Exercises on Optimisation in Euclidean Spaces II borrowed from Further Mathematics for Economic Analysis Knut Sydsaeter, Peter Hammond, Atle Seierstad and Arne Strom, with additional ones from the lectures notes of J.M. Bonnisseau*

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1 Optimization with inequality constraints

Optimization problem with inequality constraints. Necessary conditions for optimality: Kuhn-Tucker Theorem.

Exercise 1 (**SHSS 3.5, 1**) Solve the problem $\max 1 - x^2 - y^2$ subject to $x \geq 2$ and $y \geq 3$ by a direct argument, and then see wht the Kuhn-Tucker conditions have to say about the problem.

Exercise 2 (**SHSS 3.5, 2**) (a) Consider the nonlinear programming problem (where c is a positive constant)

$$\text{maximize } \ln(x+1) + \ln(y+1) \text{ subject to } \begin{cases} x + 2y \leq c \\ x + y \leq 2 \end{cases}$$

Write down the necessary Kuhn-Tucker conditions for a point (x, y) to be a solution of the problem.

(b) Solve the problem for $c = 5/2$.

(c) Let $V(c)$ denote the value function. Find the value of $V'(5/2)$.

Exercise 3 (**SHSS 3.5, 3**) Solve the following problem (assuming it has a solution)

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minimize $4\ln(x^2+2)+y^2$ subject to $x^2+y\geq 2$, $x\geq 1$
(*Hint:* Reformulate it as a standard Kuhn-Tucker maximization problem.)

Exercise 4 (**SHSS 3.5, 4**) Solve the problem $\max -(x-a)^2-(y-b)^2$ subjet to $x\leq 1$, $y\leq 2$, for all possible values of the constants a and b. (A good check of the results is to use a geometric interpretation of the problem.)

Exercise 5 (**SHSS 3.5, 5**) Consider the problem $\max f(x,y)=xy$ subject to $g(x,y)=(x+y-2)^2\leq 0$. Explain why the solution is $(x,y)=(1,1)$. Verify that the Kuhn-Tucker conditions are not satisfied for any λ, and that the Constraint Qualification does not hold at $(1,1)$.

Exercise 6 (**SHSS 3.5, 6**)
(a) Find the only possible solution to the nonlinear programming problem
maximize x^5-y^3 subject to $x\leq 1$, $x\leq y$
(b) Solve the problem by using iterated optimization: Find first the maximum value $f(x)$ in the problem of maximizing x^5-y^3 subject to $x\leq y$, where x is fixed and y varies. Then maximize $f(x)$ subject to $x\leq 1$.

Exercise 7 (**SHSS 3.6, 1**) Solve the problem $\max 1-(x-1)^2-e^{y^2}$ subject to $x^2+y^2\leq 1$.

Exercise 8 (**SHSS 3.6, 2**) Solve the problem $\max xy+x+y$ subject to $x^2+y^2\leq 2$, $x+y\leq 1$.

Exercise 9 (**B 6, 88**) Let $u=(u_1,u_2)$ be a non zero vector of $\mathbb{R}^2$ and let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x)=u\cdot x=u_1x_1+u_2x_2$. Using simple argument and the sign of u_1 and u_2, find the solution of the following optimisation problem:
1) $\min\{f(x)\mid x_1\geq 0, x_2\geq 0\}$;
2) $\min\{f(x)\mid x_1\geq 0, x_2\geq 0, x_1+x_2\leq 1\}$;
3) $\min\{f(x)\mid x_1\geq 0, x_2\geq 0, x_1+x_2=1\}$;
4) $\min\{f(x)\mid x_1\in[-1,2], x_2\in[0,1]\}$;
5) $\min\{f(x)\mid |x_1|+|x_2|\leq 1\}$;
6) $\min\{f(x)\mid \max\{|x_1|,|x_2|\}\leq 1\}$;

Exercise 10 (**B 6, 90**) Solve the following optimisation problem:

$$\begin{cases}\text{Minimise } x^2+y^2\\ 2x+y\leq -4\end{cases}$$

Let us consider the following optimisation problem:

$$\begin{cases}\text{Maximise } 3x_1x_2-x_2^3\\ x_1\geq 0,\ x_2\geq 0\\ x_1-2x_2=5\\ 2x_1+5x_2\geq 20\end{cases}$$

Draw the feasible set and show that the positivity constraints are non binding at the solution. Write the KKT conditions and find the solution.

Exercise 11 (**B 6, 91**) Solve the following optimisation problems:

$$\begin{cases} \text{Maximise } \ln(x_1x_2x_3) \\ x_1^2 + x_2^2 + x_3^2 \leq 4 \\ x_1 + x_2 + x_3 = 3 \\ x_1 > 0, x_2 > 0, x_3 > 0 \end{cases}$$

$$\begin{cases} \text{Minimise } x_1^2 + x_2^2 \\ x_1 + x_2 \geq 1 \\ x_1 \geq 0, x_2 \geq 0 \end{cases}$$

Exercise 12 (**B 6, 93**) Let $p \in \mathbb{R}^n_{++}$ and $w > 0$. We consider the following problem:

$$\begin{cases} \text{Maximise } f(x_1, x_2, x_3) = x_1x_2 \dots x_n \\ \sum_{i=1}^n p_i x_i \leq w \\ x \in \mathbb{R}^n_+ \end{cases}$$

1) Show that there exists an element $x \in \mathbb{R}^n_{++}$ such that $\sum_{i=1}^n p_i x_i \leq w$.
2) Show that there exists a least one solution.
3) Show that if $\bar{x}$ is a solution, then $\bar{x} \in \mathbb{R}^n_{++}$.
4) Show that if $\bar{x}$ is a solution, then $\sum_{i=1}^n p_i \bar{x}_i = w$.
5) Write the KKT conditions and find the unique solution of the problem.
6) If we denote by $\bar{x}(p, w)$ the optimal solution, compute $v(p, w) = f(\bar{x}(p, w))$ and compute its partial derivatives. Show the link between the partial derivative with respect to w and the KKT multipliers.

Exercise 13 (**B 6, 95**) We are looking for the closest point for the Euclidean norm to the point $(10, 10)$ in the closed unit ball.
1) Explain that this question is equivalent to solve the following problem:

$$\begin{cases} \text{Minimise } (x - 10)^2 + (y - 10)^2 \\ x^2 + y^2 \leq 1 \end{cases}$$

1) Show that this problem is a convex optimisation problem.
2) Show that this problem has a unique solution.
3) Find the solution of this problem.

Exercise 14 (**B 6, 96**) Let f from $\mathbb{R}^n$ to $\mathbb{R}$ defined by $f(x) = \exp(\|x\|^2) + a \cdot x$ where a is a given vector of $\mathbb{R}^n$ and $\| \cdot \|$ is the Euclidean norm.
1) Show that f is convex.
2) Find the solution of the following problem:

$$\begin{cases} \text{Minimise } f(x) \\ \|x\| \leq r \end{cases}$$

2 Comparative Statics

Comparative statics. Envelope result.

Exercise 15 (**SHSS 3.7, 1**)

(a) Solve the nonlinear programming problem (a and b are constants)

maximize $100 - e^{-x} - e^{-y} - e^{-z}$ subject to $x + y + z \leq a$, $x \leq b$

(b) Let $f^*(a, b)$ be the (optimal) value function. Comûte the partial derivatives of f^* with respect to a and b, and relate them with the Lagrange multipliers.

(c) Put $b = 0$, and show that $F^*(a) = f^*(a, 0)$ is concave in a.

Exercise 16 (**SHSS 3.7, 2**)

For $r = 0$ the problem

$$\max_{x \in [-1,1]} (x - r)^2$$

has two solutions, $x = \pm 1$. For $r \neq 0$, there is only one solution. Show that the value function $f^*(r)$ is not differentiable at $r = 0$.

Exercise 17 (**SHSS 3.7, 3**)

(a) Consider the problem

$$\max(\min) x^2 + y^2 \text{ subject to } r^2 \leq 2x^2 + 4y^2 \leq s^2$$

where $0 < r < s$. Solve the maximization problem and verify the Envelope Theorem in this case.

(c) Can you give a geometric interpretation of the problem and its solution?

3 Nonnegative constraints

Remarks on nonnegativity assumptions. First order conditions and nonnegative variables in Unconstrained optimization. First order conditions and nonnegative variables in Equality constrained optimization. First order conditions and nonnegative variables in Inequality constrained optimization. The general case: mixed constraints.

Exercise 18 (**SHSS 3.8, 1**) Solve the problem $\max 1 - x^2 - y^2$ subject to $x \geq 0$, $y \geq 0$, by (a) a direct argument and (b) using the Kuhn-Tucker conditions.

Exercise 19 (**SHSS 3.8, 2**) Solve the following nonlinear programming problems:

(a) $\max xy$ subject to $x + 2y \leq 2$, $x \geq 0$, $y \geq 0$

(b) $\max x^\alpha y^\beta$ subject to $x + 2y \leq 2$, $x > 0$, $y > 0$, where $\alpha > 0$ and $\beta > 0$, and $\alpha + \beta \leq 1$.

Exercise 20 (**SHSS 3.8, 3**)

(a) Solve the following problem for all values of constant c:

$\max f(x, y) = cx + y$ subject to $g(x, y) = x^2 + 3y^2 \leq 2 \leq 2$, $x \geq 0$, $y \geq 0$

(b) Let $f^*(c)$ denote ghe value function. Verify taht it is continuous. Check if the Envelope Theorem holds.

Exercise 21 (**SHSS 3.8, 4**)

(a) Write down the necessary Kuhn-Tucker conditions for the problem

$$\max \ln(1+x) + y \text{ subject to } px + y \leq m, x \geq 0, y \geq 0$$

(b) Find the solution whenever $p \in (0,1]$ and $m > 1$.

Exercise 22 (**SHSS 3.8, 5**) A model for studying the export of gas from Russia to the rest of Europe involves the following optimization problem:

$$\max[x+y-\frac{1}{2}(x+y)^2-\frac{1}{4}x-\frac{1}{3}y] \text{ subject to } x \leq 5, y \leq 3, -x+2y \leq 2, x \geq 0, y \geq 0$$

Sketch the admissible set S in the xy -plane, and show that the maximum cannot occur at an interior point of S. Solve the problem.

Exercise 23 (**SHSS 3.8, 6**) (Harder) With reference to problem

$$(1) \max f(\mathbf{x}) \text{ subject to } g_j(\mathbf{x}) \leq b_j, j = 1, \ldots, m, x_1 \geq 0, \ldots, x_n \geq 0$$

define $\widehat{\mathcal{L}}(\mathbf{x}, \lambda) = f(\mathbf{x}) - \sum_{j=1}^{m} \lambda_j (g_j(\mathbf{x}) - b_j)$. We say that $\widehat{\mathcal{L}}$ has a saddle point at $(\mathbf{x}^*, \lambda^*)$ with $\mathbf{x}^* \geq 0$, $\lambda^* \geq 0$, if

$$\widehat{\mathcal{L}}(\mathbf{x}, \lambda^*) \leq \widehat{\mathcal{L}}(\mathbf{x}^*, \lambda^*) \leq \widehat{\mathcal{L}}(\mathbf{x}^*, \lambda) \quad \text{for all } \mathbf{x} \geq \mathbf{0} \text{ and all } \lambda \geq \mathbf{0} \quad (*)$$

(a) Show that if $\widehat{\mathcal{L}}$ has a saddle point at $(\mathbf{x}^*, \lambda^*)$, then $\mathbf{x}^*$ solves problem (1). (*Hint:* Use the second inequality in $(*)$ to show that $g_j(\mathbf{x}^*) \leq b_j$ for $j = 1, \ldots, m$. Show next that $\sum_{j=1}^{m} \lambda_j (g_j(\mathbf{x}^*) - b_j) = 0$. Then use the first inequality in $(*)$ to finish the proof.)

(b) Suppose that there exist $\mathbf{x}^* \geq \mathbf{0}$ and $\lambda^* \geq \mathbf{0}$ satisfying both $g_j(\mathbf{x}^*) \leq b_j$ and $g_j(\mathbf{x}^*) = b_j$ whenever $\lambda_j^* > 0$ for $j = 1, \ldots, m$, as well as $\widehat{\mathcal{L}}(\mathbf{x}, \lambda^*) \leq \widehat{\mathcal{L}}(\mathbf{x}^*, \lambda^*)$ for all $\mathbf{x} \geq 0$. Show that $\widehat{\mathcal{L}}(\mathbf{x}, \lambda)$ has a saddle point at $(\mathbf{x}^*, \lambda^*)$ in this case.

4 Concavity and convexity in Optimization Theory

Exercise 24 (**B 5, 64**) Let a be a real number and f be a function from $\mathbb{R}^3$ to $\mathbb{R}$ defined by:

$$f(x, y, z) = 3x^2 + 2y^2 + z^2 + axy + 2yz + 2xz$$

For which values of a, the function f is convex?

Exercise 25 (**B 5, 67**) We consider the function f from $\mathbb{R}^2$ to $\mathbb{R}$ defined by:

$$f(x, y) = \sqrt{x^2 + y^2} - x$$

1) For all $(x, y) \in \mathbb{R}^2 \setminus \{0\}$, compute the gradient of the function f.
2) Show, without any computation, that the function f is convex.
3) Show that $f(x, y) \geq 0$ for all $(x, y) \in \mathbb{R}^2$.
4) Compute $f(0, 0)$ and give the minimum of f on $\mathbb{R}^2$.
5) Give all minima of f on $\mathbb{R}^2$.

We consider the sequence $(x^n, y^n) = (n, 1)$.

6) Show that the sequence $\nabla f(x^n, y^n)$ converges to a limit and compute this limit. Show that the sequence $f(x^n, y^n)$ converges to the minimal value of f on $\mathbb{R}^2$. Show that the sequence (x^n, y^n) does not converge to a minimum of f on $\mathbb{R}^2$.

Exercise 26 (**B 5, 70**) Let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by :

$$f(x, y) = x^2y^2 - 4x^2 - y^2$$

We are looking for the extremum of this function.

1) Compute the gradient vector and Hessian matrix of f at any point (x, y) of $\mathbb{R}^2$.
2) Find the points for which the gradient vanishes.
3) By studying the sign of the Hessian matrix at the points found above, find the local maximum and minimum of f and the critical points which are neither a local minimum nor a local maximum.
4) Show that the function f has neither a global maximum nor a global minimum on $\mathbb{R}^2$.

Exercise 27 (**B 5, 71**) Let f be the function from $\mathbb{R}^3$ to $\mathbb{R}$ defined by :

$$f(x, y, z) = x^2 + xy + y^2 + 2z^4 - z^2$$

We consider the following optimisation problem:

$$(\mathcal{P}) \begin{cases} \min & f(x, y, z) \\ s.c. & (x, y, z) \in \mathbb{R}^3 \end{cases}$$

1) Compute the gradient vector and the Hessian matrix of f at any point (x, y, z) of $\mathbb{R}^3$.
2) Find the points where the gradient vanishes.
3) By studying the sign of the Hessian matrix at the points found above, find the local maximum and minimum of f and the critical points which are neither a local minimum nor a local maximum.
4) By studying $f(-x, -y, -z)$, what can we say about the uniqueness of a solution?

Exercise 28 (**B 5, 72**) Let f be the function from $\mathbb{R}^2$ to $\mathbb{R}$ defined by $f(x, y) = x^4 + y^4 - (x - y)^2$.

1) Compute the points where the gradient of f vanishes and study the sufficient second order conditions at these points.
2) Show that f is coercice.
3) Show that f has a minimum on $\mathbb{R}^2$ and give this minimum.

Exercise 29 (**B 5, 73**) Let f be the function from $\mathbb{R}^n_{++}$ to $\mathbb{R}$ defined by $f(x) = \sum_{i=1}^n x_i \ln\left(\frac{1}{x_i}\right)$. Show that this function has a maximum on $\mathbb{R}^n_{++}$.