Exercises in Linear Algebra

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Contents

1 Basic notions

Exercise 1.1. Check if the following families of vectors of $\mathbb{R}^3$ are linearly independent:

1. $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\\2\end{pmatrix}, \begin{pmatrix}-1\\2\\7\end{pmatrix}.$

2. $\begin{pmatrix}-1\\0\\1\end{pmatrix}, \begin{pmatrix}2\\3\\0\end{pmatrix}, \begin{pmatrix}0\\4\\1\end{pmatrix}.$

Exercise 1.2. Show that $\left\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}-1\\0\\2\end{pmatrix}, \begin{pmatrix}1\\2\\0\end{pmatrix}\right\}$ is basis of $\mathbb{R}^3$.

Exercise 1.3. Show that span $\left\{\begin{pmatrix}3\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\-2\\-3\end{pmatrix}, \begin{pmatrix}4\\0\\1\end{pmatrix}\right\} = \mathbb{R}^3$.

Exercise 1.4. Consider the following matrices

$$A = \begin{bmatrix}4 & 3 & 2\\0 & 1 & 2\end{bmatrix}, \ B = \begin{bmatrix}-2 & 0 & 1\\1 & 2 & 3\end{bmatrix}, \ C = \begin{bmatrix}1 & 2\\0 & -1\\1 & 1\end{bmatrix}.$$

Compute, whenever possible, $A+B$, $A+C$, $B+C$, AB, AC, BC, CA, CB.

Exercise 1.5. Let $x^T = \begin{pmatrix}1 & 0 & 2 & 3\end{pmatrix}$. Compute $x^T x$ and xx^T.

Exercise 1.6. Compute, whenever possible, the determinant of the following matrices:

$$A = \begin{bmatrix}1\end{bmatrix}, \ B = \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \ C = \begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}, \ D = \begin{bmatrix}1 & 0 & 3\\0 & 1 & 1\\2 & -1 & 0\end{bmatrix},$$

$$E = \begin{bmatrix}1 & 0 & 2\\0 & 1 & -1\\3 & 1 & 0\end{bmatrix}, \ F = \begin{bmatrix}1 & 1 & 3\\0 & 1 & -1\\1 & 0 & 4\end{bmatrix}$$

Exercise 1.7. Put in RREF the matrices

$$\begin{bmatrix}1 & -1 & 1\\2 & 0 & 2\\3 & 2 & 0\end{bmatrix}, \ \begin{bmatrix}1 & 2 & 3\\0 & -1 & 2\\1 & 0 & 7\end{bmatrix}.$$

Exercise 1.8. Give the rank and nullity of the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 2 & 4 & 5 & 6 \\ 1 & 3 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 1 & 0 \\ 4 & 3 & 1 \\ 5 & 0 & 0 \\ 6 & 1 & 1 \end{bmatrix}, \ E = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 2 & -2 \end{bmatrix}.$$

2 Linear systems

Exercise 2.1. Solve by the Gauss-Jordan elimination method:

$$\begin{cases} x - 2y + z = 3 \\ 2x + y - z = 2 \\ x + 3y - 2z = -1 \end{cases}$$

Exercise 2.2. Solve by the Gauss-Jordan elimination method:

$$\begin{cases} x + y + z + t + u = 1 \\ 2x - y - z - 2t - u = 2 \\ 2y + z - t + 2u = 0 \end{cases}$$

Exercise 2.3. Solve by the Gauss-Jordan elimination method:

$$\begin{cases} x + 2y + z = 3 \\ 2x - y + z = 2 \\ x + y - z = 1 \\ 4x + 2y + z = 6 \\ x - 2y + 2z = a \end{cases}$$

with $a \in \mathbb{R}$, and distinguish according to the value of a.

3 Eigenvalues and eigenvectors

Exercise 3.1. Find the eigenvalues, their algebraic multiplicities, geometric multiplicities, and their eigenspaces, of the following matrices:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 4 & 9 \end{bmatrix}, \ B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{bmatrix}, \ D = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}.$$

Which ones are diagonalizable?