Introductory Finance

Mid-term, 4 November 2024

M1 MAEF   DU MMEF   QEM1

English or French can be used.

Please mention your degree on your sheet, and do not copy the questions.

I. We invest an amount M_0 in a risk-free investment that is paying an interest m times per year, with a constant interest rate and compounded interests.
We denote by r_d the total interest rate paid over one year and by $M(t)$ the wealth at a time $t \in [0, T]$. You will define precisely any additional notation you use.

1. Prove that $t \mapsto M(t)$ is a stepwise (staircase) function on $[0, T]$ (describe the intervals on which it is constant and the values taken).

2. Prove that $t \mapsto M(t)$ matches an exponential function at some dates.

3. At what dates of $[0, T]$, is the equality $M(t) = M_0(1 + r_d)^t$ valid?

II. We consider a bond maturing at time T, paying known cash-flows: C_t at time t, with $C_t > 0$, for $t = 1, ..., T$. We denote by ρ its yield-to-maturity at time 0.

1. a. Recall the definition of ρ and its financial meaning, stating the assumptions if any.

b. How does ρ change when the overall interest rates decrease? (explain what happens financially).

2. a. Recall the definition of the duration at time 0.

b. Prove that it can be linked to the first derivative of $P(\rho)$, the price as a function of ρ.

3. We assume that a translation of the rate curve makes the yield-to-maturity move from ρ to $\rho + \Delta\rho$.

a. Write the 2nd order approximation of the corresponding variation of price. By what curve (give its name) are you approximating the price curve $\rho \mapsto P(\rho)$ when you use it.

b. Explain if the 1st order approximation of the rate of change of the price overvalues or undervalues the effect of a rate drop (this involves a property of the price that you will prove).

III.1. Compare the price sensitivity to parallel changes in the term structure of interest rates for a zero-coupon bond and for a coupon-bearing bond with the same maturity.

2. A sovereign issuer with a known 0-coupon yield curve $t \mapsto r(0, t)$ wants to issue at par a coupon-bearing bond with maturity T. What coupon rate should this new issue propose?

I. (6 pts)

1. Let r_m be the annualised rate for 1 period, i.e. over one period, the wealth gets multiplied by $1+\frac{r_m}{m}$. The rate r_m is determined by $(1+\frac{r_m}{m})^m = 1+r_d$.

After k periods, we get $M(\frac{k}{m}) = M_0(1+\frac{r_m}{m})^k$ then $M(\cdot) = M_0(1+r_d)^{\frac{k}{m}}$ on $[\frac{k}{m}, \frac{k+1}{m}[$.

2. $1+\frac{r_m}{m} = e^{\frac{1}{m}\ln(1+r_d)} = e^{\frac{r}{m}}$ where $r = \ln(1+r_d)$.

Then $M(\frac{k}{m}) = M_0(1+\frac{r_m}{m})^k = M_0 e^{r\frac{k}{m}}$ and $M(t) = M_0e^{rt}$ at any $t = \frac{k}{m}, k \in \{0, ..., mT\}$.

3. $M(\frac{k}{m}) = M_0(1+r_d)^{\frac{k}{m}}$ hence satisfied at any $t = \frac{k}{m}, k \in \{0, ..., mT\}$.

II. (11 pts)

1.a For an observed price P at time 0, ρ is defined as the unique (as P decreasing) rate satisfying $P(\rho) = \sum_{t=1}^{T} \frac{C_t}{(1+\rho)^t}$: it is the interest rate that equates the present value of (promised) future payments received from the bond with its value today.

The yield-to-maturity corresponds to the average rate of return on the life of the bond, under the assumption that the bond is kept until its maturity and that the payments are sure.

b. Rates decrease $\Rightarrow$ existing bonds more interesting than the new ones, their prices go up, then, from the equation, the YTM goes down.

2. a The duration is the average life of the bond, it is computed by weighting each payment date by the present value of the corresponding (promised) future payment divided by the price:

$$D = \frac{1}{P}\sum_{t=1}^{T} \frac{tC_t}{(1+\rho)^t}, \text{ where } C_t \text{ is the cashflow at } t\ .$$

b. $P(\rho) = \sum_{t=1}^{T} \frac{C_t}{(1+\rho)^t}$, then $P'(\rho) = -\sum_{t=1}^{T} \frac{tC_t}{(1+\rho)^{t+1}} = -\frac{1}{1+\rho}\sum_{t=1}^{T} \frac{tC_t}{(1+\rho)^t} = -\frac{DP(\rho)}{1+\rho}$.

3.a Using the 2nd order approximation $\Delta P \sim -SP\Delta\rho + \frac{1}{2}P''(\rho)(\Delta\rho)^2$ means that we are approximating the curve $\rho \mapsto P(\rho)$ with a polynomial of degree 2 (parabola).

b. P is a convex function of ρ, indeed:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+\rho)^t} \Rightarrow P'(\rho) = -\sum_{t=1}^{T} \frac{tC_t}{(1+\rho)^{t+1}} \Rightarrow P''(\rho) = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+\rho)^{t+2}} > 0 \text{ for any } \rho > 0.$$

Then its curve is above its tangent. Consequence (draw the curve): the 1st order approximation undervalues the effect of a rate drop.

III. (4 pts)

1. The sensitivity is proportional to the duration ($1+\rho$ is close to 1).

The duration of the coupon-bearing bond is strictly smaller than T, which is the 0-coupon bond duration, so the 0-coupon bond is more sensitive.

2. With N the face value and $B(0,t)$ the discount factor between 0 and T, the coupon rate c satisfies (it is the par rate for the maturity T):

$$\sum_{t=1}^{T} cNB(0,t) + NB(0,T) = N, \text{ therefore: } c = \frac{1-B(0,T)}{\sum_{t=1}^{T} B(0,t)} = \frac{1-\frac{1}{(1+r(0,T))^T}}{\sum_{t=1}^{T}\frac{1}{(1+r(0,t))^t}}.$$