Logic and Sets
Final exam 2023 (2h)

Name: QEM/MMEF

Exercise 1 (5pts)

Indicate for each of the following assertions if they are true (T) or false (F).

1. The strong induction principle implies the weak induction principle but not the converse.
2. $B \cup (\cap_{i\in I} A_i) = \cap_{i\in I}(B \cup A_i)$
3. $A \times B = \emptyset$ is equivalent to $(A = \emptyset \vee B = \emptyset)$.
4. $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$
5. $f(A \cap B) = f(A) \cap f(B)$
6. The function $f :]0, \infty[\to \mathbb{R}$ defined by $f(x) = \log x$ is a bijection.
7. The function $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$ is a surjection.
8. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4 + 4x^2 + 2$ is an injection.
9. Let $f : A \to B$, $g : B \to C$. If $g \circ f$ is bijective, then f, g are bijective.
10. Let $f : X \to Y$. Then $f^{-1}(A)$ exists for any $A \subseteq X$ if and only if f is a bijection.
11. The relation $a\mathcal{R}b$ (a divides b, with $a, b \in \mathbb{Z}$) is reflexive, complete and transitive.
12. Countable unions of countable sets are countable.
13. $\mathbb{Z}$ is equipotent with $\mathbb{N}$.
14. $\mathbb{R}$ is equipotent with $\mathbb{Q}$.
15. $\mathbb{N}^k$ is countable for all $k \in \mathbb{N}$.
16. The set of irrational numbers is countable.
17. $2^{\mathbb{N}}$ is uncountable.
18. $\mathbb{R}^2$ is equipotent with $\mathbb{R}$.
19. It is possible that an infimum exists but no minimal element exists.
20. It is possible that a minimal element exists but no infimum exists.

Exercise 2 (4 pts)

Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

1. (3pts) Find its eigenvalues, together with their algebraic and geometric multiplicities, amd find the eigenvectors.
2. (1pt) Is the matrix diagonalizable? Justify your answer.

Exercise 3 (3pts)

Let E be a set and $\mathcal{R}$ the relation on the power set $\mathcal{P}(E)$ defined by

$$A\mathcal{R}B \text{ if } \{A = B \text{ or } A = B^c\}$$

for any $A, B \subseteq E$, where B^c is the complement of B in E.

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Take $E = \{1, 2, 3\}$. Write $\mathcal{P}(E)$ and the equivalence classes of $\mathcal{R}$ on $\mathcal{P}(E)$.

Exercise 4 (3 pts)

Let $f : X \to Y$. Show that $f(f^{-1}(B)) \subseteq B, \forall B \subseteq Y$, and show by an example that equality may not hold. Under which additional condition on f do we have equality?

Exercise 5 (4pts)

Determine, if they exist, the set of lower bounds, upper bounds, the minimal and maximal elements, the infimum and supremum of the following subsets of $\mathbb{R}$:

1. $\left\{ \dfrac{1}{1+x^2} \, : \, x \in \mathbb{R} \right\}$
2. $\{e^n \, : \, n \in \mathbb{N}\}$

Same question for the following subsets of $\mathbb{Q}$:

1. $\left\{ \dfrac{n}{n^2+1} \, : \, n \in \mathbb{N} \right\}$
2. $\{x \in \mathbb{Q} \, : \, x > \sqrt{2}\}$

Note: $0 \notin \mathbb{N}$.

Question (1pt)

Explain what is the continuum hypothesis.