Logic and Sets
Final exam 2023 (2h)

Name: QEM/MMEF

Exercise 1 (5pts)

Indicate for each of the following assertions if they are true (T) or false (F).

1. The strong induction principle implies the weak induction principle but not the converse. F
2. $B \cup (\cap_{i\in I} A_i) = \cap_{i\in I}(B \cup A_i)$ T
3. $A \times B = \emptyset$ is equivalent to $(A = \emptyset \vee B = \emptyset)$. T
4. $f^{-1}(B \cup C) = f^{-1}(B) \cup f^{-1}(C)$ T
5. $f(A \cap B) = f(A) \cap f(B)$ F
6. The function $f :]0, \infty[\to \mathbb{R}$ defined by $f(x) = \log x$ is a bijection. T
7. The function $f : \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$ is a surjection. T
8. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^4 + 4x^2 + 2$ is an injection. F
9. Let $f : A \to B$, $g : B \to C$. If $g \circ f$ is bijective, then f, g are bijective. F
10. Let $f : X \to Y$. Then $f^{-1}(A)$ exists for any $A \subseteq X$ if and only if f is a bijection. F
11. The relation $a\mathcal{R}b$ (a divides b, with $a, b \in \mathbb{Z}$) is reflexive, complete and transitive. F
12. Countable unions of countable sets are countable. T
13. $\mathbb{Z}$ is equipotent with $\mathbb{N}$. T
14. $\mathbb{R}$ is equipotent with $\mathbb{Q}$. F
15. $\mathbb{N}^k$ is countable for all $k \in \mathbb{N}$. T
16. The set of irrational numbers is countable. F
17. $2^{\mathbb{N}}$ is uncountable. T
18. $\mathbb{R}^2$ is equipotent with $\mathbb{R}$. T
19. It is possible that an infimum exists but no minimal element exists. T
20. It is possible that a minimal element exists but no infimum exists. T

Exercise 2 (4 pts)

Consider the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix}$$

1. (3pts) Find its eigenvalues, together with their algebraic and geometric multiplicities, amd find the eigenvectors.
2. (1pt) Is the matrix diagonalizable? Justify your answer.

Exercise 3 (3pts)

Let E be a set and $\mathcal{R}$ the relation on the power set $\mathcal{P}(E)$ defined by

$$A\mathcal{R}B \text{ if } \{A = B \text{ or } A = B^c\}$$

for any $A, B \subseteq E$, where B^c is the complement of B in E.

1. Show that $\mathcal{R}$ is an equivalence relation.
2. Take $E = \{1, 2, 3\}$. Write $\mathcal{P}(E)$ and the equivalence classes of $\mathcal{R}$ on $\mathcal{P}(E)$.

Exercise 4 (3 pts)

Let $f : X \to Y$. Show that $f(f^{-1}(B)) \subseteq B, \forall B \subseteq Y$, and show by an example that equality may not hold. Under which additional condition on f do we have equality?

Exercise 5 (4pts)

Determine, if they exist, the set of lower bounds, upper bounds, the minimal and maximal elements, the infimum and supremum of the following subsets of $\mathbb{R}$:

1. $\left\{\frac{1}{1+x^2} : x \in \mathbb{R}\right\}$
2. $\{e^n : n \in \mathbb{N}\}$

Same question for the following subsets of $\mathbb{Q}$:

1. $\left\{\frac{n}{n^2+1} : n \in \mathbb{N}\right\}$
2. $\{x \in \mathbb{Q} : x > \sqrt{2}\}$

Note: $0 \notin \mathbb{N}$.

Question (1pt)

Explain what is the continuum hypothesis.

Ex 2

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & 2 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 2 & 0 & -1 \end{bmatrix} \rightarrow \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 & 0 \\ 3 & 2-\lambda & 0 & 0 \\ 0 & -1 & 1-\lambda & 1 \\ 0 & 2 & 0 & -1-\lambda \end{bmatrix} = (2-\lambda)^2(1-\lambda)(-1-\lambda)$$

The roots of the characteristic polynomial are

1. $\lambda_1 = 2$ with algebraic multiplicity 2

$\lambda_2 = 1$, ______________________ 1

$\lambda_2 = -1$ ______________________ 1.

- geometric multiplicity of λ_1, eigenvector: solve $A - 2I = 0$

$$\begin{cases} 3u_1 = 0 & \rightarrow \boxed{u_1 = 0} \\ -u_2 - u_3 + u_4 = 0 & \rightarrow u_3 = -\frac{1}{2}u_4 \\ 2u_2 - 3u_4 = 0 & \rightarrow u_2 = \frac{3}{2}u_4 \end{cases}$$

Hence the set of solutions is $\{(0, \frac{3}{2}\alpha, -\frac{1}{2}\alpha, \alpha), \alpha \in \mathbb{R}\}$, which has dimension 1 (= geom. multiplicity). An eigenvector is $(0, \frac{3}{2}, -\frac{1}{2}, 1)$.

- geometric multiplicity of λ_2, eigenvector:

$$\begin{cases} u_1 = 0 \\ 3u_1 + u_2 = 0 \\ -u_2 + u_4 = 0 \\ 2u_2 - 2u_4 = 0 \end{cases} \rightarrow u_1 = u_2 = u_4 = 0$$

The set of solutions is $\{(0, 0, \alpha, 0), \alpha \in \mathbb{R}\}$ which has dimension 1 (= geometric multiplicity). An eigenvector is $(0, 0, 1, 0)$.

- geometric multiplicity of λ_3, eigenvector:

$$\begin{cases} 3u_1 = 0 \\ 3u_1 + 3u_2 = 0 \\ -u_2 + 2u_3 + u_4 = 0 \\ 2u_2 = 0 \end{cases} \rightarrow \begin{cases} u_1 = u_2 = 0 \\ u_4 = -2u_3 \end{cases}$$

The set of solutions is $\{(0, 0, \alpha, -2\alpha), \alpha \in \mathbb{R}\}$, of dimension 1 (= geom. mult.). Eigenvector $(0\ 0\ 1, -2)$.

2. Not diagonalizable as sum of geom. multiplicities is $3 < 4$.

Ex 3

1) $\mathcal{R}$ reflexive: $A \mathcal{R} A$ since $A = A$.

$\mathcal{R}$ symmetric: $A \mathcal{R} B \Rightarrow A = B$ or $A = B^c$. In the latter case, $A^c = B$ and so we get $B \mathcal{R} A$.

-) $\mathcal{R}$ transitive: suppose $A \mathcal{R} B$ and $B \mathcal{R} C$. The different cases are:

$$\left.\begin{array}{lll} A = B,\ B = C & \Rightarrow & A = C \\ A = B,\ B = C^c & \Rightarrow & A = C^c \\ A = B^c,\ B = C & \Rightarrow & A = C^c \\ A = B^c,\ B = C^c & \Rightarrow & A = C \end{array}\right\} \Rightarrow A \mathcal{R} C.$$

2) $\mathcal{P}(E) = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\} \}$

equivalence classes: $\{\emptyset, E\}$, $\{\{1\},\{2,3\}\}$, $\{\{2\}, \{1,3\}\}$, $\{\{3\}, \{1,2\}\}$.

Ex 4

- Take $y \in f(f^{-1}(B))$. Then $\exists x \in f^{-1}(B)$, $f(x) = y$. $x \in f^{-1}(B) \Rightarrow$ $f(x) \in B$. Therefore $y \in B$.
- Take $f(x) = a$ $\forall x$. Then with $B = Y$, $f(f^{-1}(Y)) = f(X) = \{a\} \neq Y$.
- surjection.

Ex. 5

	min. el.	max el.	lower bd	upper bd	inf	sup
$\{\frac{1}{1+x^2}, x \in \mathbb{R}\}$	$\emptyset$	1	$]-\infty, 0]$	$[1, +\infty[$	0	1
$\{e^n, n \in \mathbb{N}\}$	e	$\emptyset$	$]-\infty, e]$	$\emptyset$	e	$\emptyset$
$\{\frac{n}{n^2+1}, n \in \mathbb{N}\} \cap \mathbb{Q}$	$\emptyset$	$\frac{1}{2}$	$]-\infty, 0] \cap \mathbb{Q}$	$[\frac{1}{2}, +\infty[\cap \mathbb{Q}$	0	$\frac{1}{2}$
$\{x \in \mathbb{Q}, x > \sqrt{2}\}$	$\emptyset$	$\emptyset$	$]-\infty, \sqrt{2}] \cap \mathbb{Q}$	$\emptyset$	$\emptyset$	$\emptyset$