Exercises on Optimisation in Euclidean Spaces borrowed from Further Mathematics for Economic Analysis Knut Sydsaeter, Peter Hammond, Atle Seierstad and Arne Strom, with additional ones from the lectures notes of J.M. Bonnisseau*

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September 12, 2022

1 Introduction, basic concepts of topology of $\mathbb{R}^n$

Basic concepts of Topology of Rn: distance and norm; open sets; closed sets; neighborhood; frontier, closure, interior of a set; bounded and compact sets; convex combination and convex sets.

Upper and lower bounds, supremum, infimum; maximum and minimum. Continuity. Differentiability; partial, directional derivatives. A representation theorem.

Eigenvalues and eigenvectors. Diagonalization. Quadratic forms. Definiteness and semidefiniteness.

Exercise 1 (**SHSS 13.1, 5**) Sketch the set $S = \{(x,y)\mathbb{R}^2 \mid x > 0, w \geq 1/x\}$ in the plane. Is S closed?

Exercise 2 (**SHSS 13.1, 6**) (a) Let E be the subset in $\mathbb{R}^2$ consisting of the point $(0,0)$ and all point of the form $(1/n, 1/m)$ for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$ Is E closed?

(b) Let F be the subset in $\mathbb{R}^2$ defined by $F = \{(0,0)\} \cup \{(1/n, 1/n) \mid n = 1, 2, \ldots\}$. Is F closed?

*EMJMD QEM, First semester, first year, 2021-2022

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Exercise 3 (**SHSS 13.1, 7**) Consider the following three subsets of $\mathbb{R}^2$:

$A = \{(x, y) : y = 1, x \in \cup_{n=1}^{\infty}(2n, 2n + 1)\}$
$B = \{(x, y) : y \in (0, 1), x \in \cup_{n=1}^{\infty}(2n, 2n + 1)\}$
$C = \{(x, y) : y = 1, x \in \cup_{n=1}^{\infty}[2n, 2n + 1]\}$
For each of these sets determine whether it is open, closed, or neither.

Exercise 4 (**SHSS 13.1, 10**) Let S be a subset of $\mathbb{R}^n$, and let $\mathcal{U} = \{U \subset \mathbb{R}^n \mid U \subset S \text{ and } U \text{is open}\}$ be the family of all open subset of S. Similarly, let $\mathcal{F} = \{F \subset \mathbb{R}^n \mid S \subset F \text{ and } F \text{is closed}\}$ be the family of all closed supersets of S.

(a) Show that $\text{int}(S) = \cup_{U \in \mathcal{U}} U$. Thus $\text{int}(S)$ is the largest open subset of S.

(b) Show that $\text{cl}(S) = \cap_{F \in \mathcal{F}} U$. Thus $\text{cl}(S)$ is the smallest closed set containing S.

Exercise 5 (**SHSS 13.1, 11**) Show by an example that the union of infinitely manyu closed sets need not be closed.

Exercise 6 (harder) (**SHSS 13.1, 14**) Prove that the empty set $\emptyset$ and the whole space $\mathbb{R}^n$ are the only sets in $\mathbb{R}^n$ that are both open and closed.

Exercise 7 (**SHSS 2.2, 2**) Determine which of the following sets are convex by drawing each in the plane.

(a) $\{(x, y) \mid x^2 + y^2 < 2\}$;
(b) $\{(x, y) \mid x \geq 0, y \geq 0\}$;
(c) $\{(x, y) \mid x^2 + y^2 > 8\}$;
(d) $\{(x, y) \mid x \geq 0, y \geq 0, xy \geq 1\}$;
(e) $\{(x, y) \mid xy \leq 1\}$;
(f) $\{(x, y) \mid \sqrt{x} + \sqrt{y} \leq 2\}$;

Exercise 8 (**SHSS 2.2, 3**) Let S be the set of all poins $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ that satisfy all the m inequalities

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ \ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \end{cases}$$

and moreover are such that $x_1 \geq 0, \ldots, x_n \geq 0$. Show that S is a convex set.

Exercise 9 (**SHSS 2.2, 4**) If S and T are two sets in $\mathbb{R}^n$ and a and b are scalars, let $W = aS + bT$ denote the set of all points of the $a\mathbf{x} + b\mathbf{y}$, where $\mathbf{x} \in S$ and $\mathbf{y} \in T$. (Then W is called a linear combination of the two sets.) Prove that if S and T are both convex, then so is $W = aS + bT$.

Exercise 10 (**SHSS 2.2, 6**) (a) Let $S = \{\mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x}\| \leq r\}$ be the closed n-dimensional bal centred at the origin and with radius $r > 0$. Prove that S is convex.

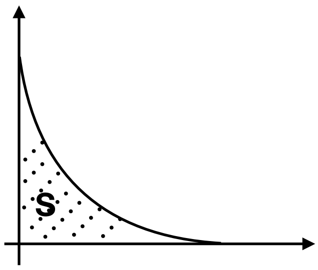

Figure (c)

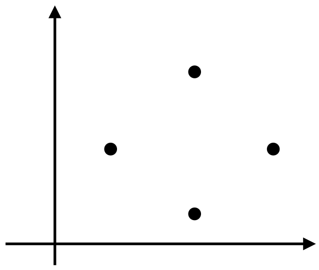

Figure (d)

(b) If we replace $\leq$ with $<$, $=$, or $\geq$ in the definition of S, we get three nex sets S_1, S_2, and S_3. Which of them are convex?

Exercise 11 (harder) (**SHSS 2.2, 7**) (a) Let S be a set of real numbers with the property that if $x_1, x_2 \in S$, then the midpoint $\frac{1}{2}(x_1 + x_2)$ also belongs to S. Show by an example that S is not necessarily convex.

(b) Does it make any difference if S is closed?

Exercise 12 (harder) (**SHSS 13.5, 2**) Determine co(S) in the cases shown in Figures (c) and (d). (In (d), S consists of the four dots.)

Exercise 13 (harder) (**SHSS 13.5, 3**) Suppose that N units of a commoditiy (50 000 barrels of oil, for example) are spread out overt points represented by a two-dimensional coordinate system so that n_1 units are to be found at the point $\mathbf{x}_1$, n_1 units at $\mathbf{x}_2$, ..., n_m units at $\mathbf{x}_m$, where $\sum_{i=1}^{m} n_i = N$. Explain why $\mathbf{z} = (1/N)(n_1\mathbf{x}_1 + n_2\mathbf{x}_2 + \ldots + n_m\mathbf{x}_m)$ is a convex combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$. What is a common name for the point $\mathbf{z}$?

Exercise 14 (harder) (**SHSS 13.3, 1**) Prove that the set $S = \{(x, y) \mid 2x - y < 2 \text{ and } x - 3y < 5\}$ is open in $\mathbb{R}^2$.

Exercise 15 (harder) (**SHSS 13.3, 2**) Prove tha the set $S = \{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \leq 0, j = 1, \ldots, m\}$ is closed if the functions g_j are all continuous.

Exercise 16 (harder) (**SHSS 13.3, 3**) Give examples of subsets S of $\mathbb{R}$ and continuous function $f : \mathbb{R} \to \mathbb{R}$ such that

(a) S is closed , but $f(S)$ is not closed.

(b) S is open, but $f(S)$ is not open.

(c) S is bounded, but $f(S)$ is not bounded.

Exercise 17 (harder) (**SHSS 13.3, 4**) For a fixed $\mathbf{a}$ in $\mathbb{R}^n$, prove that the function $f : \mathbb{R}^n \to \mathbb{R}$, defined by $f(\mathbf{x}) = d(\mathbf{x}, \mathbf{a})$ is continuous.

Exercise 18 (harder) (**SHSS 2.1, 2**) Let $f(t)$ be a $\mathcal{C}^1$ function of t with $f'(t) \neq 0$.

(a) Put $F(x, y) = f(x^2 + y^2)$. Find the gradient ∇F at an arbitrary point and show that it is parallel to the straight line segment joining the point and the origin.

(b) Put $G(x, y) = f(y/x)$. FInd ∇G at an arbitraty point where $x \neq 0$, and show that it is orthogonal to the straight line segment joining the point and the origin.

Exercise 19 (harder) (**SHSS 2.1, 3**) Compute the directional derivatives of the following functions at the given points and in the given directions.

(a) $f(x, y) = 2x + y - 1$ at $(2, 1)$, in the direction given by $(1, 1)$.

(b) $g(x, y, z) = xe^{xy} - xy - z^2$ at $(0, 1, 1)$, in the direction given by $(1, 1, 1)$.

Exercise 20 (harder) (**SHSS 2.1, 6**) Suppose that $f(x, y)$ has continuous partial derivatives. Suppose too that the maximum directional derivative of f at $(0, 0)$ is equal to 4, and that it is attained in the direction given by the vector from the origin to the point $(1, 3)$. Find $\nabla f(0, 0)$.

Exercise 21 (harder) (**SHSS 2.1, 9**) (a) Prove that if F is $\mathcal{C}^2$ and $F(x, y) = C$ defines y as a twice differentiable function of x, then

$$y'' = -\frac{1}{(F_2')^3}[F_{11}''(F_2')^2 - 2F_{12}''F_1'F_2' + F_{22}''(F_1')^2] = -\frac{1}{(F_2')^3}\begin{vmatrix} 0 & F_1' & F_2' \\ F_1' & F_{11}'' & F_{12}'' \\ F_2' & F_{21}'' & F_{22}'' \end{vmatrix}$$

(b) Let $F(x, y) = x^2y$ and $C = 8$. Use the formula in (a) to compute y'' at $(x, y) = (2, 2)$. Check the result by differentiating $y = 8/x^2$ twice.

Exercise 22 (**SHSS 2.9, 1**) (a) Let f be defined for all (x, y) by $f(x, y) = \frac{xy^2}{x^2+y^4}$ and $f(0, 0) = 0$. Show that $f_1'(x, y)$ and $f_2'(x, y)$ exist for all (x, y).

(b) Show that f has a directional derivative in every direction at every point.

(c) Show that f is not continuous at $(0, 0)$. (*Hint:* Consider the behaviour of f along the curve $x = y^2$.) Is f differentiable at $(0, 0)$.

Exercise 23 (**SHSS 1.5, 1**) For the following matrices, find the eigenvaleues and also those eigenvectors that corresdpond to the real eigenvalues:

(a) $\begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 4 \\ -2 & 6 \end{pmatrix}$

(c) $\begin{pmatrix} 1 & 4 \\ 6 & -1 \end{pmatrix}$

(d) $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

(e) $\begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$

(f) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

Exercise 24 (**SHSS 1.5, 2**) (a) Compute $\mathbf{X}'\mathbf{A}\mathbf{X}$, $\mathbf{A}^2$, and $\mathbf{A}^3$ when $\mathbf{A} = \begin{pmatrix} a & a & 0 \\ a & a & 0 \\ 0 & 0 & b \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

(b) FInd all the eigenvalues of $\mathbf{A}$.

(c) The characteristic polynomlal $p(\lambda)$ of $\mathbf{A}$ is a cubic function of λ. Show that if we replace λ by $\mathbf{A}$, then $p(\mathbf{A})$ is the zero matrix. (This is a special case od the Cayley-Hamilton theorem.)

Exercise 25 (**SHSS 1.5, 5**) Let $\mathbf{A} = \begin{pmatrix} -2 & -1 & 4 \\ 2 & 1 & -2 \\ -1 & -1 & 3 \end{pmatrix}$, $\mathbf{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

(a) Verify that $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are eigenvectors of $\mathbf{A}$, and find the associated eigenvalues.

(b) Let $\mathbf{B} = \mathbf{A}\mathbf{A}$. Show that $\mathbf{B}\mathbf{x}_2 = \mathbf{x}_2$ and $\mathbf{B}\mathbf{x}_3 = \mathbf{x}_3$. Is $\mathbf{B}\mathbf{x}_1 = \mathbf{x}_1$?

(c) Let $\mathbf{C}$ be an arbitrary $n \times n$ matrix such that $\mathbf{C}^3 = \mathbf{C}^2 + \mathbf{C}$. Prove that if λ is an eigenvalue for $\mathbf{C}$, then $\lambda^3 = \lambda^2 + \lambda$. Show that $\mathbf{C} + \mathbf{I}_n$ has an inverse.

Exercise 26 (**SHSS 1.6, 2**) (a) Let the matrices $\mathbf{A}_k$ and $\mathbf{P}$ be given by

$$\mathbf{A}_k = \begin{pmatrix} 1 & k & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \qquad \mathbf{P} = \begin{pmatrix} 1/\sqrt{10} & -3/\sqrt{35} & 3/\sqrt{14} \\ 0 & 5/\sqrt{35} & 2/\sqrt{14} \\ 3/\sqrt{10} & 1/\sqrt{35} & -1/\sqrt{14} \end{pmatrix}$$

Find the characteristic equation of $\mathbf{A}_k$ and determine the values of k that make all the eigenvalues real. What are the eigenvalues if $k = 3$?

(b) Show that columns of $\mathbf{P}$ are eigenvectors of $\mathbf{A}_3$, and compute the matrix product $\mathbf{P}'\mathbf{A}_3\mathbf{P}$. What do you see?

Exercise 27 (**SHSS 1.6, 3**) (a) Prove that if $\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$, where $\mathbf{P}$ and $\mathbf{D}$ are $n \times n$ matrices, then $\mathbf{A}^2 = \mathbf{P}\mathbf{D}^2\mathbf{P}^{-1}$.

(b) Show by induction that $\mathbf{A}^m = \mathbf{P}\mathbf{D}^m\mathbf{P}^{-1}$ for every positive integer m.

Exercise 28 (**SHSS 1.7, 5**) Using a result of the course, determine the definiteness of

(a) $Q = x_1^2 + 8x_2^2$

(b) $Q = 5x_1^2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 + 4x_3^2$

(c) $Q = -(x_1 - x_2)^2$

(d) $Q = -3x_1^2 + 2x_1x_2 - x_2^2 + 4x_2x_3 - 8x_3^2$

Exercise 29 (**SHSS 1.7, 6**) Let $\mathbf{A} = (a_{ij})_{n\times n}$ be symmetric and positive semidefinite. Prove that $\mathbf{A}$ is positive definite if and only if $|\mathbf{A}| \neq 0$.

Exercise 30 (**SHSS 1.7, 7**) (a) For what values of c is the quadratic form

$$Q(x, y) = 3x^2 - (5 + c)xy + 2cy^2$$

(i) positive definite, (ii) positive semidefinite, (iii) indefinite?

(b) Let $\mathbf{B}$ be an $n \times n$ matrix. Show that the matrix $\mathbf{A} = \mathbf{B}'\mathbf{B}$ is positive semidefinite. Can you find a necessary and sufficient condition on $\mathbf{B}$ for $\mathbf{A}$ to be positive definite, not just semidefinite?

Exercise 31 (**SHSS 1.7, 8**) Show that if $Q = \mathbf{x}'\mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is an $n\times n$ matrix, is positive definite, then

(a) $a_{ii} > 0$, $i = 1, \ldots n$, (b) $\begin{vmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{vmatrix} > 0$, $i, j = 1, \ldots, n$

Exercise 32 (**SHSS 1.7, 9**) Let $\mathbf{A}$ be a symmetric matrix. Write its characteristic polynomial as

$$\varphi(\lambda) = (-1)^n(\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_1\lambda + a_0)$$

Prove that $\mathbf{A}$ is negative definite if and only if $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

Exercise 33 (**SHSS 13.6, 4**) If S is a set in $\mathbb{R}^n$ and $\mathbf{y}$ is a boundary point of S, is $\mathbf{y}$ necessarily a boundary point $\overline{S}$? (*Hint:* The irrational number $\sqrt{2}$ is a boundary point of the set $\mathbb{Q}$ of rational numbers, but what is $\overline{\mathbb{Q}}$? If S is *convex*, then it is true that a boundary point of S is also a boundaty of point of $\overline{S}$.)

Exercise 34 (**SHSS 13.6, 5**) Some books in economics have suggested the following generalisation of the Minkowski's separating hyperplane Theorem: Two convex sets in $\mathbb{R}^n$ with only one point in common can be separated by a hyperplane. Is this statement correct? What about the assertion taht two convex sets in $\mathbb{R}^n$ with disjoint interiors can be separated by a hyperplane?

Exercise 35 (**SHSS 2.3, 1**) Which of the functions whose graphs are shown in the figure (e) are (presumably) convex/concave, strictly concave/strictly convex?

Exercise 36 (**SHSS 2.3, 2**) (a) Let f be definedfor all x, y by $f(x, y) = x - y - x^2$. Show that f is concave using different results of the course.

(b) Show that $-e^{-f(x,y)}$ is concave.

Exercise 37 (**SHSS 2.3, 3**) Show that $f(x, y) = ax_2 + 2bxy + cy^2 + px + qy + r$ is strictly concave if $ac - b^2 > 0$ and $a < 0$, whereas it is strictly convex is $ac - b^2 > 0$ and $a > 0$.

(b) Find necessary and sufficient condition for $f(x, y)$ to be concave/convex.

Exercise 38 (**SHSS 2.3, 4**) For what values of the constant a is the following function concave/convex?

$$f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$$

Exercise 39 (**SHSS 2.3, 5**) Examine the convexity/concavity of the following functions:
(a) $z = x + y - e^x - e^{x+y}$ (b) $z = e^{x+y} + e^{x-y} - \frac{1}{2}y$ (c) $w = (x+2y+3z)^2$

Exercise 40 (**SHSS 2.3, 6**) Supppose $y = f(\mathbf{x})$ is a production function determining output y as a function of the vector $\mathbf{x}$ of nonnegative factor inputs, with $f(\mathbf{0}) = 0$. Show that:
(a) If f is concave, then $f''_{ii}(\mathbf{x}) \leq 0$ (so each marginal product $f'_i(\mathbf{x})$ is decreasing).
(b) If f is concave, then $f(\lambda\mathbf{x})/\lambda$ is decreasing as a function of λ.
(c) If f is homogeneous of degree1 (constant return to scale), then f is not strictly concave.

Exercise 41 (**SHSS 2.3, 7**) Let f be defined for all $\mathbf{x}$ in $\mathbb{R}^n$ by $f(\mathbf{x}) = \|\mathbf{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$. Prove that f is convex. Is f strictly convex? (*Hint:* Use the triangular inequality for the norm.)

Exercise 42 (**SHSS 2.3, 8**) Show that the CES function f defined for $v_1 > 0$, $v_2 > 0$ by
$f(v_1, v_2) = A(\delta_1 v_1^{-\rho} + \delta_2 v_2^{-\rho})^{-1/\rho} \quad (A > 0,\ \rho \neq 0,\ \delta_1, \delta_2 > 0)$
is concave for $\rho \geq -1$ and convex for $\rho \leq -1$, and that it is strictly concave if $\rho > -1$.

Exercise 43 (**SHSS 2.3, 9**) (a) The Cobb-Douglas function $z = f(\mathbf{x}) = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ ($a_1 > 0$, …, $a_n > 0$) is defined for all $x_1 > 0$, …, $x_n > 0$. Prove that the kth leading principal minor of the Hessian $\mathbf{f}''(\mathbf{x})$ is

$$D_k = \frac{a_1 \ldots a_k}{(x_1 \ldots x_k)^2} z^k \begin{vmatrix} a_1 - 1 & a_1 & \ldots & a_1 \\ a_2 & a_2 - 1 & \ldots & a_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_k & a_k & \ldots & a_k - 1 \end{vmatrix}$$

(b) Prove that $D_k = (-1)^{k-1}(\sum_{i=1}^k a_i - 1) z^k \frac{a_1 \ldots a_k}{(x_1 \ldots x_k)^2}$. (*Hint:* Add all the other rows to the first row, extract the common factor $\sum_{i=1}^k a_i - 1$, and then subtract the first column in the new determinant for all the other columns.)
(c) Prove that the function is strictly concave if $a_1 + \ldots + a_n < 1$.

Exercise 44 (**SHSS 2.4, 1**) Prove that $f(x,y) = 1 - x^2 - y^2$ defined in $\mathbb{R}^2$ is concave by showing that the gradient is a supergradient.

Exercise 45 (**SHSS 2.4, 2**) Use the Jensen's inequality to $f(x) = \ln(x)$, with $\lambda_1 = \ldots = \lambda_n = 1/n$ to prove that

$$\sqrt[n]{x_1 x_2 \ldots x_n} \leq \frac{1}{n}(x_1 + x_2 + \ldots + x_n) \text{ for } x_1 > 0, \ldots, x_n > 0$$

Exercise 46 (**SHSS 2.4, 6**) Prove that $f(x, y) = x^4 + y^4$ defined in $\mathbb{R}^2$ is strictly convex by showing that the gradient is a subgradient.

Exercise 47 (**B chap 2, 11**) Let $x \in \mathbb{R}^n$.

1) Show that if for all $y \in \mathbb{R}^n$, $x \cdot y \leq 0$, then $x = 0$.

2) Show that if for all $y \in \mathbb{R}^n$, $x \cdot y \geq 0$, then $x = 0$.

3) Show that there exists a real number a such that for all $y \in \mathbb{R}^n$, $x \cdot y \geq a$, then $x = 0$.

4) Show that there exists a real number a such that for all $y \in \mathbb{R}^n$, $x \cdot y \leq a$, then $x = 0$.

Exercise 48 (**B chap 2, 15**) Let (u_ν) and (v_ν) be two sequences. We assume that (u_ν) is convergent. Show that if the set $\{n \in \mathbb{N} \mid u_\nu \neq v_\nu\}$ is finite, then, (v_ν) is convergent and has the same limit as (u_ν).

We assume that (u_ν) is not convergent. Show that if the set $\{\nu \in \mathbb{N} \mid u_\nu \neq v_\nu\}$ is finite, then, (v_ν) is not convergent.

Exercise 49 (**B chap 2, 18**) Give the closure and the interior of the following subsets of $\mathbb{R}^n$.

$\mathbb{R}^n$;

$\mathbb{R}^n_+$;

$\mathbb{R}^n_{++}$;

a linear subspace of $\mathbb{R}^n$ different from $\mathbb{R}^n$;

$\bar{B}(x, r)$ for $x \in \mathbb{R}^n$ and $r > 0$;

$B(x, r)$ for $x \in \mathbb{R}^n$ and $r > 0$;

in $\mathbb{R}^2$, $]0, 1]^2 \cup ([1, 2] \times \{0\})$;

in $\mathbb{R}^2$, $\{(x, y) \in \mathbb{R}^2 \mid x + y \geq 0, x^2 + y^2 \leq 1\}$;

in $\mathbb{R}^2$, $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0, xy \geq 1\}$.

Exercise 50 (**B chap 2, 26**) We consider the linear space $\mathbb{R}^n \times \mathbb{R}^p$. We define the mapping N from $\mathbb{R}^n \times \mathbb{R}^p$ to $\mathbb{R}_+$ by $N(x, y) = \max\{\|x\|_n, \|y\|_p\}$. Show that N is a norm on $\mathbb{R}^n \times \mathbb{R}^p$ and that it is equivalent to the Euclidean norm $\|(x, y)\| = \sqrt{\sum_{i=1}^n (x_i)^2 + \sum_{j=1}^p (y_j)^2}$.

Exercise 51 (**B chap 2, 27**) We consider the linear space $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ with the norm $N_{\mathcal{L}}$ and f an element of $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$. We consider the mapping Φ from $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ to itself defined by $\Phi(g) = g \circ f$.
1) Show that Φ is a linear mapping. Show that it is Lipschitz continuous with a coefficient $N_{\mathcal{L}}(f)$.
Same question with Ψ defined by $\Psi(g) = f \circ g$.

Exercise 52 (**B chap 3, 33**) Let M be a $p \times n$ matrix. Let P be the $p \times p$ matrix defined by $P = MM^t$.
1) Show that P is a symmetric positive semi-definite matrix.
2) Show that if the rank of M is equal to p, then P is positive definite.
Let N be a $n \times n$ symmetric positive definite matrix. Same questions with $Q = MNM^t$.

Exercise 53 (**B chap 3, 34**) Let $a \in \mathbb{R}$ and q be a quadratic function define on $\mathbb{R}^3$ as:

$$q(x, y, z) = x^2 + (1 + a)y^2 + (1 + a + a^2)z^2 + 2xy - 2ayz$$

1) Compute the bilinear form φ associated to q.
2) Give the matrix of q in the canonical basis of $\mathbb{R}^3$.
3) For which values of a, φ is positive definite?

Exercise 54 (**B chap 3, 38**) Let N be a norm on $\mathbb{R}^n$. Show that N is not differentiable at 0.

Exercise 55 (**B chap 3, 39**) Let f be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}$. Show that f is differentiable on $\mathbb{R}^n$ and $Df(x) = f$ for all $x \in \mathbb{R}^n$.

Exercise 56 (**B chap 3, 43**) Let f be a mapping from $\mathbb{R}^n$ to $\mathbb{R}$. We assume that there exists $c \in \mathbb{R}_+$ and $\alpha > 0$ such that for all $(x, y) \in (\mathbb{R}^n)^2$,

$$|f(y) - f(x)| \leq c\|y - x\|^{1+\alpha}$$

1) Show that the partial derivatives of f at each point of $\mathbb{R}^n$ are vanishing.
2) Deduce that f is constant.

Exercise 57 (**B chap 3, 47**) Let f be a differentiable mapping from an open subset U of $\mathbb{R}^n$ to $\mathbb{R}^p$. We assume that f is k Lipschitz continuous on U, i.e., $\exists k > 0$, $\forall x, y \in U^2$, $\|f(x) - f(y)\|_p \leq k\|x - y\|_n$. Show that for all x in U, $\|Df(x)\|_{\mathcal{L}} \leq k$.

2 On optimization

Optimization in Economics: examples. Existence result: the Weierstrass theorem.

Exercise 58 (**B chap 1, 3**) Let f be a function defined on C. Let us suppose that $\varphi : X \subset \mathbb{R} \to \mathbb{R}$ is an increasing function and $f(c) \in X$ for all $c \in C$.

$$(\mathcal{P}_1)\left\{\begin{array}{l}\max f(x)\\ x \in C\end{array}\right. \quad (\mathcal{P}_2)\left\{\begin{array}{l}\max \varphi(f(x))\\ x \in C\end{array}\right. \quad (\mathcal{P}_3)\left\{\begin{array}{l}\min -\varphi(f(x))\\ x \in C\end{array}\right.$$

1) Prove that the three following problems are equivalents, that is that their sets of solutions are the same.
2) Prove that if φ is continuous and $\mathrm{val}(\mathcal{P}_1) \in X$, then $\mathrm{val}(\mathcal{P}_2) = \varphi(\mathrm{val}(\mathcal{P}_1))$.
3) Show that if there exists a solution, then $\mathrm{val}(\mathcal{P}_2) = \varphi(\mathrm{val}(\mathcal{P}_1))$.
4) Let us consider $f(x) = x$, $C =]0, 1[$ and φ equal to the ceiling function, that is$\varphi(x)$ is the smallest element of $\mathbb{Z}$ greater or eqal to x, or $\varphi(x) = \min\{z \in \mathbb{Z} \mid x \leq z\}$. Compute $\mathrm{val}(\mathcal{P}_1)$, $\mathrm{val}(\mathcal{P}_2)$ and $\varphi(\mathrm{val}(\mathcal{P}_1))$.

Exercise 59 (**B chap 1, 5**)
1) Prove that the function $x \to ax^2 + bx + c$ with $a > 0$ is coercive.
2) Prove that the function $x \to ax^3 + bx^2 + cx + d$ with $a \neq 0$ is not coercive.

Exercise 60 (**B chap 1, 6**) Let f be a coercive function from $\mathbb{R}$ to $\mathbb{R}$. Let g be a function from $\mathbb{R}$ to $\mathbb{R}$. We assume that there exists $r > 0$ such that for all $x \in]-\infty, -r] \cup [r, +\infty[$, $f(x) \leq g(x)$. Show that g is coercive.

Exercise 61 (**B chap 1, 7**) Let f be a continuous function from $\mathbb{R}$ to $\mathbb{R}$. We consider the above minimisation problem $(\mathcal{P})$ above and we assume that C is closed. Show that the problem $(\mathcal{P})$ has a solution if there exists $\bar{c} \in C$ such that the set $\{c \in C \mid f(c) \leq f(\bar{c})\}$ is bounded.

Exercise 62 (**B chap 1, 8**) We consider the following maximisation problem:

$$(\mathcal{P}(\alpha))\left\{\begin{array}{l}\text{Maximise } ax^2 + bx + c\\ x \geq \alpha\end{array}\right.$$

For which values of (a, b, c) this problem has a solution? For which values of (a, b, c) this problem has a finite value?

When a solution exists, compute the solution and give the value of the problem.

Exercise 63 (**B chap 1, 10**) Find the solution(s) of the following maximisation problem when it exists and compute the value of the problem:

$$(\mathcal{P}(\alpha))\left\{\begin{array}{l}\text{Maximise } \sqrt{x} + 2\sqrt{c - x}\\ x \in [0, c]\end{array}\right.$$

where c is a positive real number.

$$(\mathcal{P}(\alpha))\left\{\begin{array}{l}\text{Maximise } x^2 + 2(c - x)\\ x \in [0, c]\end{array}\right.$$

where c is a positive real number.

$$(\mathcal{P}(\alpha))\begin{cases}\text{Maximise } ax - e^x \\ x \in \mathbb{R}\end{cases}$$

where a is a positive real number.

Exercise 64 (**B chap 2, 28**) Let C be a closed subset of $\mathbb{R}^n$ and $\bar{x}$ an element of $\mathbb{R}^n$. Show that the following minimisation problem has a solution:

$$(\mathcal{P})\begin{cases}\text{Minimise } \|x - \bar{x}\| \\ x \in C\end{cases}$$

Exercise 65 (**B chap 2, 29**) Let f be a coercive function from $\mathbb{R}^n$ to $\mathbb{R}$. Let g be a function from $\mathbb{R}^n$ to $\mathbb{R}$. We assume that there exists $r > 0$ such that for all x satisfying $\|x\| \geq r$, $f(x) \leq g(x)$. Show that g is coercive.

Exercise 66 (**B chap 2, 31**) Let f be a continuous function from $\mathbb{R}^n_{++}$ to $\mathbb{R}$. We assume that for all $x \in \mathbb{R}^n_{++}$, the set $A = \{x' \in \mathbb{R}^n_{++} \mid f(x') \geq f(x)\}$ is closed in $\mathbb{R}^n$. Show that for all closed subset C of $\mathbb{R}^n$ such that $C \cap \mathbb{R}^n_{++}$ is nonempty and bounded, the problem

$$(\mathcal{P})\begin{cases}\text{Maximise } f(x) \\ x \in C \cap \mathbb{R}^n_{++}\end{cases}$$

has a solution.

3 Unconstrainded optimisation

Looking for unconstrained optima: FOC; SOC.

Exercise 67 (**SHSS 3.1, 4**) Find the functions $x^*(r)$ and $y^*(r)$ such that $x = x^*(r)$ and $y = y^*(r)$ solve the problem

$$\max_{x,y} f(x, y, r) = -x^2 - xy - 2y^2 + 2rx + 2ry$$

Exercise 68 (**SHSS 3.1, 5**) Find the solutions $x^*(r, s)$ and $y^*(r, s)$ of the problem

$$\max_{x,y} f(x, y, r, s) = r^2x^2 + 3s^2y - x^2 - 8y^2$$

Exercise 69 (**SHSS 3.2, 1**) The function

$$f(x_1, x_2, x_3) = x_1^2 + x_2^2 + 3x_3^2 - x_1x_2 + 2x_1x_3 + x_2x_3$$

defined on$\mathbb{R}^3$ has only one stationary point. Show that it is a local minimum point.

Exercise 70 (**SHSS 3.2, 2**) (a) Let f be defined for all (x, y) by $f(x, y) = x^3 + y^3 - 3xy$. Show that $(0, 0)$ and $(1, 1)$ are the only stationary points, and compute the quadratic form associated to the Hessian matrix of f at the stationary points.

(b) Check the definiteness of the quadratic form at the stationary points.

(c) Classify the stationary points, local minimum, local maximum, saddle point.

Exercise 71 (**SHSS 3.2, 3**) Classify the stationary points of

(a) $f(x, y, z) = x^2 + x^2y + y^2z + z^2 - 4z$

(b) $f(x_1, x_2, x_3, x_4) = 20x_2 + 48x_3 + 6x_4 + 8x_1x_2 - 4x_1^2 - 12x_3^2 - x_4^2 - 4x_2^3$

Exercise 72 (**SHSS 3.2, 4**) Suppose $f(x, y)$ has only one stationary point (x^*, y^*) which is a local minimum point. Is (x^*, y^*) necessarily a global minimum point? It may be surprising that the answer is no. Prove this by examining the function defined for all (x, y) by $f(x, y) = (1 + y)^3x^2 + y^2$. (*Hint:* Look at $f(x, -2)$ as $x \to \infty$.)

Exercise 73 (**B chap 3, 48**) For the following functions, find the critical points where the gradient vanish.

1) $f(x, y) = \ln(1 + xy)$, $(x, y) \in \{(x', y') \in \mathbb{R}^2 \mid xy > -1\}$

2) $f(x, y) = xy^2 + xy - 2x - 12y$

3) $f(x, y, z) = -2x^2 - 2xy - xz - \frac{1}{2}y^2 + 2xz - 2z^2 + x - 2y - z$

4) $f(x, y) = x^2y^2 - 4x^2 - y^2$

5) $f(x, y) = 2x^4 + 2x^2y + y^2 - 2x^2 + 1$

6) $f(x, y) = \frac{1}{\sqrt{x^2+y^2}} + \frac{1}{\sqrt{(x-1)^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0, 0), (1, 0)\}$

7) $f(x, y) = x(2\ln(x) - y - 1) + e^y$ on $\mathbb{R}_+^* \times \mathbb{R}$

8) $f(x, y) = (x^2 + (x - 1)^2)(y - 3)^2 + y$

9) $f(x, y) = x^4 - 2x^2y + x^2 + 3y^2 - 2xy - 2y + 3$

10) $f(x, y) = x^4 - x^2 + y^4 + 2xy^2 - 2y^2 - 2x + 2$

11) $f(x, y) = 14x^2 - 6xy + 6y^2$

Exercise 74 (**B chap 3, 50**) Let f be a continuously differentiable mapping from U an open subset of $\mathbb{R}^n$ to $\mathbb{R}$. Let $y \in \mathbb{R}^n$. Show that if the following optimisation problem

$$\begin{cases} \text{Maximise } x \cdot y - f(x) \\ x \in U \end{cases}$$

has a solution $\bar{x}$ then $y = \nabla f(\bar{x})$.

Exercise 75 (**B chap 3, 51**) Check for the critical points of Exercise **??** if they satisfy the second order necessary optimality condition.

4 Optimisation with equality constraints

Optimization problem with equality constraints. Necessary conditions for optimality: Theorem of Lagrange. The Lagrangian function: interpretation of the Lagrange multipliers.

Equality constraints: Second order conditions. Sufficient conditions for local optimality. Sufficient conditions for global optimality.

Exercise 76 (**SHSS 3.3, 1**) (a) Solve the problem $\max -x^2 - y^2 - z^2$ subject to $x + 2y + z = a$.

(b) Compute the optimal value function $f^*(a)$ and verify that the derivative of the value function is equal to the multiplier.

Exercise 77 (**SHSS 3.3, 2**) (a) Solve the problem

$\max x + 4y + z$ subject to $x^2 + y^2 + z^2 = 216$ and $x + 2y + 3z = 0$

(b) Change the first constraint to $x^2 + y^2 + z^2 = 215$ and the second to $x + 2y + 3z = 0.1$. Estimate the corresponding change in the maximum value by using that the partial derivatives of the value function are equal to the multipliers.

Exercise 78 (**SHSS 3.3, 3**) (a) Solve the problem

$$\max e^x + y + z \text{ subject to } \begin{cases} x + y + z = 1 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

(b) Replace the constraints by $x + y + z = 1.02$ and $x^2 + y^2 + z^2 = 0.98$. What is the approximate change in optimal value of the objective function?

Exercise 79 (**SHSS 3.3, 4**) (a) Solve the utility maximizing problme (assuming $m \geq 4$)

$\max U(x_1, x_2) = \frac{1}{2}\ln(1 + x_1) + \frac{1}{4}\ln(1 + x_2)$ subject to $2x_1 + 3x_2 = m$

(b) With $U^*(m)$ as indirect utility function, show that $dU^*/dm = \lambda$.

Exercise 80 (**SHSS 3.3, 5**) (a) Solve the problem $\max 1 - rx^2 - y^2$ subject to $x + y = m$, with $r > 0$.

(b) Find the value function $f^*(r, m)$ and compute $\partial f^*/\partial r$ and $\partial f^*/\partial m$ and verify that they are equal to the partial derivative of the Lagrangian computed at the solution.

Exercise 81 (**SHSS 3.3, 6**) (a) Solve the problem

$\max x^2 + y^2 + z^2$ subject to $x^2 + y^2 + 4z^2 = 1$ and $x + 3y + 2z = 0$

(b) Suppose we change the first constraint to $x^2 + y^2 + 4z^2 = 1.05$ and the second constraint to $x + 3y + 2z = 0.05$. Estimate the corresponding change in the value function.

Exercise 82 (**SHSS 3.3, 7**) (a) Let $U(\mathbf{x}) = \sum_{j=1}^n \alpha_j \ln(x_j - a_j)$, where α_j, a_j, p_j, and m are all positive constants with $\sum_{j=1}^n \alpha_j = 1$, and with $m > \sum_{i=1}^n p_i a_i$. Show that if $\mathbf{x}^*$ solves

$\max U(\mathbf{x})$ subject to $\mathbf{p} \cdot \mathbf{x} = m$, $\mathbf{x} \geq \mathbf{0}$

then the expenditure on good j is the following linear function of prices and income

$p_j x_j^* = \alpha_j m + p_j a_j - \alpha_j \sum_{i=1}^n p_i a_i$, $j = 1, 2, \ldots, n$

(b) Let $U^*(\mathbf{p}, m) = U(\mathbf{x}^*)$ denote the indirect utility function. Verify Roy's identity:

$$\frac{\partial U^*}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = -\lambda x_i^*, \quad i = 1, \ldots, n$$

Exercise 83 (**SHSS 3.3, 8**) (a) Find the solution of the following problem by solving the constraints for x and y:

minimize $x^2 + (y-1)^2 + z^2$ subject to $x + y = \sqrt{2}$ and $x^2 + y^2 = 1$

(b) Note that there are three variables and two constraints (z does not appear in the constraints). Show that the condition on the matrix of the partial derivatives of the constraints are not satisfied, and that there are no Lagrange multipliers for which the Lagrangian is stationary at the solution point.

Exercise 84 (**SHSS 3.3, 9**)

$$Q(x_1, \ldots, x_n) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j, \quad S = \{(x_1, \ldots, x_n) \mid x_1^2 + \ldots + x_n^2 = 1\}$$

Assume that the coefficient matrix $\mathbf{A} = (a_{ij})$ of the quadratic form Q is symmetric and prove that Q attains maximum and minimum values over the set S which are equal to the largest and smallest eigenvalues of $\mathbf{A}$. (*Hint:* Consider first the case $n = 2$. Write $Q(\mathbf{x})$ as $Q(\mathbf{x} = \mathbf{x}'\mathbf{A}\mathbf{x}$. The first-order conditons give $\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$.)

Exercise 85 (**SHSS 3.3, 10**) Consider the problem

$\max \mathbf{x}, \mathbf{r} f(\mathbf{x}, \mathbf{r})$ subject to $\begin{cases} g_j \mathbf{x}, \mathbf{r}) = 0, \ j = 1, \ldots, m \\ r_i = b_{m+i}, \ i = 1, \ldots, k \end{cases}$

where f and $g_1, \ldots, g_m$ are fiven fucntions and $b_{m+1}, \ldots, b_{m+k}$ are fixed parameters. (We maximize f w.r.t. both $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{r} = (r_1, \ldots, r_k)$, but with $r_1, \ldots, r_k$ completely fixed.) Define $\tilde{\mathbf{b}} = (0, \ldots, 0, b_{m+1}, \ldots, b_{m+k})$ (there are m zeros). Prove that the partial derivative of the value function with respect to r_i is equal to the Lagrangian with respect to r_i computed at the solution for $i = m+1, \ldots, m+k$ by using the fact that the multiplier is equal to the partial derivative of the value function for right-hand perturbation and those first-order condition for the optimisation problem that refer to the variables r_i.

Exercise 86 (**SHSS 3.4, 1**)

(a) Find the four points that satisfy the first-order conditions for the problem $\max(\min) x^2 + y^2$ subject to $4x^2 + 2y^2 = 4$

(b) Compute $B_2(x, y)$ the determinant of the bordered Hessian of order 2 at the four points found in (a). What can you conclude?

(c) Can you give a geometric interpretation of the problem?

Exercise 87 (**SHSS 3.4, 2**) Compute the B_2 and B_3 the determinant of the bordered Hessian of order 2 and 3 for the problem

max(min)$x^2 + y^2 + z^2$ subject to $x + y + z = 1$

Show that the second-order conditions for a local minimum are satisfied.

Exercise 88 (**SHSS 3.4, 3**) Use the second order sufficient conditions to classify the candidates for optimality in the problem

local max(min)$x + y + z$ subject to $x^2 + y^2 + z^2 = 1$ and $x - y - z = 1$

Exercise 89 (**B chap 3, 54**)

Let $\alpha \in \mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid x_i > 0, \forall i = 1, \ldots, n\}$. The function f from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ is defined by

$$f(x) = \sum_{i=1}^{n} \alpha_i \ln(x_i)$$

where $\ln(x_i)$ is the standard logarithm function of x_i. Let $\beta \in \mathbb{R}^n_{++}$. We consider the following optimisation problem:

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ \sum_{i=1}^{n} \beta_i x_i = 1 \\ x \in \mathbb{R}^n_{++} \end{array} \right.$$

Compute the unique point satisfying the first order necessary condition. Are the second order necessary condition satisfied at this point?

Exercise 90 (**B chap 3, 55**) Let us consider the following optimisation problem:

$$(P) \left\{ \begin{array}{l} \text{Minimise } 5x^2 + 4xy + y^2 \\ 3x + 2y = 5 \end{array} \right.$$

1) First method: solve the problem by reducing it to a one dimensional optimisation problem.
2) Second method: write the first order necessary condition and find the solutions and the multipliers.

Exercise 91 (**B chap 4, 56**) Let $U = \{x \in \mathbb{R}^n \mid \forall i = 1, \ldots, n, x_i > -1\}$. The function f from U to $\mathbb{R}$ is defined by

$$f(x) = \sum_{i=1}^{n} \ln(x_i + 1)$$

where $\ln(x_i + 1)$ is the natural logarithm of $x_i + 1$. we consider the following optimisation problem :

$$(\mathcal{P}) \left\{ \begin{array}{l} \text{Maximise } f(x) \\ \sum_{i=1}^{n} x_i = 0 \\ x \in U \end{array} \right.$$

Show that there exists a unique point satisfying the first order necessary condition.

Exercise 92 (**B chap 4, 57**) Let (f^i) be n differentiable functions from $\mathbb{R}$ to $\mathbb{R}$. Let E be the linear subspace of $\mathbb{R}^n$ defined by:

$$E = \{x \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 0\}$$

Let $\bar{x}$ be a solution of the following optimisation problem:

$$\begin{cases} \text{Minimise } \sum_{i=1}^{n} f^i(x_i) \\ x \in E \end{cases}$$

Show that for all $i = 2, \ldots, n$, $(f^i)'(\bar{x}_i) = (f^1)'(\bar{x}_1)$.

Exercise 93 (**B chap 4, 60**) For the following problem, find the points satisfying the first order necessary conditions (minimum or maximum):

$$\begin{cases} \text{Optimise } \frac{1}{3}x - \frac{1}{4}y \\ x^2 - 2x + y^2 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } \ln x + \ln y + \ln z \\ x^2 + y^2 + z^2 = 3 \\ x > 0, y > 0, z > 0 \end{cases}$$

$$\begin{cases} \text{Optimise } 4x^2 + y^2 \\ xy + 2 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } xy \\ x^2 + 4y^2 - 8 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } 2y^4 - 2xy^2 + x^2 - 4y^2 + 2x + 2 \\ -x + y^2 - 2 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } x + 3y - z \\ x^2 + 3y^2 + z^2 - 2\sqrt{x^2 + 3y^2} - 4 = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } x^2 - \frac{3}{2}x + y^2 - \frac{3}{2}y \\ x^2 + y^2 - 2xy - x - y = 0 \end{cases}$$

$$\begin{cases} \text{Optimise } 4x + y + 2 \\ \ln x + 2\ln y = 0 \\ x > 0, y > 0 \end{cases}$$

$$\begin{cases} \text{Optimise } -\frac{2}{3}xy + \frac{5}{2}y + \frac{8}{3}x - \frac{11}{6} \\ x^2 + y - 1 = 0 \end{cases}$$

Exercise 94 (**B chap 4, 61**) For the above optimisation problems, write explicitly the associated Lagrangian mapping and check if the second order necessary condition is satisfied or not at the points satisfying the first order necessary condition.