**Introductory Finance - Tutorial corrections** - Isabelle Nagot
M1 MAEF - DU MMEF - QEM1

**Exercise 1. a.** See lecture notes, the continuous rate is obtained as $\ln 1.1 = 0.0953$.

$(1+\frac{r_m}{12})^{12} = 1.1$ then $\frac{r_m}{12} = 1.1^{\frac{1}{12}} - 1 = 0.79741\%$. Annualised: $r_m = 9.569\%$.

**b.** $x \mapsto (1+\frac{x}{m})^m - (1+x)$ is strictly increasing on $\mathbb{R}^{*+}$ and worth 1 at 0.

**c.** and **d.** See lecture notes.

**e.** Over $[\frac{k-1}{m}, \frac{k}{m}[$, investor A holds $M(1+\frac{r_m}{m})^{k-1} = Me^{r\frac{k-1}{m}}$ while investor B holds $Me^{r\frac{k}{m}}$ at time $\frac{k}{m}$ (which is more). The maximal difference is then the maximum over $k$ of: $M[e^{r\frac{k}{m}} - e^{r\frac{k-1}{m}}]$ (left limit at $\frac{k}{m}$), i.e. (for $k = m-1$): $Me^r(1-e^{-\frac{r}{m}})$.

**Exercise 2.** Present Value: $PV = \dfrac{100}{1.04} + \dfrac{200}{1.042^2} + \dfrac{300}{1.045^3} = 543.244919$ M€ $= 543,244,919.17$€.

Cf Excel file:

| | | | |
|---|---|---|---|
| *Year* | 1 | 2 | 3 |
| *Rate* (%) | 4 | 4.2 | 4.5 |
| *Cash Flow* ($M^{on}$) | 100 | 200 | 300 |
| *Discount Factor* | 0.96154 | 0.92101 | 0.87630 |

**Exercise 3.**

**a.** The present value at 0 of an annuity with maturity date $T$ is: $PV = \sum_{t=1}^{T} \dfrac{A}{(1+r)^t}$.

Using: $\sum_{t=1}^{T} x^t = x\dfrac{1-x^T}{1-x}$ we get: $\sum_{t=1}^{T} \dfrac{1}{(1+r)^t} = \dfrac{1}{1+r}\dfrac{1-\dfrac{1}{(1+r)^T}}{1-\dfrac{1}{1+r}} = \dfrac{1}{r}\left[1-\dfrac{1}{(1+r)^T}\right]$.

Then $PV = \dfrac{A}{r}\left[1-\dfrac{1}{(1+r)^T}\right]$.

With $T = 10$, $r = 0.05$ and $A = 100$ k€, we get: $PV = 772.173$ k€ $= 772,173$€.

**b.** We take the limit when $T \to +\infty$ in $\sum_{t=1}^{T} \dfrac{1}{(1+r)^t} = \dfrac{1}{r}\left[1-\dfrac{1}{(1+r)^T}\right]$, assuming $r > 0$.

We obtain: $\boxed{\sum_{t=1}^{+\infty} \dfrac{1}{(1+r)^t} = \dfrac{1}{r}}$, therefore $PV = \dfrac{A}{r}$.

With $r = 0.05$ and $A = 100$€, we get: $PV = 2000$€.

**Exercise 4.**

**a.** $\dfrac{P}{N} = c\sum_{t=1}^{T} \dfrac{1}{(1+\rho)^t} + \dfrac{1}{(1+\rho)^T} = f(\rho)$, can be written $f(\rho) \overset{(*)}{=} \dfrac{c}{\rho}\left[1-\dfrac{1}{(1+\rho)^T}\right] + \dfrac{1}{(1+\rho)^T}$ as well.

**b.** $f(c) = 1$ (we saw $P = N \iff \rho = c$). $f$ decreasing then $\rho < c \iff P > N$.

**c.** $\dfrac{C}{P} = \dfrac{5.5\%}{1.2665} = 4.343\%$.

Note: YTM=3.231658, duration= 10.880707, convexity=144.014 (see later).

**d.1** $P > N \Rightarrow \frac{C}{P} < \frac{C}{N} = c$.

**d.2** 1st investment: return $= \rho$. 2nd investment: return $= \frac{C}{P}$. 2nd is better, then $\dfrac{C}{P} > \rho$.
Note that the 2nd investment is equivalent to investing $P$ every year at rate $\frac{C}{P}$.

By direct computation: from (*), $\dfrac{\rho P}{cN} = 1 + \dfrac{1}{(1+\rho)^T}\left(\dfrac{\rho}{c} - 1\right) < 1$ from $\rho < c$. Then $\rho P < C$.

**e.** It is easy to reverse the argument when the bond trades at a discount.

At par: YTM$= c$ and $\frac{C}{P} = \frac{C}{N} = c$.

**Exercise 5.** cf Excel file **a.** $\frac{P}{N}$ and $\rho$ are linked through $\frac{P}{N} = \frac{1}{(1+\rho)^T}$.
Here $T = 10$ at issue date, then $5 + \frac{10{,}2}{12}$ on 09-oct-23 and $5 + \frac{10{,}02}{12}$ on 11-oct-24
(gives the price actually observed on the market for these dates).

**b.** $\dfrac{P}{N}$ and $\rho$ are linked through $\dfrac{P}{N} = \dfrac{c}{\rho}\left[1 - \dfrac{1}{(1+\rho)^T}\right] + \dfrac{1}{(1+\rho)^T}$ (*).
From an observed $P$, $\rho$ can be computed through a numerical method like the NewtonRaphson method.
If $\rho$ is given, we check the relationship.

**Exercise 6.**
**a.** $P = 95.6705\%$, $D = 4,702$ years for $c = 3\%$ and $D = 4,62$ years for $c = 4\%$ (cf Excel file).

To compute the price, the formula (*) in Exercise 5. can also be used).

Intuitively, we can imagine that the duration diminishes when the coupon rate increases (more of the investment gets reimbursed earlier) - and this is satisfied in this example -, but this has to be checked more precisely as the price changes in that case. This is the purpose of exercise 11.

**b.** The yield-to-maturity of this annuity is $r$, as the payment is risk-free and the price is computed by using the discount rate $r$ for any payment date (as the YTM for a sovereign bond is an average of the prevailing yield curve).

We have: $D = \dfrac{r}{A}\displaystyle\sum_{t=1}^{+\infty}\dfrac{tA}{(1+r)^t} = r\sum_{t=1}^{+\infty}\dfrac{t}{(1+r)^t}$.

But $\displaystyle\sum_{t=1}^{+\infty}\dfrac{1}{(1+r)^t} = \dfrac{1}{r} \overset{(*)}{\Rightarrow} \sum_{t=1}^{+\infty}\dfrac{-t}{(1+r)^{t+1}} = -\dfrac{1}{r^2} \Rightarrow \sum_{t=1}^{+\infty}\dfrac{t}{(1+r)^t} = \dfrac{1+r}{r^2}$ then $D = \dfrac{1+r}{r}$ (not $+\infty$).

N.A.: 21 years. (*): as a power series, the lefthandside quantity is infinitely differentiable on its area of convergence, and one can differentiate under the sign $\sum$.

Notes: - we will see in chapter II, section 3. of the course that the duration is linked to the first derivative of the price w.r.t the yield-to-maturity, which is what above calculus uses.

- We saw that for $x \in ]0,1[$, $X = \displaystyle\sum_{t=1}^{+\infty} t x^t$ can be computed by taking the first derivative in $\displaystyle\sum_{t=1}^{+\infty} x^t = \dfrac{x}{1-x}$.

Other method:

$$X(1-x) = \sum_{t=1}^{+\infty} t x^t - \sum_{t=1}^{+\infty} t x^{t+1} = \sum_{t=0}^{+\infty}(t+1)x^{t+1} - \sum_{t=1}^{+\infty} t x^{t+1} = \sum_{t=0}^{+\infty} x^{t+1} = \frac{x}{1-x} \text{ then } X = \frac{x}{(1-x)^2}.$$

**Exercise 7.**
**a.** The bond paying annually a coupon at rate $c(0,T)$ with a $N$ face value quotes at par if and only if $c(0,T)$ satisfies:

$$\sum_{t=1}^{T} c(0,T)NB(0,t) + NB(0,T) = N, \text{ therefore: } c(0,T) = \frac{1-B(0,T)}{\sum_{t=1}^{T} B(0,t)} = \frac{1-\frac{1}{(1+r(0,T))^T}}{\sum_{t=1}^{T}\frac{1}{(1+r(0,t))^t}}.$$

The par rate for the maturity $T$, $c(0,T)$, is an "average" of the 0-coupon rate curve between 0 and $T$, for this issuer. Note that $c(0,1) = r(0,1)$.

Note that the notion of par yield is used in constructing interest rate swaps.

**b.** For a bond at par, the YTM equals the coupon rate, hence the result. The par rate for the maturity $T$, $c(0,T)$, is the coupon rate of a bond which would be issued today, for maturity $T$.

**c.** $c(0,T)$ is the yield of a coupon-bearing bond, it is then affected by the fact that the holder gets some payments before $T$, at times corresponding to lower rates. $c(0,T)$ is then an "average" of the $r(0,t)$ for $t$ between 0 and $T$, hence below $r(0,T)$.

Equivalently, the 0-coupon rate $r(0,T)$ corresponds to a longer maturity than $c(0,T)$.
Note that in that case, the par yield curve itself is increasing as well.

Mathematical proof:
from b., $c(0,T)$ is the YTM and the coupon rate of a bond which would be issued today, for maturity $T$.

Therefore: $c(0,T)\sum_{t=1}^{T}\frac{1}{(1+c(0,T))^t}+\frac{1}{(1+c(0,T))^T} = c(0,T)\sum_{t=1}^{T}\frac{1}{(1+r(0,t))^t}+\frac{1}{(1+r(0,T))^T}$

With $c(0,T) \geq r(0,T)$, we would have $\forall t = 1,...,T$, $c(0,T) > r(0,t)$, and the above inequality would not be possible (the left-hand side term would be lower).

**Exercise 8.** $T$ being the time-to-maturity of the bond, we need to discount payments done at dates $\frac{k}{m}$, for $k = 1, 2, ..., mT$.
Let $\rho$ be the yield-to-maturity for the asset corresponding to this payment frequency $m$
(we could denote it by $\rho_m$ to recall its compounding frequency per year).

**a.** Between two payment dates (spaced by $\frac{1}{m}$ years), the discount factor in the bond price computation is $\frac{1}{1+\frac{\rho}{m}}$ ($\rho$ is annualised as usual).

We get $P = \sum_{k=1}^{mT}\frac{C_k}{(1+\frac{\rho}{m})^k}$, where $C_k$ is the $k^{th}$ payment.

Note that a payment $C$, done at $t \in \{\frac{1}{m}, \frac{2}{m}, ..., T\}$, has a present value equal to $\frac{C}{(1+\frac{\rho}{m})^{mt}}$.

For example, for a standard bond, $C_k = \frac{cN}{m}$ or $\frac{cN}{m} + N$, hence

$$\frac{P}{N} = \frac{c}{m}\sum_{k=1}^{mT}\frac{1}{(1+\frac{\rho}{m})^k} + \frac{1}{(1+\frac{\rho}{m})^{mT}} = \frac{c}{m}\frac{1}{1+\frac{\rho}{m}}\frac{1-\frac{1}{(1+\frac{\rho}{m})^{mT}}}{1-\frac{1}{1+\frac{\rho}{m}}} + \frac{1}{(1+\frac{\rho}{m})^{mT}}$$
$$= \frac{c}{\rho}\left[1-\frac{1}{(1+\frac{\rho}{m})^{mT}}\right] + \frac{1}{(1+\frac{\rho}{m})^{mT}}.$$

**b.** The $k^{th}$ payment is done at time $\frac{k}{m}$, hence the duration is: $D = \frac{1}{P}\sum_{k=1}^{mT}\frac{C_k\frac{k}{m}}{(1+\frac{\rho}{m})^k}$.

**c.** We have $P'(\rho) = -\sum_{k=1}^{mT}\frac{C_k\frac{k}{m}}{(1+\frac{\rho}{m})^{k+1}} = -\frac{D}{1+\frac{\rho}{m}}P$. Hence $\frac{\Delta P}{P} \sim -\frac{D}{1+\frac{\rho}{m}}\Delta\rho$.

Remark:
The rate defined above is called the Bond Equivalent Yield (BEY). Dividing by $m$, we get the periodic yield-to-maturity, $\frac{\rho_{\text{BEY}}}{m}$.
The bond equivalent yield (BEY) allows fixed-income securities whose payments are not annual to be compared with securities with annual yields. The BEY is a calculation for restating semi-annual, quarterly or monthly discount bond or note yields into an annual yield, and is the yield stated in the quotations.
The rate $\rho$ such that $(1 + \frac{\rho_{\text{BEY}}}{m})^m = 1 + \rho$ is called the Effective Annual Yield.

**Exercise 9.**

**1.** The risk is that interest rates go up: in that case bond prices go down as existing bonds become less attractive compared to newly issued bonds having a higher coupon.

**2. a.** The value at $T-1$ of $1 + r(T-1, T)$ at time $T$ is 1 $\quad (r(T-1,T)$ is known at $T-1)$,
the value at $T-2$ of $1 + r(T-2, T-1)$ at time $T-1$ is 1, ... continuing backward, we get the present value at 0, which is 1 as well.
Multiplying by $N$, we get that the value of the FRB just after any coupon payment date is $N$.
Note that the FRB is equivalent to $N$ invested at for 1 year at 0 (then at rate $r(0,1)$), then $N$ reinvested at time 1 (rate $r(1,2)$),...

**b.** Between 2 coupons dates: $t-1 < s < t$. Just before $t$, price is still $N$ (as the accrued coupon is equal to the whole coupon). No cash-flow between $s$ and $t$.
Then price at $s = NB(s,t) + \{\text{accrued coupon between } t-1 \text{ and } s\} = NB(s,t) + [s-(t-1)]r(t-1,t)N$ (remember that the dates are in years). $B(s,t)$ is the discount factor between $s$ and $t$.

**Exercise 10. 1.** Slope $= P'(\rho) = -SP(\rho)$.

**2.** Taking the 1st order approximation, means that we are approximating the curve $\rho \mapsto P(\rho)$ by its tangent at $(\rho, P(\rho))$. Because of the convexity of $\rho \mapsto P(\rho)$, the curve is above its tangent.
Consequence (draw the curve):
1st order only: overvalues (in absolute value) the effect of a rate increase.
undervalues the effect of a rate drop.

Ie: asymmetry of the effect of a rate increase / drop: for a same absolute variation of the rate $|\Delta\rho|$, the variation $|\Delta P|$ for a rate increase is smaller than the variation for a rate drop (while it is the same if we look at 1st order only).

**Exercise 11.**

- <u>1st order approximation</u>: $\Delta P(\rho) \sim P'(\rho)\Delta\rho$, where $P(\rho) = \dfrac{N}{(1+\rho)^T}$. Then $P'(\rho) = \dfrac{-NT}{(1+\rho)^{T+1}}$.

We get: $\Delta P(\rho) \sim -\dfrac{NT}{(1+\rho)^{T+1}}\Delta\rho$.

N.A.: If the rate goes from 10% to 9%, $\Delta\rho$= -0.01. $P'(\rho)\Delta\rho = \frac{-100\times 10}{(1.1)^{11}}(-0.01) = \frac{10}{(1.1)^{11}} = 3.504939$ M€.

Then $\Delta P(\rho) \sim 3,504,939$€.

<u>Other method</u>: using the formula involving the sensitivity:
$\dfrac{\Delta P}{P} \sim -S\Delta\rho$ where $S = \dfrac{D}{1+\rho}$, with $D$ the duration.

Then $\Delta P(\rho) \sim -SP\Delta\rho$ which gives, as $D = T$: $\quad \Delta P(\rho) \sim -\dfrac{TN}{(1+\rho)^{T+1}}\Delta\rho$, same result.

• <u>2nd order approximation</u>: $\Delta P \sim P'(\rho)\Delta\rho + \frac{1}{2}P''(\rho)(\Delta\rho)^2$.

We have $P''(\rho) = \dfrac{NT(T+1)}{(1+\rho)^{T+2}}$ then $P''(\rho)(\Delta\rho)^2 = \frac{100\times 10\times 11}{(1.1)^{12}}(0.01)^2 = \frac{10\times 10\times 1.1}{(1.1)^{12}} \times 0.01 = \frac{1.1}{(1.1)^{12}}$. Then

$\Delta P \sim \frac{10}{(1.1)^{11}} + \frac{1}{2}\frac{1}{(1.1)^{11}}$. Note that the 2nd term of the same is 20 times smaller than the 1st one.

$\Delta P \sim 3,504,939(1+\frac{1}{20}) = 3,680,186$€.

• <u>Exact computation</u>: $\Delta P =$ new price - previous price $= \dfrac{100}{(1+0.09)^{10}} - \dfrac{100}{(1+0.1)^{10}} = 3,686,752$€. Obviously, the approximation is indeed better with the 2nd order term.

**Exercise 12.**

$P = \frac{A}{\rho}$. If the YTM goes from $\rho$ to $\rho'$, the relative variation of price is: $\frac{\Delta P}{P} = \frac{\frac{A}{\rho'} - \frac{A}{\rho}}{\frac{A}{\rho}} = \frac{\rho}{\rho'} - 1$.

Rate increase of 1%: $\rho' = 11\% \Rightarrow \frac{\Delta P}{P} = \frac{0.1}{0.11} - 1 = -9.09\%$.

Rate drop of 1%: $\rho' = 9\% \Rightarrow \frac{\Delta P}{P} = \frac{0.1}{0.09} - 1 = 11.11\%$.

If we want to use the approximation of the price change, we will compute the following parameters:

$D = \frac{1+\rho}{\rho}$, $S = \frac{1}{\rho}$, $\mathcal{C} = \frac{P''(\rho)}{P(\rho)} = \frac{2A}{\rho^3}\frac{\rho}{A} = \frac{2}{\rho^2}$ (with $\rho = 10\%$, $S = 10$ and $\mathcal{C} = 200$).

**Exercise 13. 1.** $\displaystyle\sum_{t=1}^{T} \frac{C}{(1+\rho)^t} + \frac{N}{(1+\rho)^T} = \sum_{t=1}^{T} \frac{C}{(1+r_t)^t} + \frac{N}{(1+r_T)^T}$ then

$$c\sum_{t=1}^{T} \frac{1}{(1+\rho)^t} + \frac{1}{(1+\rho)^T} \overset{(*)}{=} c\left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} + \dots \frac{1}{(1+r_T)^T}\right] + \frac{1}{(1+r_T)^T}.$$

**2.** Yield-to-maturity for a 0-coupon maturing at $T$: $r_T$.

**3.** $0 < r_1 < r_2 < \dots < r_T \Rightarrow f(r_1) > c\left[\frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} + \dots \frac{1}{(1+r_T)^T}\right] + \frac{1}{(1+r_T)^T} > f(r_T)$

ie $f(r_1) > f(\rho) > f(r_T)$. Then $r_1 < \rho < r_T$ as $f$ is a decreasing function.

**4.** A greater proportion of total payments comes on the shortest maturities, where the rates are lower.

Quantitative argument: equation (*) defines the yield-to-maturity $\rho$ from the issuer's yield curve. The corresponding equation for any annuity of this issuer is, $\rho_A$ being the yield-to-maturity of the annuity:

$$\sum_{t=1}^{T} \frac{1}{(1+\rho_A)^t} \overset{(**)}{=} \frac{1}{1+r_1} + \frac{1}{(1+r_2)^2} + \dots \frac{1}{(1+r_T)^T}.$$