Microeconomics 1. (TA1) October 25, 2023.

(P1). $f(z) = \alpha z$. $\alpha > 0$ $z \geq 0$.

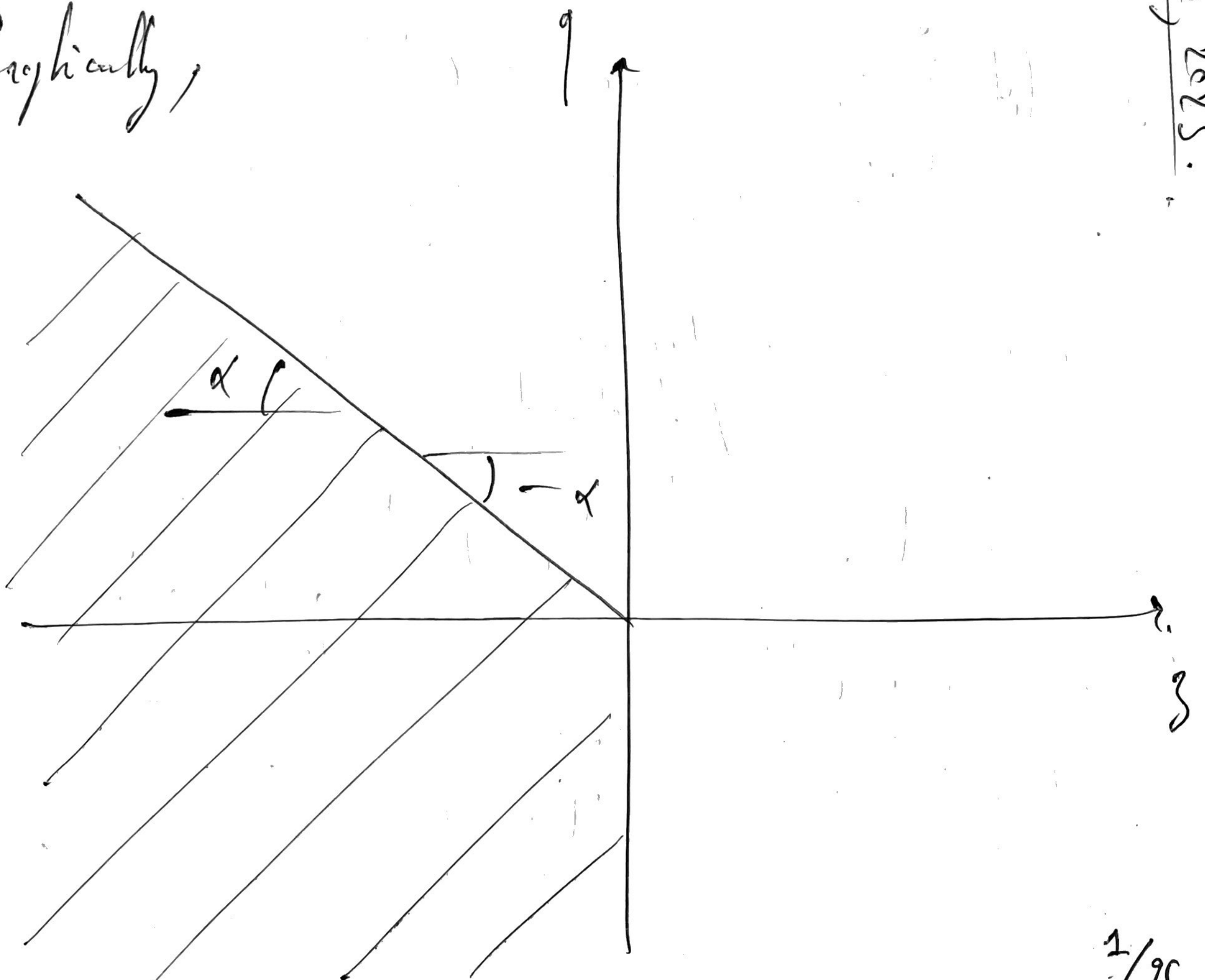
a). The production "function" is

$$Y^* = \{(-z, q) \,;\, q = f(z) \text{ and } z \geq 0\}.$$

and because of free dispose, the production set is:

$$Y = \{(-z, q) \,;\, q - f(z) \leq 0 \,,\, z \geq 0\}.$$

Graphically,



b). * possibility of inaction $\equiv$ $\boxed{0 \in Y.}$

$\longrightarrow$ yes, because. $\underset{q.}{0} - f(\underset{z \geq 0.}{0}) = 0 \leq 0.$ $\square$.

* $\boxed{\text{closedness}}$. YES, because.
Y is closed,
$(-z, q) \longmapsto q - f(z)$.
is continuous,

so. $Y = \{ q - f(z) \leq 0 \}$.
is of closed-form. $\square$.

* impossibility of free production:

i.e. $\boxed{y \in Y \text{ and } y \geq 0. \Longrightarrow y = 0.}$

YES because. $y \geq 0 \iff \begin{matrix} q \geq 0. \\ z \leq 0. \end{matrix}$ and. $z \geq 0.$

$\Longrightarrow z = 0. \longrightarrow q \leq f(z) = 0.$

$\Downarrow$ and $q = 0.$

$\square$.

* free disposal:

$\boxed{Y - \mathbb{R}^L_+ \subset Y.}$

YES, because...

Let $y \in Y$, i.e. $y = (-z, q)$, $z \geq 0$ and $q \leq f(z)$.

Let $y' = y - \delta$, $\delta \in \mathbb{R}_+^2$ i.e. $\delta = (\delta_1, \delta_2) \geq 0$.

$= (-z', q')$.

Then. $z' = z + \delta_1 \geq 0$. and.

$q' = q - \delta_2 \leq q$.

and.

$$f(z') = \alpha(z + \delta) = f(z) + \overset{\geq 0}{\widehat{\alpha \delta_2}} \geq f(z).$$

So. $q' \leq q \underset{\text{because } y \in Y}{\leq} f(z) \leq f(z')$.

$z' \geq 0$ and $q' - f(z') \leq 0$.

so $y' = (-z', q') \in Y$. $\square$.

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* Irreversibility $\equiv$.

$$\boxed{y \in Y \text{ and } y \neq 0 \implies -y \notin Y.}$$

by contradiction, assume $\underset{\substack{\shortparallel \\ (z,q)}}{y} \in Y$ and $\underset{\substack{\shortparallel \\ (z,-q)}}{-y} \in Y$.

$$(z,q) \Longrightarrow (z,-q).$$

then $z \leq 0$ and $-z \leq 0 \longrightarrow z = 0$.

$$\Longrightarrow q \leq f(z) = 0.$$

so $y = 0$. $\square$.

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* Convexity. | Y convex.
  i.e. $y, y' \in Y$ and $\alpha \in [0,1]$.
  $\Longrightarrow \alpha y + (1-\alpha) y' \in Y$.

let $y, y' \in Y$ and $\alpha \in [0,1]$.

$\underset{\substack{\shortparallel \\ (-z,q)}}{y}, \underset{\substack{\shortparallel \\ (-z',q')}}{y'}$. then $z, z' \geq 0$.

so $\alpha z + (1-\alpha) z' \geq 0$.

and $q \leq f(z)$ and $q' \leq f(z')$.

so

$$\alpha q + (1-\alpha) q' \leq \alpha f(z) + (1-\alpha) f(z')$$

$$= f(\alpha z + (1-\alpha) z').$$

$\square$.

using that $f : z \longmapsto \alpha z$ is linear.

* . Return to scale.

Let $y \in Y$ and $\beta \geq 0$

is $\beta y \in Y$?

$y \in Y$ iff.

$y = (-z, q)$, $z \geq 0$ and $q \leq f(z)$.

$\beta y = (-\beta z, \beta q)$; $\beta z \geq 0$ using $\beta \geq 0$. and. $\beta q \leq \beta f(z)$ $= f(\beta z)$ using linearity of f.

So, yes, Y exhibits CONSTANT RETURN TO SCALE.

i.e both decreasing and increasing return to scale. $\square$.

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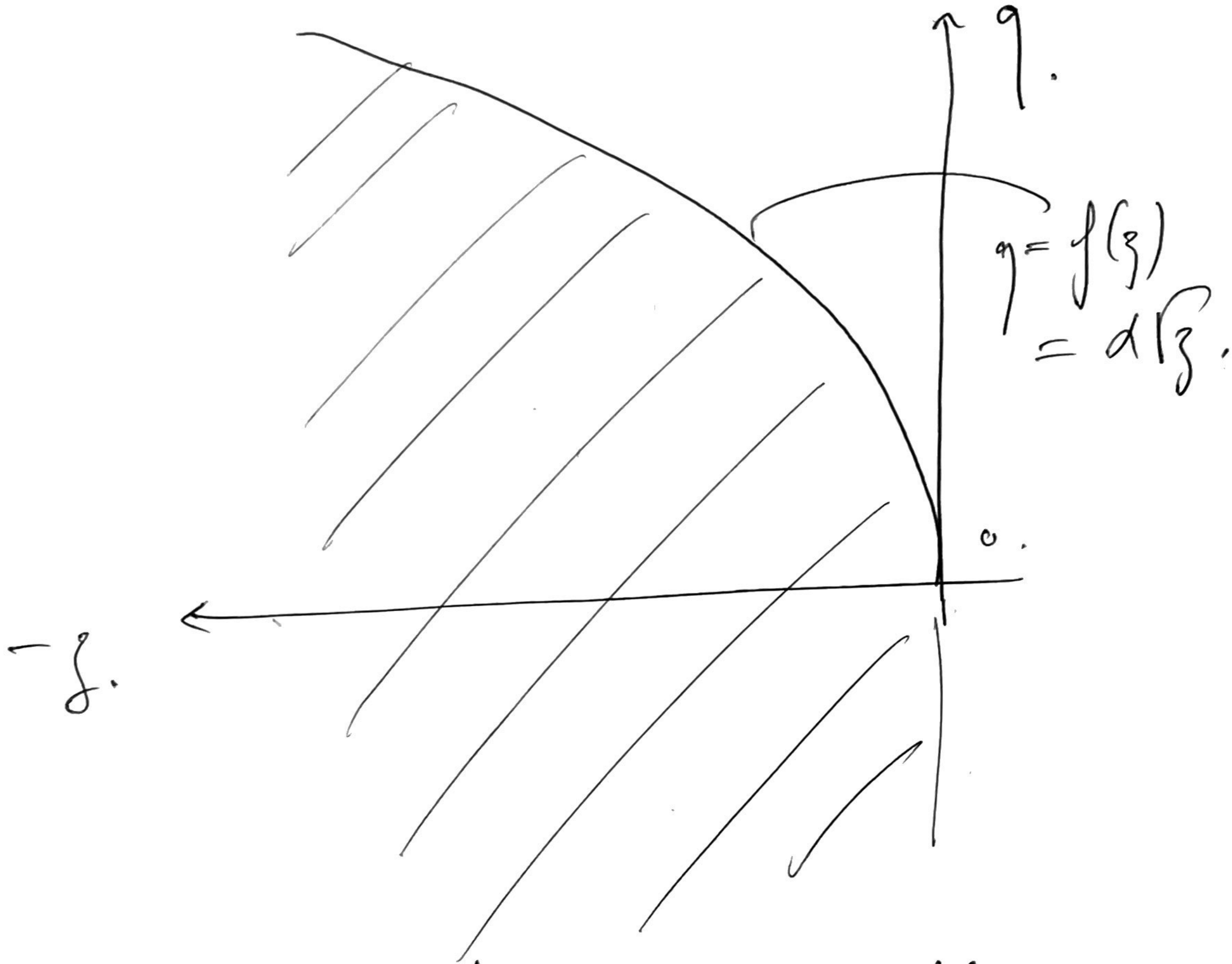
* (P1). for. $\boxed{f(z) = \alpha\sqrt{z} \ ; \ \alpha > 0 \quad z \geq 0.}$

a). $Y = \{(-z, q) \ ; \ q - f(z) \leq 0 \text{ and } z \geq 0\}.$



b). * possibility of inaction. i.e $0 \in Y$.

Yes, because $0 - f(0) \leq 0$.

* closedness. Yes, because Y is closed-form. (it includes its boundary).

* "no free lunch" i.e. $y \in Y$ and $y \geq 0 \implies y = 0$.

let $y \in Y$ and $\overset{(-z,q)}{\overset{\shortparallel}{y}} \geq 0$.

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\begin{pmatrix} z \geq 0 \\ \text{and} \\ q \leq f(z) \end{pmatrix} \qquad \begin{pmatrix} z \leq 0 \\ \text{and} \\ q \geq 0 \end{pmatrix} \implies \underline{z = 0}.$$

$$\Downarrow$$

$$q \leq f(0) = 0 \implies q = 0$$

□.

so $y = 0$.

* free disposal. i.e. $Y - \mathbb{R}^2_+ \subset Y$.

let $y \in Y$ and $\underset{(\delta_1, \delta_2)}{\underset{\shortparallel}{\delta}} \in \mathbb{R}^2_+$, show $y - \delta = y' \in Y$...

$$y' =: (-z', q') \implies z' = \underbrace{z}_{\substack{\geq 0 \\ \text{because} \\ y \in Y}} + \underbrace{\delta_1}_{\geq 0 \text{ by hyp.}} \geq 0.$$

and

$$q' = q - \delta_2 \underset{y \in Y}{\leq} f(z) - \delta_2 \underset{\delta_2 \geq 0}{\leq} f(z) \underset{\text{using } z' \geq z \text{ and } f \text{ increasing}}{\leq} f(z')$$

$\implies$ so $y' \in Y$. □.

* irreversibility: $y \in Y$ and $y \neq 0$ $\Longrightarrow$ $-y \notin Y$.

by contradiction
let $y \in Y$ and $-y \in Y$.

$$y = (-z, q) \qquad -y = (z, -q).$$

$$\Updownarrow \qquad\qquad \Updownarrow$$

$$\left(z \geq 0 \text{ and } q \leq f(z) \right) \qquad \left(-z \geq 0 \text{ and } -q \leq f(-z) \right)$$

$$f(z) = 0, \quad f(-z) = 0$$

$\Longrightarrow z = 0$

$\Longrightarrow q = 0$

$\Longrightarrow y = 0$. $\square$.

* convexity: let $y, y' \in Y$ and $\alpha \in [0,1]$.

we have: $\alpha z + (1-\alpha) z' = z'' \geq 0$
[using $z \geq 0$ and $z' \geq 0$.

and

$$q'' = \alpha q + (1-\alpha) q' \leq \alpha f(z) + (1-\alpha) f(z')$$
$$\leq f(\alpha z + (1-\alpha) z')$$
$$= f(z'').$$

using f concave.

$\Longrightarrow$ so $y'' \in Y$ $\Longrightarrow$ Y convex $\square$.

* return to scale.

let $y \in Y$ and $\beta \geq 0$.

$$y' = \beta y = (-\beta z, \beta q).$$

$\Updownarrow$

$(z \geq 0$ and $q \leq f(z).)$

we have $z' = \beta z \geq 0$. since $\beta \geq 0$ and $z \geq 0$.

$$q' = \beta q \underset{\substack{\uparrow \\ y \in Y}}{\leq} \beta f(z) \underset{\substack{\uparrow \\ \text{iff } \beta \in [0,1]}}{\leq} f(\beta z) = f(z').$$

$\Longrightarrow$ $y' = \beta y \in Y$ iff $\beta \in [0,1]$.

(using $\beta \leq \sqrt{\beta}$ $\Longleftrightarrow$ $\beta^2 \leq \beta$ $\forall \beta \in [0,1]$)

$\Longrightarrow$ Y exhibit decreasing (nonincreasing) return to scale. $\square$

(P1) for. $\boxed{f(z) = \alpha z^2 + \beta z \quad ; \quad \alpha, \beta > 0. \quad z \geq 0.}$

a)

$$f(z) = \alpha z^2 + \beta z = 0 \quad \text{iff} \quad z = 0$$

$$= z(\alpha z + \beta). \qquad \text{or} \quad z = -\beta/\alpha.$$

(not in Y since $y \in Y \Rightarrow z \geq 0$.)

$$df/dz = 2\alpha z + \beta = 0 \quad \text{iff} \quad z = -\beta/2\alpha.$$

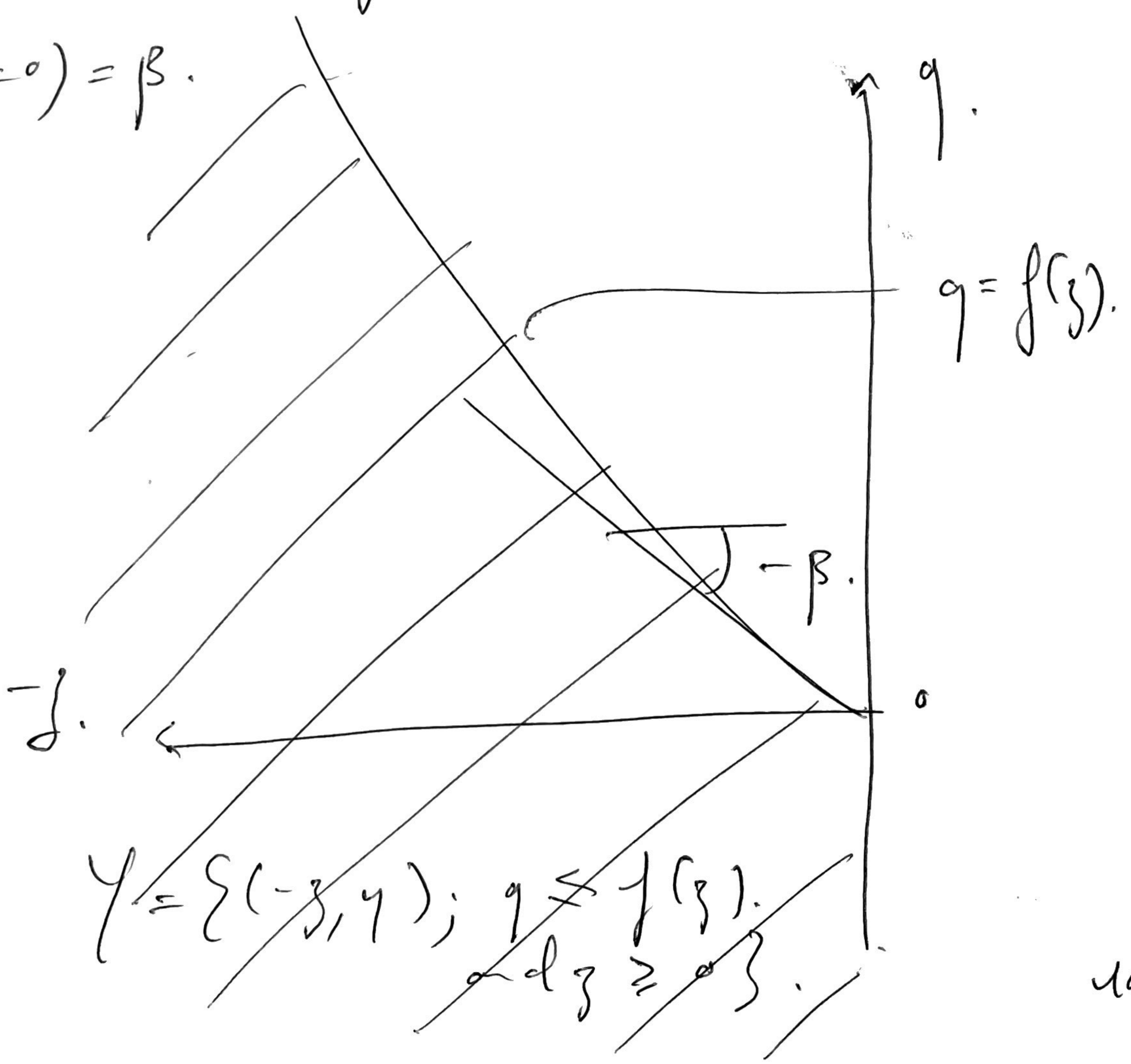
not in Y.

$$d^2f/dz^2 = 2\alpha > 0.$$

$\Longrightarrow$ so f CONVEX.

$$df/dz\,(z=0) = \beta.$$

$\Longrightarrow$



$$Y = \{(-z, q) \,;\, q \leq f(z). \text{ and } z \geq 0\}.$$

b) * <u>possibility of inaction</u>:

Yes, $0 - f(0) = 0 \leq 0 \Longrightarrow 0 \in Y$. □.

* <u>closedness</u>: Yes, Y closed-form.

* <u>impossibility of free production</u>.

let $y \in Y$ and $y \geqslant 0$

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$ $z \leq 0$ and $q \geqslant 0$. $\Longrightarrow z = 0$.

$(z \geq 0$ and $q \leq f(z))$.

$\qquad\qquad \overset{\|}{0}$ $\Longrightarrow q = 0$.

$\Longrightarrow y = 0$. □.

* <u>free-disposal</u>: let $y \in Y$ and $\delta \in \mathbb{R}^2_+$.

$$y' = y - \delta = (-z - \delta_1, q - \delta_2).$$

$$\Longrightarrow z' = z + \delta_1 \geq 0 \quad \text{(} z \geq 0 \text{ and } \delta_1 \geq 0\text{)}$$

and

$$q' = q - \delta_2 \leq q \leq f(z) \leq f(z').$$

$\delta_2 \geq 0$ $\qquad$ using $z' \geq z$ (from $\delta_1 \geq 0$).

and f increasing $\forall z \geq 0$.

(since $df/dz = 2\alpha z + \beta \geq 0 \ \forall z \geq 0$).

* irreversibility:

let $y, -y \in Y \longrightarrow z = 0 \ldots$

$\Downarrow \qquad \Downarrow$

$q \leq \underbrace{f(0)}_{=0} \qquad -q \leq f(-z) = f(0) = 0 \longrightarrow q = 0.$

$\longrightarrow y = 0. \qquad \square.$

* convexity:

let $y, y' \in Y \qquad \alpha \in [0,1]$.

$y'' = \alpha y + (1-\alpha) y'$.

we have $z'' \geq 0$.

BUT suppose $q = f(z)$ and $q' = f(z')$.

i.e. y and y' are on the "production frontier"

$\equiv$ the boundary of Y.

we have $q'' = \alpha q + (1-\alpha) q'$

$= \alpha f(z) + (1-\alpha) f(z')$.

$\geq f(\alpha z + (1-\alpha) z') = f(z'')$.

(using f CONVEX, from $d^2f/dz^2 = 2\alpha > 0 \quad \forall z \geq 0$

$\longrightarrow$ so $y'' \notin Y$

$\Longrightarrow$ Y not CONVEX! $\qquad \square.$

* <u>returns to scale</u>.

let $y \in Y$ $\beta \geq 0$.

$\beta y = y' \in Y$ iff.

$$q' = \beta q \leq f(z') = f(\beta z).$$

knowing $q \leq f(z)$

$$\Rightarrow \beta f(z) \leq f(\beta z).$$

$$\Leftrightarrow \beta(\alpha_2 z^2 + \cancel{\alpha_0 z}) \leq \beta^2 \alpha_2 z^2 + \cancel{\beta \alpha_0 z}.$$

$$\Leftrightarrow \underbrace{(\beta^2 - \beta)}_{\beta(\beta-1)} \alpha_2 z^2 \geq 0. \quad \forall z \geq 0.$$

$$\Leftrightarrow \underline{\beta \geq 1}.$$

$\Rightarrow$ Y exhibits increasing (nondecreasing) returns to scale. A.

(P2) let. $\overset{\text{production fct.}}{q = f(z)}$. $z \in \mathbb{R}^L_+$ MWG. 5.B.3.

$Y = \{(-z, q) \,;\, z \geq 0 \text{ and } q - f(z) \leq 0\}$.

Show. $\boxed{Y \text{ convex} \iff f \text{ concave.}}$

$(\Longrightarrow)$. Y convex. iff. (let $\overset{(-z,q)}{y}, \overset{(-z',q')}{y'} \in Y$, $\alpha \in [0, 1]$.

$\Longrightarrow \alpha y + (1-\alpha) y' \in Y$.)

i.e. ($f(z) \geq q$ and $f(z') \geq q'$

$\Longrightarrow f(\alpha z + (1-\alpha) z') \geq \alpha q + (1-\alpha) q'$)

So as a special case, if $q = f(z)$ and $q' = f(z')$ $\Longrightarrow$ $f(\alpha z + (1-\alpha) z') \geq \alpha f(z) + (1-\alpha) f(z')$

i.e. f concave.

$(\Longleftarrow)$ let f concave.

i.e. $\forall z, z'$, $\alpha \in [0, 1]$.

$f(\alpha z + (1-\alpha) z')$
$\geq \alpha f(z) + (1-\alpha) f(z')$.

so $\forall q, q'$ with. $f(z) \geq q$ and. $f(z') \geq q'$.

we have. $f(\alpha z + (1-\alpha) z') \geq \alpha q + (1-\alpha) q'$.

So. $\forall y, y' \in Y$, $\alpha \in [0,1]$.

$$\Longrightarrow \alpha y + (1-\alpha) y' \in Y$$

ie Y convex. $\square$.

(P3) Let $z \in \mathbb{R}_+^{L-1}$ and $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ concave.

Show that the transformation fct.

$F(y) := y_L - f(z)$ is quasiconvex on the convex set $A = \{(-z, y_L) = y \; ; \; z \geq 0, \; y_L \geq 0\}$.

Def. $F(y)$ is QUASI CONVEX iff its lower contour sets; ie $\forall t \in \mathbb{R}$.

$L(t) = \{ y \in A \; ; \; F(y) \leq t \}$ are convex.

ie

$\forall t$, $F(y) \leq t$ and $F(y') \leq t$.

$$\Longrightarrow F(\alpha y + (1-\alpha) y') \leq t \; , \; \alpha \in [0,1].$$

let. t

$$f(y) \leq t \text{ and. } f(y') \leq t.$$

let $y'' = \alpha y + (1-\alpha) y' \qquad \alpha \in [0,1].$

we have.

$$f(y'') = g'' - f(z'')$$

$$\leq g'' - (\alpha f(z) + (1-\alpha) f(z')).$$

$$= \alpha f(y) + (1-\alpha) f(y').$$

using that f concave.

i.e. $f(z'') \geq \alpha f(z) + (1-\alpha) f(z')$

$$\leq \alpha t + (1-\alpha) t = t.$$

so f is <u>quasiconvex</u>. $\square$

(P4) $L = 3$. $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$

$\alpha, \beta > 0 \quad z_1, z_2 \geq 0$.

a) Write the production set.

$$Y = \{(-z, q) ; z = (z_1, z_2) \geq 0 \text{ and } q - f(z) \leq 0.\}.$$

b). Verify that it satisfies:

* possibility of inaction, i.e. $0 \in Y$.
  Yes, because. $0 - f(0) = 0 - 0 \leq 0$.

* closedness, Yes because.
  Y of closed-form because.
  $(z, q) \longmapsto q - f(z)$
  continuous because f continuous.

* no-free lunch.

Yes because. $y \in Y$.
and $y \geq 0$. $\Longrightarrow$ $-z \geq 0$ and $z \geq 0$

$y = (-z, q)$. $\Longrightarrow z = 0. \Longrightarrow f(z) = 0.$
$\Longrightarrow q = 0.$
$\Big\} \Longrightarrow y = 0.$

* free disposal. let $y \in Y$. let $\delta \geq 0$.

$$y' = y - \delta = (-z_1 - \delta_1, -z_2 - \delta_2, q - \delta_3).$$

We have that $\overset{z'_1}{z_1 + \delta_1}$ and $\overset{z'_2}{z_2 + \delta_2} \geq 0$.

and

$$q' = q - \delta_3 \leq q \leq f(z) \quad \text{using } y \in Y.$$

$\delta \geq 0$.

and $f(z') = f(z + \underbrace{(\delta_1, \delta_2)}_{\geq 0}) \geq f(z)$.

using that $f : z \longmapsto z_1^{\alpha} z_2^{\beta}$ is increasing for $\alpha, \beta > 0$

So

$$q' = q - \delta_3 \leq f(z').$$

$\Longrightarrow y' \in Y \; \forall \delta \geq 0$

$$\Longrightarrow Y - \mathbb{R}^3_+ \subset Y. \quad \square$$

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* <u>irreversibility</u>. let $y \in Y$ and $-y \in Y$.

$\Longrightarrow$ then we have that $z = 0$.

$\Longrightarrow q = 0$ (because $f(0) = 0$).

$\Longrightarrow y = 0. \quad \square$

* convexity.

ii (P2) we've shown

$$\boxed{\Psi \text{ convex iff } f \text{ concave.}}$$

iii (P3) we've shown.

$$\boxed{f \text{ concave} \Longrightarrow F \text{ quasi convex.}}$$

In fact, we can show.

$$\boxed{\begin{aligned} \Psi \text{ convex} &\Longleftrightarrow f \text{ concave.} \\ &\Longleftrightarrow F \text{ quasi convex.} \end{aligned}}$$

Proof: We are left to prove

F quasi convex $\Longrightarrow$ Ψ convex.

Suppose f quasi-convex.

$\Longrightarrow$ ($\forall t$ $y, y' \in L(t) := \{y : f(y) \le t\}$.

$\Longrightarrow y'' = \alpha y + (1-\alpha) y' \in L(t)$

$\forall \alpha \in [0,1]$).

it is true $\forall t$ so it is true in particular for . $t = 0$.

and we have. $L(t=0) = Y$.

i.e $L(t)$ CONVEX $\forall t$.

$\Longrightarrow$ so Y is CONVEX , $\boxed{}$.

... back to proving Y convex when $f(z) = z_1^{\alpha} z_2^{\beta}$.

The easiest way is to prove that.

$f(z) = z_1^{\alpha} z_2^{\beta}$ is concave

iff $\alpha + \beta \le 1$.

<u>Proof:</u> $f(z) = z_1^{\alpha} z_2^{\beta}$.

$$\Rightarrow \quad Hf(z) = \begin{pmatrix} \frac{\alpha(\alpha-1)}{z_1^2} f(z) & \frac{\alpha\beta}{z_1 z_2} f(z) \\ \frac{\alpha\beta}{z_1 z_2} f(z) & \frac{\beta(\beta-1)}{z_2^2} f(z) \end{pmatrix}.$$

<u>Reminder:</u>
$$\mathrm{Tr} = \lambda_1 + \lambda_2.$$
$$\det = \lambda_1 \lambda_2.$$

so if $\det \geq 0$ then λ_1 and λ_2 have the <u>same sign</u>:

and this sign is given by the sign of Tr!

$$\det Hf(z) = \Big[\underbrace{\alpha(\alpha-1)\beta(\beta-1) - (\alpha\beta)^2}\Big] \frac{f(z)^2}{z_1^2 z_2^2}$$

$$\cancel{(\alpha\beta)^2} - \alpha^2\beta - \alpha\beta^2 + \alpha\beta - \cancel{(\alpha\beta)^2}.$$
$$= \alpha\beta\left(1 - (\alpha+\beta)\right).$$

$$\Rightarrow \det Hf(z) = \overbrace{\alpha\beta}^{\geq 0}\left(1-(\alpha+\beta)\right)\underbrace{\left(\frac{f(z)}{z_1 z_2}\right)^2}_{\geq 0}.$$

$\alpha, \beta \geq 0$ by assumption.

≥ 0 iff $\alpha+\beta \leq 1$.

$\Downarrow$

$\alpha \leq 1$ and $\beta \leq 1$ given that $\alpha, \beta \geq 0$.

and

$$\operatorname{Tr} Hf(z) = \overbrace{\frac{\alpha(\alpha-1)}{z_1^2}}^{\leq 0}\underbrace{f(z)}_{\geq 0} + \overbrace{\frac{\beta(\beta-1)}{z_2^2}}^{\leq 0}\underbrace{f(z)}_{\geq 0}.$$

$\Rightarrow \operatorname{Tr} Hf(z) \leq 0.$

$\lambda_1, \lambda_2 \leq 0$ i.e.

$\Rightarrow$ Hence $Hf(z)$ semidefinite negative $\forall z$ iff $\underline{\alpha+\beta \leq 1}$.

ie Y CONVEX iff $\alpha+\beta \leq 1$. . A.

c) Returns to scale.

Let. $y \in Y$. Then $\lambda y \in Y$.

iff. $\lambda z \geq 0$. $\xRightarrow[\text{knowing } z \geq 0.]{} \lambda \geq 0$.

and.

We want to determine under what conditions does.

$$q - f(z) \leq 0 \Longrightarrow \lambda q - f(\lambda z) \leq 0.$$

$$\Updownarrow$$

$$\lambda q - \lambda f(z) \leq 0$$

given that. $\lambda \geq 0$.

So this implication holds

iff. $f(\lambda z) \geq \lambda f(z)$

$$\lambda^{\alpha+\beta} \underbrace{f(z)}_{\geq 0} \geq \lambda \underbrace{f(z)}_{\geq 0}$$

iff. $\boxed{\lambda^{\alpha+\beta-1} \geq 1.}$

So, if $\underline{\alpha+\beta > 1}$..

then $\lambda^{\alpha+\beta-1} \geq 1$. $\forall \lambda \geq 1$.

i.e. $\lambda y \in Y$ $\forall \lambda \geq 1$ and $y \in Y$.

i.e. Y exhibits increasing return to scale !

if $\underline{\alpha+\beta < 1}$.

then $\lambda y \in Y$ $\forall y \in Y$ and $\forall \lambda \in [0, 1]$.

$\Longleftarrow$ decreasing return to scale.

if

$\underline{\alpha+\beta = 1}$.

then $\lambda y \in Y$. $\forall y \in Y$ and $\forall \lambda \geq 0$.

(since $\forall \lambda > 0$. $\lambda^0 = 1 \geq 1$.

and. $0 \in Y$).

$\Rightarrow$ i.e. Y exhibits CONSTANT - RETURN TO SCALE.

$\square$.

(PS) Show that (in general ?) if the production set exhibit nondecreasing (i.e. increasing...) return to scale, then either

$$\pi(p) \leq 0 \quad \text{or} \quad \pi(p) = +\infty.$$

$p \gg 0$ price system.

Def. $\pi(p) = \max\{p \cdot y \; ; \; y \in Y\}$. is the "profit fct". ← production set.

Suppose Y exhibits increasing return to scale.
i.e. $\forall y \in Y, \alpha \geq 1 \implies \alpha y \in Y$.

So... either
$\exists y \in Y$ s.th. $p \cdot y > 0$, in which case
$\forall \alpha \geq 1, \alpha y \in Y \implies p \cdot \alpha y \geq p \cdot y$.

$\implies \boxed{\pi(p) = +\infty.}$

or $\nexists y \in Y$ s.th. $p \cdot y > 0$.
i.e. $\forall y \in Y \;\; p \cdot y \leq 0$.

$\implies \boxed{\pi(p) \leq 0}$