{CML} = \sqrt{\mu^t \Sigma^{-1} \mu}.$$

Sharpe ratio of portfolios on efficient frontier

Theorem

For any portfolio on the efficient frontier with return $R^\star$ and risk $\sigma^\star$, it holds

$$\mathcal{S}_{Eff} = \frac{R^\star}{\sigma^\star} \leq \sqrt{\mu^t \Sigma^{-1} \mu}.$$

In particular the highest Sharpe ratio of the efficient frontier is achieved by the market portfolio. Proof:

$$\begin{aligned}
(R^\star)^2 - (\mu^t\Sigma^{-1}\mu)(\sigma^\star)^2 &= \frac{(\mu^t\Sigma^{-1}\mathbf{1})^2}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2} + \frac{1}{\lambda^2}(\mu^t\Sigma^{-1}\mu)^2 + \frac{1}{\lambda^2}\frac{(\mu^t\Sigma^{-1}\mathbf{1})^4}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^4} \\
&- \frac{2}{\lambda^2}\frac{(\mu^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mu)^2}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} + \frac{2}{\lambda}\frac{\mu^t\Sigma^{-1}\mathbf{1}}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}}\mu^t\Sigma^{-1}\mu - \frac{2}{\lambda}\frac{(\mu^t\Sigma^{-1}\mathbf{1})^3}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2} \\
&- \frac{\mu^t\Sigma^{-1}\mu}{\mathbf{1}^t\Sigma^{-1}\mathbf{1}} - \frac{1}{\lambda^2}(\mu^t\Sigma^{-1}\mu)^2 + \frac{1}{\lambda^2}\frac{(\mu^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mu)^2}{\mathbf{1}\Sigma^{-1}\mathbf{1}} \\
&= \frac{1}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2}\Big((\mu^t\Sigma^{-1}\mathbf{1})^2 - (\mu^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mathbf{1})\Big) \\
&+ \frac{(\mathbf{1}^t\Sigma^{-1}\mu)^2}{\lambda^2(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2}\Big((\mu^t\Sigma^{-1}\mathbf{1})^2 - (\mu^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mathbf{1})\Big) \\
&+ \frac{2\mu^t\Sigma^{-1}\mathbf{1}}{\lambda(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2}\Big((\mu^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mathbf{1}) - (\mu^t\Sigma^{-1}\mathbf{1})^2\Big) \\
&= \frac{(\mu^t\Sigma^{-1}\mathbf{1})^2 - (\mu^t\Sigma^{-1}\mu)(\mathbf{1}^t\Sigma^{-1}\mathbf{1})}{(\mathbf{1}^t\Sigma^{-1}\mathbf{1})^2}\left(1 - \frac{1}{\lambda}(\mathbf{1}^t\Sigma^{-1}\mu)\right)^2 \leq 0.
\end{aligned}$$

Practical implementation and limitations of Markowitz' approach

Markowitz in practice

◦ In practical implementation, the parameters μ and Σ are not known and need to be estimated. We have access to data over n days

$$R_1, \cdots, R_n \in \mathbb{R}^d$$

◦ We assume that $R_1, \cdots, R_n$ are i.i.d. $\sim \mathcal{N}(\mu, \Sigma)$. The joint density of $(R_1, \cdots, R_n)$ is

$$L_n(\mu, \Sigma) = (2\pi)^{-\frac{nd}{2}} (det(\Sigma)^{-\frac{n}{2}}) \prod_{i=1}^{n} \exp(-\frac{1}{2}(R_i - \mu)^t \Sigma^{-1} (R_i - \mu))$$

Theorem

Given $R_1, \cdots, R_n$ i.i.d. observations, the max log-likelihood estimators are

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} R_i, \quad \text{and} \quad \hat{\Sigma} = \frac{1}{n} \sum_{i=1}^{n} (R_i - \hat{\mu})(R_i - \hat{\mu})^t$$

Limitations of Markowitz portfolio theory

◦ The Gaussian assumption of returns. Usually, they are assymetric and have heavy tails.

◦ Limitations of standard deviation as a risk measure. It is symmetric and treats equally loss and gains. In practice, returns of financial assets or of many portfolios are not symmetric. Think of a portfolio strategy that consists in selling out of the money puts on some indices. You will often get the premium and pay nothing but in case of a crash, you will pay a lot.

◦ Standard deviation does not tell us much about these rare big losses which matter most for risk measurement.

◦ In practice, $\hat{\Sigma}$ is singular. $X = (R_1, \cdots, R_n)^t \in \mathbb{R}^{d\times n}$ must be injective. A necessary condition is $d \leq n$.

Thank you!