Exercises on Dynamic Programming borrowed from Further Mathematics for Economic Analysis Knut Sydsaeter, Peter Hammond, Atle Seierstad and Arne Strom, with additional ones from the lectures notes of J.M. Bonnisseau*

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October 19, 2022

1 Metric spaces, normed spaces

Exercise 1 (**B II, 1**)

Let φ be a concave, continuous, strictly increasing function from $\mathbb{R}_+$ to $\mathbb{R}_+$ satisfying $\varphi(0) = 0$.

1) Show that for all $(t, t') \in \mathbb{R}_+ \times \mathbb{R}_+$, such that $0 < t < t'$,

$$\frac{\varphi(t')}{t'} \geq \frac{\varphi(t') - \varphi(t)}{t' - t} \geq \frac{\varphi(t + t') - \varphi(t)}{t'}$$

and deduce that $\varphi(t + t') \leq \varphi(t) + \varphi(t')$.

2) Let (X, d) be a metric space. Show that $\varphi \circ d$ is a distance on X.

3) Let $\varphi(t) = \frac{t}{1+t}$ defined on $\mathbb{R}_+$. Show that φ satisfies the assumptions of the exercise.

4) Let (X, d) be a metric space. Show that δ from $X \times X$ to $\mathbb{R}_+$ defined by $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a distance on X.

Exercise 2 (**B II, 2**) Let $((X^i, d^i)_{i=1}^p$ be p metric spaces. Let N be a norm on $\mathbb{R}^p$ such that for all $(\xi, \zeta) \in \mathbb{R}^p_+ \times \mathbb{R}^p_+$, if $\xi \geq \zeta$, that is $\xi_i \geq \zeta_i$ for all

*EMJMD QEM, First semester, first year, 2022-2023

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$i = 1, \dots, p$, then $N(\xi) \geq N(\zeta)$. Show that the function δ_N defined by: for all $(x = (x^i), y = (y^i)) \in X \times X$,

$$\delta(x, y) = N\left((d^i(x^i, y^i))_{i=1}^p\right)$$

is a distance on X.

Exercise 3 (**B II, 7**) On $\mathbb{R}$, show that the distance d defined by $d(x, y) = |x - y|$ and δ defined by $\delta(x, y) = \frac{d(x,y)}{1+d(x,y)}$ are not equivalent but topologically equivalent.

Exercise 4 (**B II, 10**) Let (X, d) be a metric space and F be a nonempty subset of X. We define the function distance to F, d_F, by $d_F(x) = \inf\{d(x, y) \mid y \in F\}$. We prove that this function is Lipschitz continuous of rank 1.

Let $(x, y) \in X \times X$. We assume without any loss of generality that $d_F(x) \geq d_F(y)$.

1) Let $r > 0$. Show that there exists $\zeta \in F$ such that $d(y, \zeta) \leq d_F(y) + r$.
2) Show that $d_F(x) - d_F(y) \leq d(x, \zeta) - d(y, \zeta) + r$.
3) Deduce from the previous question that $d_F(x) - d_F(y) \leq d(x, y) + r$.
4) Conclude.

Exercise 5 (**B II, 12**) Let (E, N) be a normed linear space. We define the norm N^2 on $E \times E$ by $N^2(x, y) = N(x) + N(y)$. We define the norm $\tilde{N}$ on $\mathbb{R} \times E$ by $\tilde{N}(t, x) = |t| + N(x)$.

1) Show that the mapping Σ from $E \times E$ to E defined by $\Sigma(x, y) = x + y$ is continuous for the norms N^2 and N.
2) Show that the mapping Π from $\mathbb{R} \times E$ to E defined by $\Pi(t, x) = tx$ is continuous for the norms $\tilde{N}$ and N.

Exercise 6 (**B II, 26**) We consider the space $\mathcal{C}^1([0, 1])$ of $\mathcal{C}^1$ functions on $[0, 1]$ with the uniform norm $\| \cdot \|_\infty$. Let Φ be the derivation operator from $\mathcal{C}^1([0, 1])$ to $\mathcal{C}([0, 1])$ defined by $\Phi(f) = f'$.

1) Show that Φ is a linear mapping.
2) Show that Φ is not continuous if $\mathcal{C}([0, 1])$ is also equipped with the uniform norm $\| \cdot \|_\infty$. Hint: consider the sequence $f_\nu(t) = \frac{1}{\nu+1} \sin(2\pi\nu t)$.

Exercise 7 (**B II, 29**) The aim of this exercise is to prove the space $\mathcal{C}([0, 1], \mathbb{R})$ with the norm $\|f\|_1 = \int_0^1 |f(t)|dt$ is not a Banach space.

Let us consider the sequence (f_ν) defined by $f_\nu(t) = 0$ for $t \in [0, \frac{1}{2} - \frac{1}{3(\nu+1)}]$, $f_\nu(t) = \frac{3(\nu+1)}{2} t + \frac{1}{2} - \frac{3(\nu+1)}{4}$ for $t \in [\frac{1}{2} - \frac{1}{3(\nu+1)}, \frac{1}{2} + \frac{1}{3(\nu+1)}]$ and $f_\nu(t) = 1$ for $t \in [\frac{1}{2} + \frac{1}{3(\nu+1)}, 1]$.

1) Show that this sequence satisfies the Cauchy Criterion for the norm $\| \cdot \|_1$. For $\nu < \mu$, note that

$$\|f_\nu - f_\mu\|_1 = \int_{\frac{1}{2} - \frac{1}{3(\nu+1)}}^{\frac{1}{2} + \frac{1}{3(\nu+1)}} |f_\nu(t) - f_\mu(t)|dt$$

Assume that this sequence has a limit $\bar{f}$ in $\mathcal{C}([0,1])$.
2) Show that for all ν,

$$\|\bar{f}-f_\nu\|_1 = \int_0^{\frac{1}{2}-\frac{1}{3(\nu+1)}} |\bar{f}(t)|dt + \int_{\frac{1}{2}-\frac{1}{3(\nu+1)}}^{\frac{1}{2}+\frac{1}{3(\nu+1)}} |\bar{f}(t)-f_\nu(t)|dt + \int_{\frac{1}{2}+\frac{1}{3(\nu+1)}}^{1} |\bar{f}(t)-1|dt$$

3) Deduce from the previous question that for all $r \in]0,1/2[$, $\int_0^{\frac{1}{2}-r} |\bar{f}(t)|dt$ and $\int_{\frac{1}{2}+r}^{1} |\bar{f}(t)-1|dt$ are equal to 0.
4) Deduce from the previous question that $\bar{f}(t)=0$ on $[0,1/2[$ and $\bar{f}(t)=1$ on $]1/2,1]$.
4) Show that we get a contradiction.

2 Sequences

Exercise 8 (**SHSS A3, 1**) Prove that the sequence (x_k) defined by $x_1 = 1$, $x_{k+1} = 2\sqrt{x_k}$ for $k \geq 1$ converges, and find its limit. (*Hint:* Prove first by induction that $x_k < 4$ for all k.)

Exercise 9 (**SHSS A3, 2**) Prove that the sequence (x_k) defined by $x_1 = \sqrt{2}$, $x_{k+1} = \sqrt{x_k + 2}$ for $k \geq 1$ satisfies $|x_{k+1}-2| < \frac{1}{2}|x_k - 2|$, and use this to prove that $x_k \to 2$ as $k \to \infty$. (*Hint:* $x_{k+1} - 2 = (x_{k+1}^2 - 4)\ (x_{k+1}+2)$.)

Exercise 10 (**SHSS A3, 3**) Let S be a nonempty set of real numbers bounded above, and $b^* = \sup S$. Show that there exists a sequence (x_n), $x_n \in S$, such that $x_n \to b^*$.

Exercise 11 (**SHSS A3, 6**) Let (x_k) be a qequence such that $|x_{k+1} - x_k| < 1/2^k$ for $k \geq 1$. Prove that (x_k) is a Cauchy sequence.

Exercise 12 (**SHSS A3, 7**) Prove that if (x_k) converges to both x and y, then $x = y$.

Exercise 13 (**SHSS A3, 9**) Prove that every suquence of real numbers has a monotone subsequence.

Exercise 14 (**SHSS 13.2, 1**) Find the limits of the following sequences in $\mathbb{R}^2$ if the limits exist.
(a) $\mathbf{x}_k = (1/k, 1+1/k)$;
(b) $\mathbf{x}_k = (k, 1+3/k)$;
(c) $\mathbf{x}_k = ((k+2)/3k, (-1)^k/2k)$;
(d) $\mathbf{x}_k = (1+1/k, (1+1/k)^k)$;

Exercise 15 (**SHSS 13.2, 2**) Prove that a squence in $\mathbb{R}^n$ cannot converge to more than one point.

Exercise 16 (**SHSS 13.2, 3**) Prove that every convergent sequence in $\mathbb{R}^n$ is a Cauchy sequence.

Exercise 17 (**SHSS 13.2, 4**) Prove that if every sequence of point is a set S in $\mathbb{R}^n$ contains a convergent subsequence, then S is bounded. (*Hint:* If S is unbounded, then for each natural number k, there exists and $\mathbf{x}_k$ in S with $\|\mathbf{x}_k\| > k$.)

Exercise 18 (**SHSS 13.2, 5**) Let $(\mathbf{x}_k)$ be a sequence of points in ca compact subset X of $\mathbb{R}^n$. Prove that if every convergent subsequence of $(\mathbf{x}_k)$ has the same limit $\mathbf{x}^0$, then $(\mathbf{x}_k)$ converges to $\mathbf{x}^0$.

Exercise 19 (**B II, 3**) Let X be a set and d be the distance defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Show that a sequence is convergent for this distance if and only if it is constant after a given rank, that is, for a sequence (u_ν), there exists $\underline{\nu} \in \mathbb{N}$ such that for all $\nu \geq \underline{\nu}$, $u_\nu = u_{\underline{\nu}}$.

Exercise 20 (**B II, 4**) Let X be a set and d be the distance defined by $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$. Show that (X, d) is a complete metric space.

Exercise 21 (**B II, 6**) Let us now consider the following norm N on ℓ^∞:

$$N((u_\nu)) = \sum_{\nu=0}^{\infty} \frac{1}{2^\nu} |u_\nu|$$

The purpose of the exercise is to show that ℓ^∞ is not complete for the norm N. Let us consider the sequence $(u^i = (u^i_\nu)_{\nu\in\mathbb{N}})_{i\in\mathbb{N}}$ of ℓ^∞ defined by: for all $i \in \mathbb{N}$,

$$u^i_\nu = \nu \text{ if } \nu \leq i, i \text{ otherwise}$$

1) Show that this sequence satisfies the Cauchy criterion for the norm N.
2) Show that for all $v \in \ell^\infty$, the real sequence $N(u^i - v)$ is bounded below by a non negative number for all i large enough and conclude that the sequence (u^i) is not convergent for the norm N.

Exercise 22 (**B II, 8**) Let X be a set and d and δ two topologically equivalent distances on X. Show that a sequence (u_ν) of X is convergent for d if and only if it is convergent for δ.

3 Fixed Points

Exercise 23 (**SHSS 14.4, 1**) Consider the function f defined for all $x \in]0, 1[$ by

$$f(x) = \frac{1}{2}(x + 1)$$

Prove that f maps $]0, 1[$ into itself, but f has no fixed point. Why does Brouwer's theorem not apply?

Exercise 24 (**SHSS 14.4, 2**) Consider the continuous transfomration $\mathbf{T} = (x,y) \to (-y,x)$ from the xy-plane into itself, consisting of a $90°$ rotation around the origin. Define the set

$$E = \{(x,y) \mid x^2 + y^2 = 1\}, \quad B = \{(x,y) \mid x^2 + y^2 \leq 1\}$$

Are these sets compact? $\mathbf{T}$ induces continuous maps $\mathbf{T}_E : E \to E$ and $\mathbf{T}_B : B \to B$. Does either transformation have a fixed point? Explain the results in the light of Brouwer's theorem.

Exercise 25 (**SHSS 14.4, 3**) Let $\mathbf{A} = (a_{i,j})$ be an $n \times n$ matrix whose elements all satisfy $a_{ij} \geq 0$. Assume that all comun sums are 1, so that $\sum_{i=1}^{n} a_{ij} = 1$, $(j = 1, \dots, n)$. Prove that if $\mathbf{x} \in \Delta^{n-1}$, then $\mathbf{Ax} \in \Delta^{n-1}$, where Δ^{n-1} is the unit simplex defined by $\sum_{i=1}^{n} \delta_i = 1$, $\delta_i \geq 0$ for all $i = 1, \dots, n$. Hence $\mathbf{x} \to \mathbf{Ax}$ is a (linear) transofrmation of Δ^{n_1} into itself. What does Brouwer's theorem say in this case?

4 Finite Horizon Dynamic Programming

Exercise 26 (**SHSS 12.1, 1**) (a) Solve the problem

$$\max \sum_{t=0}^{2} (1 - (x_t^2 + 2u_t^2)), x_{t+1} = x_t - u_t, t = 0, 1$$

where $x_0 = 5$ and $u_t \in \mathbb{R}$. (Compute $J_s(x)$ and $u_s^*(x)$ for $s = 2, 1, 0$.)

(b) Use the difference equation in $x_{t+1} = x_t - u_t$ to compute x_1 and x_2 in terms of u_0 and u_1 (with $x_0 = 5$), and find the objective function as a function S of u_0, u_1, and u_2. Next, maximize this function and find the solution of the initial problem.

Exercise 27 (**SHSS 12.1, 2**) Consider the problem

$$\max_{u_t \in [0,1]} \sum_{t=0}^{T} \left(\frac{1}{1+r} \right)^t \sqrt{u_t x_t}, x_{t+1} = \rho(1 - u_t)x_t, t = 0, 1, \dots, T-1, x_0 > 0$$

where r is the rate of discount. Compute $J_s(x)$ and $u_s^*(x)$ for $s = T, T-1, T-2$.

Exercise 28 (**SHSS 12.1, 4**) Consider the problem

$$\max_{u_t \in [0,1]} \sum_{t=0}^{T} (3 - u_t)x_t^2, x_{t+1} = u_t x_t, t = 0, 1, \dots, T-1, x_0 \text{is given}$$

(a) Compute the value functions $J_T(x)$, $J_{T-1}(x)$, $J_{T-2}(x)$, and the corresponding control function $u_T^*(x)$, $u_{T-1}^*(x)$ and $u_{T-2}^*(x)$.

(b) Find an expression for $J_{T-n}(x)$ for $n = 0, 1, \dots, T$, and the corresponding optimal controls.

Exercise 29 (**SHSS 12.1, 5**) Solve the problem

$$\max_{u_t \in [0,1]} \sum_{t=0}^{T_1} \left(-\frac{2}{3} u_t \right) + \ln x_T, x_{t+1} = x_t(1+u_t), t = 0, 1, \ldots, T-1, x_0 > 0 \text{ given}$$

Exercise 30 (**SHSS 12.1, 7**) (a) Consider the problem

$$\max_{u_t \in \mathbb{R}} \sum_{t=0}^{T_1} \left(-e^{-\gamma u_t} \right) - \alpha e^{-\gamma x_T}, x_{t+1} = 2x_t - u_t, t = 0, 1, \ldots, T-1, x_0 \text{ given}$$

where α and γ are positive constants. Compute $J_T(x)$, $J_{T-1}(x)$, and $J_{T-2}(x)$.

(b) Prove that $J_t(x)$ written in the form $J_t(x) = -\alpha_t e^{-\gamma x}$, and find a difference equation for α_t.

Exercise 31 (**B II, 35**) Compute the optimal allocation in the following problem when $u(c) = \sqrt{c}$ and $u(c) = \ln(c)$:

$$\begin{cases} \text{Maximise } u(c_1) + \beta u(c_2) \\ c_2 = (1+r)(w_0 - c_1) \\ c_1 \geq 0, \, c_2 \geq 0 \end{cases}$$

Exercise 32 (**B II, 36**) Compute the optimal allocation in the T period problem

$$\begin{cases} \text{Maximise } \sum_{\tau=t+1}^{T-1} \beta^\tau u(c_\tau) \\ (1+r)^{T-t-1} c_{t+1} + (1+r)^{T-t-2} c_{t+2} + \ldots + (1+r) c_{T-1} \leq (1+r)^{T-t-1} w_{t+1} \\ c_\tau \geq 0, \text{ for } t = t+1, \ldots, T-1 \end{cases}$$

when $u(c) = \sqrt{c}$ and $u(c) = \ln(c)$. Compute the derivative of the value function with respect to w_0 and check that it is equal to λ_0.

Exercise 33 (**B II, 37**) Write the complete first order necessary conditions of the problem

$$\begin{cases} \text{Maximise } \sum_{t=0}^{T-1} \beta^t f_t(a_t, s_t) + \beta^T f_T(s_T) \\ s_{t+1} = g_t(a_t, s_t), \, t = 0, \ldots, T-1, \\ (a_t, s_t) \in A_t \; t = 0, \ldots, T-1 \\ s_T \in A_T \end{cases}$$

when the sets A_t are defined as follows:

$$A_t = \{(a, s) \in \mathbb{R}^2 \mid s \geq 0, a \in [\underline{\alpha}_t(s), \overline{\alpha}_t(s)]\}$$

where $\underline{\alpha}_t$ and $\overline{\alpha}_t$ are continuously differentiable functions from $\mathbb{R}_+$ to $\mathbb{R}$ satisfying $\underline{\alpha}_t(s) \leq \overline{\alpha}_t(s)$ for all $s \in \mathbb{R}_+$.

Exercise 34 (**B II, 38**) Apply the dynamical programming algorithm to the intertemporal allocation of wealth with $\beta \in]0, 1[$, the interest rate r equals to 0 and the utility function is $\ln c$.

5 Stationary Dynamic Programming

Exercise 35 (**SHSS 12.3, 1**) Consider the problem

$$\max_{u_t \in \mathbb{R}} \sum_{t=0}^{\infty} \beta^t \left(-e^{-u_t} - \frac{1}{2} e^{-x_t} \right), x_{t+1} = 2x_t - u_t, t = 0, 1, \ldots, x_0 \text{ given}$$

where $\beta \in]0,1[$. Find a constant $\alpha > 0$ such that $J(x) = -\alpha e^{-x}$ solves the Bellman equation, and show that α is unique.

Exercise 36 (**SHSS 12.3, 2**) (a) Consider the following problem with $\beta \in]0,1[$:

$$\max_{u_t \in \mathbb{R}} \sum_{t=0}^{\infty} \beta^t \left(-\frac{2}{3} x_t^2 - u_t^2 \right), x_{t+1} = x_t + u_t, t = 0, 1, \ldots, x_0 \text{ given}$$

Suppose that $J(x) = -\alpha x^2$ solves the Bellman equation. Find a quadratic equation for α. Then find the assoicated value of u^*.

(b) By looking at the objective function, show that, fiven any starting value x_0, it is reasonable to ignore any policy that fails to satisfy both $|x_t| \leq |x_{t-1}|$ and $|u_t| \leq |x_{t-1}|$ for $t = 1, 2, \ldots$ Is the instantaneous objective function $-\frac{2}{3} x_t^2 - u_t^2$ bounded on the feasible reasonable paths?

Exercise 37 *(Ramsay growth model)* (**B II, 39**)
1) Write the first order necessary conditions for the Ramsay growth model at an interior solution (c_t^*, k_t^*), that is $c_t^* \in]0, k_t^*[$ for all t.
2) Derive from these conditions the Euler equation:

$$\beta u'(c_{t+1}^*) f'(k_{t+1}^*) = u'(c_t^*)$$

3) Show that an optimal solution is always an interior solution as a consequence of the Inada condition $u'(0) = +\infty$.

Exercise 38 *Ramsay growth model:* (**B II, 41**) Check that the following assumptions are satisfied in the Ramsey growth model:

a) f_t and g_t are concave functions and increasing with respect to s;

b) A_t is convex and if $(a_t, s_t) \in A_t$ and $s_t' \geq s_t$, then $(a_t, s_t') \in A_t$.

c) At a solution (a_0^*, s_0^*) of the problem with the initial state s_0^*. We assume that the functions f_0 and g_0 are differentiable on a neighbourhood of (a_0^*, s_0^*) and $\frac{\partial g_0}{\partial a}(a_0^*, s_0^*) \neq 0$.

Show that at an interior solution:

$$V'(k_0^*) = u'(c_0^*) f'(k_0^*)$$

Exercise 39 *Steady state* (**B II, 42**) We consider the Bellman equation and we denote by $\alpha(s_0)$ the optimal solution given s_0. A fixed point s^* of $g(\alpha(\cdot), \cdot)$ is called a Steady state. Show that if $s_0 = s^*$, then the optimal solution of the problem is the constant sequence$(\alpha(s^*), s^*)_{t\in\mathbb{N}}$.

We consider the set $\mathcal{B}(I)$ of the bounded functions from I to $\mathbb{R}$ with the uniform norm. We recall that this is a complete metric space. We define an operator T from $\mathcal{B}(I)$ to itself as follows:

$$Th(s) = \sup\{f(a,s) + \beta h(g(a,s)) \mid (a,s) \in A\}$$

Remark 1 Th is well defined since h bounded, $g(a,s) \in I$ by assumption, the set of a such that $(a,s) \in A$ is compact and f is continuous and upper-bounded.

Exercise 40 (**B II, 43**) Show that if $I = \mathbb{R}_+$ and A is defined by

$$A = \{(a,s) \in \mathbb{R}^2 \mid s \geq 0, a \in [\underline{\alpha}(s), \overline{\alpha}(s)]\}$$

where $\underline{\alpha}$ and $\overline{\alpha}$ are continuous functions from $\mathbb{R}_+$ to $\mathbb{R}$ satisfying $\underline{\alpha}(s) \leq \overline{\alpha}(s)$ for all $s \in \mathbb{R}_+$, then Th defined by

$$Th(s) = \sup\{f(a,s) + \beta h(g(a,s)) \mid (a,s) \in A\}$$

is continuous if h is a continuous function of $\mathcal{B}(I)$, the set of bounded functions from I to $\mathbb{R}$ with the uniform norm.

Exercise 41 (**B II, 44**) We consider the stationary Ramsay growth model. We consider the Bellman equation and for all $k \geq 0$, we denote by $\alpha(k)$ the optimal solution. Let $\varphi(k) = f(k) - \alpha(k)$.
1) Show that α and φ are continuous.
2) Show that if $k > 0$, then $\varphi(k) > 0$ and $\alpha(k) > 0$.
3) Show that φ is increasing.
4) Show that $f - \varphi$ is increasing.

Exercise 42 (**B II, 45**) We consider the stationary Ramsay growth model.
1) Show that if we choose an interval $I = [0, \hat{k}]$ with $\hat{k} \geq \bar{k}$, where $\bar{k}$ is the fixed point of f, then Assumption B is satisfied.
2) Show that the optimal capital stock (k_t^*) is monotonic;
3) Show that if $f'(0) \leq \frac{1}{\beta}$, then the optimal capital stock (k_t^*) converges to 0;
3) Show that if $f'(0) > \frac{1}{\beta}$, then the optimal capital stock (k_t^*) converges to a steady state K which is strictly positive and satisfies $f'(K) = \frac{1}{\beta}$.