Université Paris 1 Panthéon Sorbonne
Exam M2 IRFA - MMMEF, 2023-2024
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Exam: Market Risk Measures

18th December 2023

- Documents, calculator as well as cell phone are prohibited.
- The duration of the exam is **2h00**.
- It is forbidden to leave without returning the copy of your exam sheet.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from previous questions.

Name: **IRFA/MMMEF:**

Exercice 1 (Questions on the lectures (9 pts))

1. (2 pts) For a given loss L and confidence level $\alpha \in (0,1)$, recall the definitions of the Value-at-Risk at level α of L denoted by $\mathrm{VaR}_\alpha(L)$ and the Expected Shortfall, also known as Conditional Value-at-Risk, at level α of L denoted by $\mathrm{ES}_\alpha(L)$.
2. (1 pt) Show that the $\mathrm{ES}_\alpha(L)$ is given by the solution to the convex minimization problem
$$\mathrm{ES}_\alpha(L) = \min_{\xi \in \mathbb{R}} \left\{ \xi + \frac{1}{1-\alpha} \mathbb{E}[(L-\xi)_+] \right\}.$$
3. (1 pt) Show that the map $(0,1) \ni \alpha \mapsto \mathrm{VaR}_\alpha(L)$ is non-decreasing, namely, if $\alpha_1 \leqslant \alpha_2$ then one has
$$\mathrm{VaR}_{\alpha_1}(L) \leqslant \mathrm{VaR}_{\alpha_2}(L).$$
4. (1 pt) Using the result of question 2, prove that $(0,1) \ni \alpha \mapsto \mathrm{ES}_\alpha(L)$ is also a non-decreasing map.
5. (1 pt) What are the axioms/properties that a risk measure ρ should satisfy in order to be *coherent*?
6. (1 pt) Are these axioms satisfied by the standard deviation risk measure $\rho_{\mathrm{SD}}(L) = \sqrt{\mathrm{var}(L)}$, where $\mathrm{var}(L)$ stands for the variance of L? Justify your answer. In particular, if you think this is the case, you need to prove it, whereas if it isn't, you should say precisely why.
7. (1 pt) Are these axioms satisfied by the VaR_α risk measure? Justify again your answer as in the previous question.
8. (1 pt) Prove that for any $\alpha \in (0,1)$, it holds
$$\mathrm{VaR}_\alpha(-L) = -\mathrm{VaR}_{1-\alpha}(L).$$

Exercice 2 (Computation of the Value-at-Risk of a portfolio loss (5 pts + Bonus 1 pt))

We consider a portfolio consisting in a *long position* of $\beta = 5$ shares of a stock with initial price $S_0 = 100$. The intraday log-returns of the asset $Y_{t+1} = \log(S_{t+1}/S_t)$, $t \geqslant 0$, are assumed to be an i.i.d. sequence with common law being given by a Gaussian distribution with mean 0 and standard deviation $\sigma = 0.1$. We denote by Φ the cumulative distribution function of the normal distribution $\mathcal{N}(0,1)$ and by φ its density function.

1. (2 pts)
   (a) Provide the expression of the loss random variable L_1.
   (b) Using the question 8. of Exercise 1, prove that the Value-at-Risk at level α of the portfolio loss L_1 between today and tomorrow is given by
$$\mathrm{VaR}_\alpha(L_1) = 500(1 - \exp(-0.1 \times \Phi^{-1}(\alpha))), \quad \alpha \in (0,1).$$
2. (1 pt) We keep the short position during 100 days. What is the law of $\log(S_{100}/S_0)$?
3. (1 pt) Prove that the Value-at-Risk at level α of the portfolio loss L_{100} corresponding to this new position is given by
$$\mathrm{VaR}_\alpha(L_{100}) = 500(1 - \exp(-\Phi^{-1}(\alpha))), \quad \alpha \in (0,1).$$

4. (1 pt) Prove that the Expected Shortfall at level α of L_{100} satisfies the identity

$$\mathrm{ES}_\alpha(L_{100}) = 500\Big(1 - \frac{1}{1-\alpha}\int_{\Phi^{-1}(\alpha)}^{+\infty} \exp(-x)\varphi(x)\,dx\Big),$$

recalling that φ is the density function of the normal distribution $\mathcal{N}(0,1)$.

5. (1 pt) Deduce that

$$\mathrm{ES}_\alpha(L_{100}) = 500\Big(1 - \frac{\exp(\frac{1}{2})}{1-\alpha}\Phi(-1-\Phi^{-1}(\alpha))\Big).$$

Exercice 3 (Expectile risk measure (6 pts)) For $\alpha \in (0,1)$, we define the function $\ell_\alpha : \mathbb{R} \to \mathbb{R}_+$ by

$$\ell_\alpha(x) = \alpha(x_+)^2 + (1-\alpha)((-x)_+)^2 = \begin{cases} \alpha x^2 & \text{if } x \geqslant 0, \\ (1-\alpha)x^2 & \text{if } x \leqslant 0, \end{cases}$$

recalling that $x_+ = \max(0,x)$.

We consider a real-valued random variable L representing the loss of a portfolio over a fixed time horizon. It is assumed that $L \in L^2(\mathbb{P})$ and that its cumulative distribution function is continuous.

In this exercice, our aim is to study the risk measure defined as the solution to the optimization problem

$$\rho_\alpha(L) = \arg\min_{x\in\mathbb{R}} L_\alpha(x), \qquad \text{where} \qquad L_\alpha(x) := \mathbb{E}[\ell_\alpha(L-x)]. \tag{0.1}$$

1. (3 pts)
   (a) (1 pt) Prove that the function L_α defined by (0.1) is convex.
   (b) (1 pt) Prove that L_α is continuously differentiable on $\mathbb{R}$ with derivative given by

   $$L'_\alpha(x) = \mathbb{E}[-2\alpha(L-x)_+ + 2(1-\alpha)(x-L)_+], \quad x \in \mathbb{R}.$$

   (c) (1 pt) Using the fact that ℓ_α is convex and Jensen's inequality, prove that

   $$\lim_{x\to+\infty} L_\alpha(x) = \lim_{x\to-\infty} L_\alpha(x) = +\infty.$$

2. (1 pt) Deduce from question 1. that L_α admits a minimum on $\mathbb{R}$ and that $\rho_\alpha(L)$ is a solution to the following equation

   $$\alpha\mathbb{E}[(L-x)_+] = (1-\alpha)\mathbb{E}[(x-L)_+].$$

   From now on we will assume that the solution to the above equation is unique so that $\rho_\alpha(L)$ is its unique solution.

3. (1 pt) Let $\lambda > 0$. Using the above characterization for $\rho_\alpha(\lambda L)$, prove that $\rho_\alpha(\lambda L) = \lambda\rho_\alpha(L)$. What is the name of this property in the language of risk measures?

4. (1 pt) Let $d \in \mathbb{R}$. Using the same method as the previous question prove that

   $$\rho_\alpha(L+d) = \rho_\alpha(L) + d.$$

   What is the name of this property in the language of risk measures?