Introductory Finance

M1 MAEF DU MMEF QEM1 Erasmus

Abstract Financial products classification (not exhaustive) with:
- motivations
- valuation and risk, which are intimately linked.

Main function of financial markets:
First aim was: financing firms, ie provide them with money to develop their activities.
= Classical function of bond and stock markets, in complement to others parts of the financial system: banks, insurance companies.

More generally speaking, financial markets allow transfers between economic agents.
Indeed, some agents invest more than they spare and reversely.

Then financial markets perform the essential economic function of channeling funds from people who have an excess of available funds to people who have a shortage.

<u>Market transfers</u>
from individuals and firms that do not have a productive use for them
to those that do, resulting in greater economic efficiency.

It has direct effects on personal wealth and economic well-being
and on the investment behavior of businesses.

<u>Time transfers</u> ex: savings for pensions (for a given agent)

But also <u>risk transfer</u> to agents who accept to bear them.
Eg: Consider a French company due to receive M dollars at a known future time t (because it exports in the United-States). Company's costs are in euros so it will have to change dollars against euros.
If this exchange takes place at time T, the company doesn't know how many euros it will get
because the future rate \$/€ at date T is not known today.
The company is therefore exposed to a <u>foreign exchange risk</u>.
Several financial products (derivatives) bring a solution to this problem, by allowing hedging this risk (transferring it to another agent).

Financial markets are changing rapidly, new instruments appearing almost daily.
Motivation for new instrument (asset) families = improve each function of the financial market.
The Great Financial Crisis started in 2007-08 has slowed this down...

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Chapter I Basic notions

1 Classical products. Firms financing

securities (or *financial instrument*) = claims on the issuer's future income
or assets (any financial claim or piece of property that is subject to ownership).

Roughly speaking, a company that needs to raise funds to finance its activities, for example to build a new factory or develop a new product, can do so by:

1.1 issuing shares and selling them to investors. stocks also known as shares or equities.

A common stock (typically just called a stock) represents a share of ownership in a corporation.
It is a security that is a claim on the earnings and assets of the corporation.

The company is then "owned" by its shareholders; If you own one share of common stock in a company that has issued 1 million shares, you are entitled to 1 one-millionth of the firm's net income and 1 one-millionth of the firm's assets (+ some voting rights).
If the company makes a profit, part of this may be paid out to shareholders as a dividend of so much per share.

1.2 Another way to raise money is to issue a bond. This corresponds to a loan.
It may be issued by a government (*sovereign bonds*) or by a company (*corporate bonds*).

A **bond** is a debt security that promises to pay a certain stream of payments (coupons and principal) in the future, until a final date called maturity:

the basic bond pays its owner a fixed-interest payment (coupon) every period (ex: every year) until the maturity date, when a specified final amount (face value or par value) is repaid.

The coupon corresponds to the interest on the loan:
you lend N to the company. Each year, you receive C and you get back N at the maturity.

$$\begin{cases} C & \text{at date } t_1, ..., t_{p-1} \\ N + C & \text{at date } t_p = T \text{ maturity} \end{cases}$$

(implied interest rate for this loan at inception: $r = C/N =$ "coupon rate").

A zero-coupon bond (for example in dollar) with maturity date T is a contract which guarantees the holder 1$ to be paid at time T.

There are more complex bonds: floating rate notes, callable bonds...
We will study bonds in more details.

The bond market is especially important to economic activity, because it enables corporations or governments to borrow to finance their activities and because it is where interest rates are determined (see below).

Markets
Primary market: market in which new issues of a security (bond or stock) are sold to initial buyers by the corporation borrowing the funds (not well known to the public, takes place behind closed doors).

Ex: Treasury bonds (bills, notes, or bonds) are issued through auctions.

Secondary market: market in which securities that have been previously issued can be re-sold.

Eg, the stock market, in which such shares are traded, is the most widely followed financial market.

When an individual buys a security in the secondary market, the corporation that issued the stock acquires no new funds (but increases security liquidity $\Rightarrow$ makes them more desirable $\Rightarrow$ easier for the issuer to sell in the primary market).

A stock index is a value representative of a stock market (considered as a barometer of the economy). It is calculated from a basket of the principal assets (can be computed for a given sector as well). Ex: in France, the CAC-40 index is based on the 40 largest and most liquid stocks trading on the exchange. The weights of the stocks reflect the stock's total market capitalization (= stock price $\times$ number of shares outstanding).
A free-float weighting may be used: a float factor is assigned to each stock to account for the proportion of outstanding shares that are held by the general public.
Sometimes, the price of each component stock is the only consideration when determining the value of the index, a simple average is taken ("price-weighted index", ex: Dow Jones Industrial Average).
The dividends may be included (total-return index, e.g.: DAX) or not (price-return index, e.g. S&P 500 or CAC40) in the computation.

• Exchanges: Buyers and sellers meet in one central location to conduct trades.

• OTC markets The other method of organizing a secondary market is to have an Over-The-Counter market, in which dealers at different locations who have an inventory of securities stand ready to buy and sell securities "over the counter" to anyone who comes to them and is willing to accept their prices. Because OTC dealers are in computer contact and know the prices set by one another, the OTC market is very competitive and not very different from a market with an organized exchange.
Many common stocks are traded over-the-counter ("unlisted securities"), although the largest corporations usually have their shares traded at organized stock exchanges such as the New York Stock Exchange.
Ex: the US government bonds market, with a larger trading volume than the NYSE, is set up as an OTC market.

Also: dark pools (privately organized platforms for trading, not accessible by the investing public, operated by some large firms. Initially created to allow large trades without showing them and thus avoiding market impact.
Accounts for 10% to 20% of equity trading. May harms the price formation process, as the price on public exchanges no longer reflects the real value investors place on shares.

Since 2008, the regulators are trying to diminish the part of OTC products. At that time, 70% of all financial transactions were OTC.
EMIR: European Market and Infrastructure Regulation (Mifid II directive, January 2018) $\Rightarrow$
All standardized OTC derivative contracts (in particular interest rate derivatives) should be traded on exchanges or electronic trading platforms, where appropriate. Clearing houses act as the middlemen in trades and assume the default risk for both parties.

Value/Risk of both products - some elements: (for the holder)

• Stocks have a value that reflects the views of investors about the likely future dividend payments and capital growth of the company.

This value depends on the evolution of the firms and is studied through fundamental analysis and economics arguments. It is not the matter of this course.
Quotes are also affected by:
- liquidity considerations: the market price is associated to the part of the capital that is exchanged, this part can be very small, then it may be not representative of the whole value of the company.
- traders behavior (technical analysis methods, automated trading, ex: flash crash, 6 May 2010...).

• Bonds value is linked to the company's default risk,
Ie the risk that the issuer cannot reimburse some coupons or all of future payments.

In case of default, there are precise rules of priority for debt holders (eg Lehman, 15 Oct 2008).

In chapter II, we will study all the risk factors of bonds, ex: what happen if interest rates change?

Stocks: price risk much bigger, price can go to 0. The firm must pay all its debt holders before it pays its equity holders (but holders benefit directly from any increases in the firm's profits, while payments for debt holders are fixed).

2 Interest rates

An interest is the reward that investors demand for accepting delayed payment.

An **interest rate** is the cost of borrowing or the price paid for the rental of funds (usually expressed as a percentage of the rental of $100 per year, ie annualized).

There are many interest rates in the economy: central banks rates (a central bank rate is the interest rate at which a nation's central bank lends money to domestic banks, often in the form of very short-term loans), interest rates on many different types of bonds, mortgage interest rates, car loan rates...

Interest rates are important on a number of levels. They have an impact on the overall health of the economy because they affect consumers' willingness to spend or save and also businesses' investment decisions.

High interest rates, for example, could:
- deter consumers' expenses or business investments, because the cost of financing them would be high. Might cause a corporation to postpone building a new plant that would ensure more jobs.
- encourage people to save because they can earn more interest income by putting aside some of their earnings as savings.

1 euro today should be worth more than 1 euro tomorrow because the euro today can be used/ invested today. It should be better to have it today than in the future, as you can just keep it

$\Rightarrow$ an interest rate **should be positive** (normally!).

But not in particular cases (in fact negative interest rates have been observed in a lot of countries between 2015 and 2022!):

Ex: in June 2014, the European Central Bank (ECB) cut its deposit rate to -0.1%, to encourage eurozone banks to lend to small firms rather than to accumulate cash (the hope is to boost the economy). Instead of earning interest on money left with the ECB, banks are charged by the central bank to park their cash with it.

Managing the Central bank rate is a method by which central banks affect economic activity. Lower bank rates can help to expand the economy by lowering the cost of funds for borrowers, and higher bank rates increase it and dampen demand across the economy, resulting in excess supply and lower inflation. Such policies act on the short end of the yield curve.

Some regulations (e.g. for insurance companies) act on the remaining of the curve: they are not allowed to keep large amounts of cash, then it has to be invested in some secure assets.

Ex: in August 2019, German government rates up to maturity 30 years were all negative.
(a 10Y-bond has been issued in August with no coupon: "DBR 0 08/15/29", ISIN: DE0001102473
initial price = 102.64, quote 22/09/2023 = 86.17, see exercise 5).

Interest rates have a major impact on prices of financial instruments. In particular, the time value of money strongly affects assets and liabilities of some companies, in particular of insurance companies.

3 Future and present value

Note: an interest rate depends on the maturity of the payment. Here the interest rate r is supposed to be constant.

1. Future value $\rightarrow$ capitalization $\rightarrow$ compounding interest rates

Definition: the future value in n years of a capital M is the value that one obtains when investing this capital on n years.

Ex: once per year.
Consider an amount M invested at an interest rate of r_d per year (discrete rate for one year). After n years, you get: $M(1+r_d)^n$.

We consider the general case where an interest is paid several times per year and compounded: the already paid interests are themselves invested. We have m periods in 1 year, the interest is compounded m times per year. The length of one period is $1/m$ (eg, $m = 12$: monthly, $m = 24$ for the saving account "livret A" in France, i.e. interest compounded half-monthly).

We want to compute what should be the interest rate on each period, knowing that after 1 year the total interest has to correspond to the annual rate r_d, as announced by the bank.

Ex: the bank promises a rate of 10% for money invested for 1 year. But an interest is paid each month, so you wonder what the rate for a period of one month is. The answer is: 0,797%. But this number cannot be easily compared to other rates. The habit is to annualise any rate to facilitate comparison. For that you multiply by 12 above answer to know what the rate would be over a one year period (without any compounding). You get 9,569%.

General case: let r_m be the annualised rate for 1 period (eg monthly rate for $m = 12$), i.e. the interest on M invested over 1 period is $r_m M\times$ period length $= r_m M/m$, meaning that over one period, the wealth gets multiplied by $1 + \frac{r_m}{m}$.
On one year, the interest rate (obtained by compounded the intermediate interests) is r_d.

Then r_m is obtained by $\boxed{\left(1+\frac{r_m}{m}\right)^m = 1 + r_d}$, where r_d is the announced annual rate for one year.

After t years, with t integer, you get: $M\left(1+\frac{r_m}{m}\right)^{mt}$.

Exercises: 1. explain above results when the annual rate is 10%.

2. Prove that, after each interest payment, the value of the investment belongs to an exponential curve that is independent of m.

For any $k \in \mathbb{N}$, the future value of M after k sub-periods (hence k interest payments), is:
$M(1+\frac{r_m}{m})^k = M(1+r_d)^{\frac{k}{m}} = Me^{\frac{k}{m}\ln(1+r_d)} = e^{r\frac{k}{m}}$ where $\underline{r = \ln(1+r_d)}$.

That amount belongs to the curve $t \mapsto e^{rt}$, where t is the time (in years), curve that is independent of m.

Note that we have, for any m positive integer: $\left(1+\frac{r_m}{m}\right)^m \overset{(*)}{=} e^r$ (by definition of r).

Then, for a given m, for any $t \in \mathbb{N}$, the future value of M after t years is $M(1+\frac{r_m}{m})^{mt} = Me^{rt}$.

Most mathematical models (ex Black-Scholes for option pricing) are in continuous time.
$\Rightarrow$ a continuous interest rate is used (i.e. continuously compounded)

to be compared to a discrete rate as above: the interest is paid (and compounded) at discrete dates.

The continuous compounding corresponds to the limit as m goes to infinity (first approach):

Th: When $m \to +\infty$, r_m converges toward $r = \ln(1 + r_d)$.

Proof: from (*), we have $r_m = m(e^{\frac{r}{m}} - 1) \sim m(1 + \frac{r}{m} - 1) = r$, ie $\underline{r_m \to r = \ln(1 + r_d)}$.

Note that we used in the proof: $\forall x > 0, \forall y \in \mathbb{R}$, $x^y = e^{y \ln x}$ and $e^x \sim 1 + x$ when $x \to 0$.

The continuous rate r is then given by $r = \ln(1 + r_d)$.

When the number of periods m gets large, the future value dynamics is close to $t \mapsto Me^{rt}$, which is the future value dynamics for a continuous interest rate.

<u>Second approach for a continuous rate</u>: (modeling directly in continuous time)

at any period of time (even very short) an interest is paid.
$M(t)$ wealth at t.
In the continuous compound model, it is assumed that the instantaneous rate of change of M is proportional to M: $\quad \forall t \geq 0, \ \dfrac{dM(t)}{dt} = rM(t) \quad (*)$

Note that this is consistent with what we have in discrete time: on one period of length $\frac{1}{m}$, the change in the wealth corresponds to the paid interest: $\Delta M = r_m M \frac{1}{m}$ (**),
therefore the rate of change in the wealth on one period is: $\frac{\Delta M}{1/m} = r_m M$, proportional to the wealth.

Another writing for (*) is: "$dM(t) = rM(t)dt$". This is a notation for (*), that can be interpreted as follows: for an infinitely small change in t, dt, the corresponding change in the wealth at time t is $dM(t) = M(t + dt) - M(t) = rM(t)dt$. This variation corresponds to the interest paid between t and $t+dt$. That means: for $h > 0$ infinitely small, $M(t+h) - M(t) = rM(t)h$ (compare to equation (**)).

From (*) we get: $\forall t \geq 0$, $M'(t) = rM(t)$, then the relative rate of change of the function M is constant:
$\forall t \geq 0$, $\frac{M'(t)}{M(t)} = r$, which implies: $\forall t \geq 0$, $[\ln M]'(t) = r$.
Integrating between 0 and t, we obtain $\ln M(t) - \ln M(0) = rt$ thus $M(t) = M(0)e^{rt}$.

Conclusion: 1 euro today is worth at t : $\begin{cases} (1 + r)^t & \text{if } r = \text{ discrete interest rate (and } t = \text{payment date)} \\ e^{rt} & \text{if } r = \text{ continuous interest rate} \end{cases}$

Be careful: "to get an interest of 10% of your investment after 1 year, you need a discrete rate of 10%, or a continuous rate = 9,53%".

Exercise **1. a.** Explain and check this last sentence.
b. What would be the corresponding monthly rate (i.e. when an interest is paid monthly)?

2. Present value $\to$ Pricing of a future cash flow
reverse of Future Value.

An asset is hold for the future flows it will give. We need to calculate the present value of a future flow.

a. <u>Deterministic cash flow</u> (ie known at the time of pricing)
= "risk-free asset": can have only one value in the future

present value (today, at time 0) of 1 euro at $t = \begin{cases} \dfrac{1}{(1 + r)^t} & \text{if } r = \text{ discrete interest rate} \\ e^{-rt} & \text{if } r = \text{ continuous interest rate} \end{cases}$

= price of a 0-coupon bond with maturity t (a bond that pays no coupons and pays 1 euro at t)
= discount factor at 0 for time t.

$\rightarrow$ Discounting (=calculating what dollars received in the future are worth today)

allows to: bring back a future flow at date 0, compare flows at different times.

Obviously, the rate used to discount a cash-flow paid at time t is the rate that prevails, at the pricing time 0, for the maturity t. We will denote it by $r(0,t)$.

A coupon-bearing bond can be regarded as a collection of 0-coupon bonds, each maturing on one of the different coupon dates $t_1, ..., t_p$. We just need to price a 0-coupon bond, ie to know the present value of: 1\$ to be paid at time t.
For a standard bond: coupon C paid at time $1, ..., T$ and the nominal N reimbursed at T, we will get that the price at time 0 is: $P = \sum_{t=1}^{T} \frac{C}{(1+r(0,t))^t} + \frac{N}{(1+r(0,T))^T}$.

See exercises **1. 2. 3.**.

b. General case: stochastic cash flow
Imagine now that the flow is not known at the date of the pricing: its value at future date of payment is random.
Present value of a stochastic flow C: should we take: $\frac{E(C)}{(1+r)^t}$?

Imagine you can choose between two flows to be received in one year:

- first flow is worth 500 in any case,
- second flow (lottery) is worth $\begin{cases} 1000 & \text{with proba } 1/2 \\ 0 & \text{with proba } 1/2 \end{cases}$

Both flows have the same expectation but you certainly prefer the first one.
That means that you are "risk averse".

If you have to pay today to receive these flows in 1 year, you will be ready to pay more for the first one (which is worth $\frac{500}{1+r}$).
An interpretation is that your preferences (the values you give for a future flow) can be represented by a utility function U which is concave ($U(x)$ is the value attributed to the certain flow x):

To win 1 million is much better than 0 but to win 2 millions is not as much better than 1 million.
A concave utility function means risk aversion: one prefer to receive $E(C)$ for sure than to receive C. Indeed, since U is concave, the inequality of Jensen states: $E(U(C)) \leq U(E(C))$, i.e.: the expected utility of a flow is less than the utility of its expected value.
$\frac{E(C)}{(1+r)^t}$ is the PV of the cash flow $E(C)$ at t (deterministic cash flow), then the PV today of the cash flow C at t is less than $\frac{E(C)}{(1+r)^t}$.

Generally the utility function of the investor is not estimated, only summarized by a discount rate taking it into account with the investment risk as well: the Present Value of a stochastic flow C is then expressed as:

$$\begin{cases} \frac{E(C)}{(1+\mu)^t} & \text{if } \mu = \text{ discrete interest rate} \\ E(C)e^{-\mu t} & \text{if } \mu = \text{ continuous interest rate} \end{cases}$$

We have most often $\mu > r$ and $\mu - r$ increases with the risk on the asset.

The method of Discounted Cash Flows (DCF) is the classical way to evaluate companies/projects in corporate finance.

For quoted assets, the prices settle at levels that reflect an average of the preferences of the investors. The implied utility function is the one of the market (notion of "representative investor"). When the market risk aversion increases, an equity price generally goes down even if nothing has changed for the company itself.

3. Return

We will use the term "return" for the rate of return.
= gain or loss of a security in a given period (income + capital gains), divided by its initial value.
Usually it is annualised.

Let S_s be the price of a given security at time s.
⋆ At time t, we will compute its annualised return realised over the past t years as μ such that:

$$S_t = S_0(1+\mu)^t \quad \text{or} \quad S_0 = \frac{S_t}{(1+\mu)^t} \qquad (*)$$

Like for the interest rates, we may consider discrete or continuous (rate of) returns, hence we may write $S_0 = S_t e^{-\mu t}$ instead of (*).
This is an observed (or realised) return, at the end of the investment.

⋆ At time 0, we can compute an anticipated return μ by $S_0 = \dfrac{E(S_t)}{(1+\mu)^t}$ or $S_0 = e^{-\mu t}E(S_t)$.

This return corresponds to the discount rate appearing in the above discussion about the present value computation.

μ depends on the risk of the stock.
The riskier the stock is (ie the more dispersed its future values are), the larger μ:
the investors require compensation for the risk they assume by buying this stock.

More precisely, the value of μ depends on the systematic risk of the investment.

See the Capital Asset Pricing Model (2nd semester course on Portfolio Theory):
An investor should not require a higher expected return for bearing nonsystematic risk, as this risk can be almost completely eliminated by holding a well-diversified portfolio. But he generally requires a higher expected return than the risk-free interest rate for bearing positive amounts of systematic risk.

⋆ In the case of an investment with multiple future cash flows, the return of the investment will be computed as the interest rate that makes the present value of the future net cash flows equal to the initial investment (it corresponds in corporate finance to the internal rate of return, IRR): $PV = \sum_{t=1}^{T} \dfrac{C_t}{(1+\mu)^t}$, where C_t is the cash flow at time t.

⋆ In market finance, logarithmic returns are sometimes used, for more frequent observations.
For Δt small, the return on a period, $\frac{S_{t+\Delta t}-S_t}{S_t}$, can be approximated by $\ln \frac{S_{t+\Delta t}}{S_t}$ as $\ln(1+x) \sim x$.
Ex: daily returns on $[j\delta, (j+1)\delta]$, $0 \le j \le n-1$: $R_j = \frac{S_{(j+1)\delta}-S_{j\delta}}{S_{j\delta}}$ is close to $\ln \frac{S_{(j+1)\delta}}{S_{j\delta}}$.
Advantage: logarithmic returns are additive.

Chapter II Bonds and term structure of interest rates

1 Bonds prices, yields, and duration

At the issue date, an interest rate is chosen, which reflects an "average" of the risk-free yield curve at that date + the issuer's risk-premium (has to be sufficient to attract investors)
= **Coupon rate** (coupon amount per year / face value) = coupon as a percentage of the face value.

It is fixed at the issue date, and mentioned in the bond short description, ex: UKT 4 1/2 07/06/28, MEX 5.625 01/17...

Most bonds pay semi-annual coupons (ex: UK, US,...). To simplify, we will generally write prices (or other quantities) in the case of annual coupons (like in France, Germany...) - see exercise 8. for an extension to other frequencies.

Standard bond: Face value (or principal) = N, payments = $C, ...C, C + N$. coupon rate: $c = C/N$. Flows equivalent to: invest N each year at rate c.

For a standard bond the coupon rate corresponds, at the issue date, to the bond's rate of return (prove it).

Example of non-standard bond, in a environment of negative interest rates:

0-coupon bond issued at price 102€, for a nominal 100€ , with a maturity 2Y.

Indeed, because of the complexity of having a negative coupon, it is common on negative-yielding issues to sell the bonds at a cash price higher than par. This means the issuer pays back less than the amount borrowed, removing the need to figure out how to organise reverse coupon payments.

Bonds are generally quoted on a price-plus-accrued-interest basis. This means the price is quoted separately (as a percentage of the bond's nominal value, ie Price/Nominal in %) from the percentage coupon accrued from the last coupon date to the trade date. The buyer pays (or the seller receives) both the market price of the bond and accrued interest. This means that the market price quoted is clean of coupon effect, which allows comparisons between various bonds.
ex: Quote=100% (of the face value) at the issue date for standard bonds.

Accrued (part of the coupon already accrued) = $C \times \frac{nb\ of\ days}{base}$; ex: base = 360 (depends on calendar conventions).

The dirty price (or gross price) of a bond is the price to be paid to buy it

= **clean price** (the one quoted) + accrued (both as a percentage of the nominal).

Precisely, the amount paid for one bond is: clean price (in %) $\times N + \frac{nb\ of\ days}{base} \times C$.

(we can write (in %): dirty price = clean price $+\frac{nb\ of\ days}{base} \times c$).

A bond which is worth 100% (ie priced at its nominal value) (resp $>$ 100%; $<$ 100%) is called at par (resp: at a premium; at a discount).

Current yield = (coupon per year) / (current clean price). Not equal to the bond's rate of return.
P= clean price (quote = P/N); current yield = $C/P = cN/P$.
Bond at par: $P = N$ thus current yield = c (coupon rate).
Bond at a premium: $P > N$, thus current yield $< c$.

Ex 1: MEX 5.625 01/17 semi-annual coupons but 5.625% is the annual coupon rate.
Clean price 25/09/09 = 103.729. Current yield = 5.625%/1.03729 = 5.423%.

Ex 2: UKT 4.5 7 SEP 2034 (Issue date : 17/06/2009). Semi-annual: coupon dates: 7 March, 7 September. Clean price 30/09/14 = 123.658. Current yield = 4.5%/1.23658=3.64%.

To compare bonds with different maturities, coupons dates, amounts, face values..., the good notion is the **yield-to-maturity**: the interest rate ρ that equates the present value of (promised) future payments received from the bond with its value today.

Let P be the clean price of the bond, at a coupon date (to keep it simple). T the number of remaining coupons. The yield-to-maturity ρ is defined by:

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+\rho)^t} \text{ where } C_t \text{ is the cash flow a time } t \text{ (usually } C \text{ or } C+N\text{)}$$

We always assume that all the C_t are positive. Then there is exactly one solution ρ, as the righthand side term is a decreasing function of ρ (sum of decreasing functions).

Note: there exists a function yield in Excel (arguments: date, maturity, coupon rate, price, nominal, coupon frequency).

The bond price and the yield to maturity are negatively related (the bond price is a decreasing function of the YTM).

Note that ρ is computed under the assumption that the payments are sure (payments C_t are assumed to be non random). Then it **corresponds to the return** an investor will receive by holding the bond to maturity, if all payments are actually done.

Also, we can write as well $P = \sum_{t=1}^{T} \frac{C_t}{(1+r(0,t))^t}$ where $r(0,t)$ is the 0-coupon rate for maturity T for this issuer (or this class of risk). Then ρ is an average rate between 0 and T for this issuer or class of risk (obviously ρ depends also of the values $C_1, ..., C_T$).

At another date:

$$P = \sum_{t=0}^{T} \frac{C_t}{(1+\rho)^{\alpha+t}} \text{ with } C_t \text{ cash flow at time t, } \alpha \text{ time length until the next coupon date}$$

and when the remaining coupons are paid at times $0, 1, ...T$.

Example: YTM of a 0-coupon bond with maturity T = rate prevailing for maturity T (ie $r(0,T)$).

Standard bond at a coupon date: we have $\frac{P}{N} = c\sum_{t=1}^{T} \frac{1}{(1+\rho)^t} + \frac{1}{(1+\rho)^T}$.

Reminder (sum of a finite geometric series): $\boxed{\forall T \in \mathbb{N}^*, \forall x \neq 1, \sum_{t=1}^{T} x^t = x\frac{1-x^T}{1-x}}$. Indeed:

$(1+x+x^2+...+x^T)(1-x) = 1-x+x-x^2+x^2-...-x^T+x^T-x^{T+1} = 1-x^{T+1}$ then $\sum_{t=0}^{T} x^t = \frac{1-x^{T+1}}{1-x}$.

Therefore $\sum_{t=1}^{T} x^t = x\left(\sum_{t=1}^{T} x^{t-1}\right) = x\left(\sum_{t=0}^{T-1} x^t\right) = x\frac{1-x^T}{1-x}$.

We get: $$\sum_{t=1}^{T} \frac{1}{(1+\rho)^t} = \frac{1}{1+\rho}\frac{1-\frac{1}{(1+\rho)^T}}{1-\frac{1}{1+\rho}} = \frac{1}{\rho}\left[1-\frac{1}{(1+\rho)^T}\right].$$

Therefore: $$\boxed{\frac{P}{N} = \frac{c}{\rho}\left[1-\frac{1}{(1+\rho)^T}\right] + \frac{1}{(1+\rho)^T}}.$$

For $\rho = c$, we get: $\frac{P}{N} = 1$ ie the bond is at par.

Since P is a decreasing function of ρ, we conclude that:

$\rho = c \Longleftrightarrow P = N$ (bond at par), observed for example at issue time,
$\rho < c \Longleftrightarrow P > N$ (bond at a premium),
$\rho > c \Longleftrightarrow P < N$ (bond at a discount).

Duration (Macaulay)
The YTM corresponds to an average rate on the life of the bond. But what is the time length of the investment? Part of it for 1Y, 2Y,...

Definition: the average life of a given investment, called duration, is computed by weighting each payment date by the present value of the corresponding (promised) future payment divided by the price at that date.

For a standard bond, the future payments are deterministic (under assumption that the coupons are certain). To compute the duration, the payments C_t are discounted at rate ρ (like in the price formula):
$$D = \frac{1}{P}\sum_{t=1}^{T}\frac{tC_t}{(1+\rho)^t}.$$
Note that the sum of the weights is equal to 1. The weight at time t is the relative contribution of payment at t to total price P, or the proportion of PV that gets paid at t.

Remarks:

- we could have defined the duration by: $D = \frac{1}{P}\sum_{t=1}^{T}\frac{tC_t}{[1+r(0,t)]^t}$ i.e. discounting at time t with the 0-coupon rate of the issuer $r(0,t)$ instead of the yield-to-maturity ρ. The computations are simplified by using ρ.

- this definition can be extended to random cash-flows, the weight for time t remains the proportion of PV that gets paid at t (used in exercise 16).

Ex: - the duration of the zero-coupon bond maturing at T is T.

- Compute the duration for a perpetual annuity paying A yearly (Exercise 6.b).

Some terminology A debt instrument is *short-term* if its maturity is less than a year and *long-term* if its maturity is 10 years or longer. Else: intermediate-term.
Money market = market where short-term debt instrument are traded.
Longer-term and equities: *capital market*

US government bonds are called Treasuries, UK government bonds are called Gilts, German ones Bunds, French ones OAT (obligations assimilables du Trésor).

Short-term Treasuries are called Treasury bills. Intermediate-term Treasuries are called Treasury notes. Long-term Treasuries are called Treasury bonds.

Ex (see https://www.aft.gouv.fr): OATs (Obligations Assimilables du Trsor) are medium- and long-term Treasury securities with a maturity of 2 to 50 years.

The assimilation technique consists in attaching a new issue to an OAT already created and issued previously. The State thus has the possibility of issuing tranches of bonds whose issue price is adjusted by market developments, but whose characteristics remain identical to those of the initial issue. In this way, the State avoids a large number of bonds with different characteristics and ensures the liquidity of the outstanding issues. OATs are the only form of medium- and long-term financing for the State.

In 2024, there are nearly seventy OATs in circulation, which are designated by their rate and maturity date (e.g. "OAT 0.00% May 25, 2032" or "OAT 0.50% May 25, 2072") and whose characteristics are shown on its title sheet published on the AFT website (https://www.aft.gouv.fr/index.php/fr/encours-detaille-oat).

2 Term structure of interest rates

also known as the yield curve.

At a time t, a 0-coupon yield curve displays the relationship between spot rates of 0-coupon bonds and their remaining time to maturity.

Reminder: a zero-coupon bond with maturity date T is a contract which guarantees the holder 1\$ to be paid on the date T. Its price at t is denoted by $B(t,T)$, it corresponds to the discount factor between t and T.
The price at time t of the 0-coupon paying 1 (in the given currency) at time T is:

$$B(t,T) = \frac{1}{[1 + r(t,T)]^{T-t}},$$ where $r(t,T)$ is the 0-coupon (discrete) rate at time t.
for an investment until T.

The 0-coupon yield curve at a given time called 0 is the curve: $t \mapsto r(0,t)$, on an interval $[0,T]$ as long as possible.

This curve is used to price the other fixed income securities. For a coupon-bearing bond, the price is obtained by discounting each coupon or nominal payment with the rate corresponding to the payment date.
Yield curves can be drawn for a given issuer, or for a given class of risk in a given country or area.

2.1 "Risk-free" yield curve

Sovereign yield curves play an important role in the economy. They are graphed from the prices of a list of liquid government bonds and some other debt securities. In a given country, the Government yield curve is used as a benchmark for other debt in the market, such as mortgage rates or bank lending rates, and can predict changes in economic output and growth. In particular, we will see that the yield curve is a measure of the market's expectations of future interest rates given the current market conditions.

This curve is called the "risk-free" (or "default-free") yield curve, at least in low risk countries. But with the current debt levels in most countries, the existence of a "risk-free" rate is not so clear.

a. Yield curve building

We look for the curve at time 0. The curve should be graphed from 0-coupon bonds prices ($B(0,t)$ for different t), but there does not exist quoted 0-coupons bonds for all the terms.
Therefore the curve is deduced from the prices of the coupon-bearing bonds, plus potentially other fixed income instruments. The chosen securities need to be liquid, to have fixed cashflows, and no special features. Currently, swaps (see later) are frequently used, their prices are indeed linked to the 0-coupon bonds prices.

Example: the Euro area yield curve is published daily by ECB, available here:
https://www.ecb.europa.eu/stats/money/yc/html/index.en.html
"The zero coupon curve represents the yield to maturity of hypothetical zero coupon bonds, since they are not directly observable in the market for a wide range of maturities. They must therefore be estimated from existing zero coupon bonds and fixed coupon bond prices or yields."

A coupon-bearing bond can be regarded as a collection of 0-coupon bonds, each maturing on one of the different coupon-payment dates. Then its price is a linear combination of 0-coupons bonds prices, for example, assuming annual coupons and a maturity T:

$$\text{at time } 0,\ P = \sum_{t=1}^{T} CB(0,t) + NB(0,T) = \sum_{t=1}^{T} \frac{C}{(1+r(0,t))^t} + \frac{N}{(1+r(0,T))^T} \qquad (*)$$

The same can be written for any series of known cashflows. Then the yield curve can be deduced from the observed fixed income instruments prices. It will be a curve implicit to the quoted prices, as it is not straightforward to obtain the unknowns $\{r(0,t), t \in]0,T[\}$ from equations as above, written for several different securities (note that it may be easier to get $\{B(0,t), t \in]0,T[\}$, from the linearity).

Several methods are used to build this curve, all equivalent to minimize, for a given set of representative securities with maturities covering the longest period (up to 50Y), the spread between the observed prices (ie quoted) and the theoretical prices which would result from the curve (using a pricing like in above formula).

At a given time 0, we look for **the curve** $t \mapsto r(0,t)$ **achieving** (optimization program, with a least squares method):

$\min_{t \mapsto r(0,t)} \sum_{\text{bond j}} (P_j^{theo} - P_j^{obs})^2$, where P_j^{theo} is computed according to $t \mapsto r(0,t)$ or $t \mapsto B(0,t)$ like in (*).

Note that on $[0,T]$ the knowledge of $t \mapsto r(0,t)$ is equivalent to the knowledge of $t \mapsto B(0,t)$ and several methods have been developed, differing on which curve is fitted:

Ex: **1.** the discount function $t \mapsto B(0,t)$ can be built. This function is assumed to be continuously differentiable and - in normal economic environment - monotonically decreasing (due to the time value of money).
A classical method is to fit this function by means of polynomial splines of the second or third order (splines = piecewise polynomials), or by third order exponential splines (Vasicek-Fong, 1982), with smoothing constraints at the knots (ex: discount function polynomial function on $[0, 5Y]$, $[5Y, 10Y]$, and $[10Y, 30Y]$, with 1st and 2nd derivatives continuous at points 5Y and 10Y).

2. In Nelson Siegel (1987), another curve is fitted, indeed the continuously-compounded zero-coupon yield ($t \mapsto y(0,t)$, where $B(0,t) = e^{-y(0,t)t}$).
This paper proposed a 3 factors model for this curve (see p17) which is currently widely used.

b. Main patterns and interpretation
There are three main patterns created by the term structure of interest rates:

In general, interest rates increase with time. The longer the term, the bigger should be the reward (as there is more risk).

1) Normal Yield Curve:

As its name indicates, this is the yield curve shape that forms during normal market conditions, wherein investors generally believe that there will be no significant changes in the economy, such as in inflation rates, and that the economy will continue to grow at a normal rate.
During such conditions, investors expect long-term FI securities to offer higher yields than short-term FI securities. This is a normal expectation of the market because short-term instruments generally hold less risk than long-term instruments; the farther into the future the bond's maturity, the more time and, therefore, uncertainty the bondholder faces before being paid back the principal. To invest in one instrument for a longer period of time, an investor needs to be compensated for undertaking the additional risk.

For the 2 other shapes, this property is still valid, but other factors are stronger.

2) Flat or quasi-flat Yield Curve:

A flat yield curve usually occurs when the market is making a transition that emits different but simultaneous indications of what interest rates will do. In other words, there may be some signals that short-term interest rates will rise and other signals that long-term interest rates will fall. This condition will create a curve that is flatter than its normal positive slope. When the yield curve is flat, investors can maximize their risk/return tradeoff by choosing FI securities with the least risk, or highest credit quality. In the rare instances wherein long-term interest rates decline, a flat curve can sometimes lead to an inverted curve.

3) Inverted Yield Curve:

These yield curves are rare and necessarily transitory.They form during extraordinary market conditions wherein the expectations of investors are the inverse of those demonstrated by the normal yield curve. In such abnormal market environments, bonds with maturity dates further into the future are expected to offer lower yields than bonds with shorter maturities. The inverted yield curve indicates that the market currently expects interest rates to decline as time moves farther into the future, which in turn means the market expects yields of long-term bonds to decline.

Example: the U.S. Treasury Yield Curve was inverted during the first trimester 2007 (also at 2006 end), quasi-flat in April 2007, and then normal as short rates went down. See: http://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=yieldYear&year=2007. The slope of the yield curve (e.g. difference between 2 years and 10 years rates) historically has been a good recession predictor.

At mid-Oct 2024, 11 countries have an inverted yield curve. 14 have a partially inverted yield curve (Germany, US, UK, Canada...) have a partially inverted yield curve.

See for example http://www.worldgovernmentbonds.com/inverted-yield-curves/.

At mid-Oct 2023, 27 countries had an inverted yield curve (United Kingdom, U.S., Germany, France, Spain, Australia,...), but Japan had a normal curve.

You may be wondering why investors would choose to purchase long-term FI investments when there is an inverted yield curve, which indicates that investors expect to receive less compensation for taking on more risk. Some investors, however, interpret an inverted curve as an indication that the economy will soon experience a slowdown, which causes future interest rates to give even lower yields. Before a slowdown, it is better to lock money into long-term investments at present prevailing yields, because future yields will be even lower.

Classical theories of the term structure

Studying the term structure of interest rates boils down to wondering about market participants' preferences (investors, borrowers) for curve maturities. Market participants' preferences can be guided by their expectations, the nature of their liability or asset, and the level of risk premium they require for offsetting their risk aversion.

We will see that forward rates can be computed from the yield curves.

The *pure expectations theory* postulates that forward rates exclusively represent future short-term rates as expected by the market, whereas the *pure risk premium theory* (including the liquidity premium theory and the preferred habitat theory) postulates that forward rates exclusively represent the risk premium required by the market to hold longer-term bonds. The market segmentation theory postulates that each of the two main market investor categories (the one preferring short bonds, the other long bonds) is invariably located on the same curve portion (short, long). As a result, short and long curve segments are perfectly impermeable.

2.2 Credit risk

In fact, a bond is never perfectly risk-free. Its future cash flows can go to 0 in case of default.
I.e. there is a default risk (non payment or partial payment, the future flows are C and $C+N$ or only part of them).
The recovery rate is function of the issuer's rating (reputation and quality).

Compare 2 identical bonds (dates, coupons and principal) but with different issuers. If Issuer2 is riskier than Issuer1, investors will prefer the 1st bond, then Issuer2's bond has a lower price.

Equivalently, to borrow, Issuer2's has to propose a better interest rate (i.e. a higher coupon).

	Issuer	Issue date	Common Name
Examples of sovereign bonds:	ITALY	09/02/2016	BTP 2.7 01/03/47
	UK	21/09/2016	UKT 1.5 22/07/47

Default risk

Default = a failure to pay back interests or principal according to the terms of the contract (agreement)
2 sources of default:
1. the shareholders of a corporation can decide to break the debt contract (comes from their limited liability status: they are liable of the corporation's losses only up to their investment in it).
2. the creditors can prompt bankruptcy when specific debt protective clauses, known as covenants, are infringed.

In case of default, 3 eventualities:

- Default can lead to immediate bankruptcy. Depending on their debt securities' seniority and face value, creditors are fully, partially or not paid back, thanks to the sale of the firm's assets. The percentage of the interests and principal they receive, according to seniority, is called the recovery rate.

- Default can result in a reorganization of the firm within a formal legal framework (eg under Chapter 11 of the American law, corporations that are in default are granted a deadline so as to overcome their financial difficulties. This depends on the country's legislation.)

- Default can lead to an informal negotiation between shareholders and creditors. This results in an exchange offer through which shareholders propose to creditors the exchange of their old debt securities for a package of cash and newly issued securities (eg: Greece in 2011, with a haircut of 50%, see later p.30).

How to measure the risk

The task of rating agencies consists in assessing the default probability of corporations through what is known as rating.

A rating is a ranking of a bond's quality, based on criteria such as the issuer's reputation, management, balance sheet, and its record in paying interest and principal. The 3 major ones (the *Big Three*) are Moody's (created in 1909), Standard and Poor's, and Fitch. At 2015 end, they were providing 96.5% of all the ratings in the world (93% of European Union in 2022).

Known issues:

- independence : potential conflicts of interest, when issuers pay to obtain their grading,
- mistakes: issuers having (very) good grades just before a default (Vivendi Universal defaulting 2 weeks after having obtained its AAA, Lehman Brothers rated A at its default's time, other examples include Enron, Worldcom...), also, products rated AAA defaulting (subprimes).

Following the Great Financial Crisis started in 2008 (or 2007) due to these subprimes, the rating market is now more regulated: in Europe, by the European Securities and Markets Authority (ESMA), since 2011, in the US by the *Wall Street Reform and Consumer Protection Act* (called Dodd-Frank Act) of 2010.

The titles belonging to the 4 higher categories (AAA,AA,A and BBB) are called "investment grades". The lower categories are called "high yield" or "junk bonds".

S&P:	AAA,AA,A,BBB,BB,B,CCC	and AA includes AA+,AA,AA-...
Moody's:	Aaa,Aa,A,Baa,Ba,B,Caa	and Aa includes Aa1,Aa2,Aa3...

Government securities are usually considered to have no credit risk and are taken as reference to build the term structure of interest rates ("risk-free") - but in fact, states are attributed a rating as well. More recently, the debt of several countries has reached levels where the credit risk cannot be considered as negligible anymore.
For example, currently, the German bonds are used as the reference for euro-denominated bonds.

S&P rating for countries are available on www.worldgovernmentbonds.com
(world's map: www.worldgovernmentbonds.com/world-credit-ratings/).

Corporate bonds are affected by default or credit risk. Their yields contain a default premium over government bonds, accounting for total default or credit risk.

The difference between the yields on a corporate bond and a (low risk) government bond is called the credit spread (or just spread). It reflects the extra compensation investors receive for bearing credit risk. The credit spread is then greater for lower-rated bonds.

def: spread at date 0 for a given maturity T: yield-to-maturity $(0, T)$ = risk-free rate $(0, T)$ + spread $(0, T)$

(note: the spread is mainly explained by the credit risk but contains also a liquidity component).
The spread can also be defined as the average spread for the whole curve.

A yield curve can be built for a given credit-rating category (if enough prices available: enough issuers and enough maturities): the rating agencies periodically publish the yields curves for corporate bonds based on their ratings. These curves represent the average yields of the bonds belonging to a given credit-rating category.

How changes in the credit spread affect the bondholder
A credit upgrade on a specific corporate bond, say from S&P BBB to A, will narrow the credit spread for that particular bond (and increase the bond price) because the risk of default lessens. If interest rates are unchanged, the total yield on this "upgraded" bond will go down in an amount equal to the narrowing spread (and the price will increase accordingly, see below).
Downgrades will diminish the bond price.

So **credit risk** includes: total default risk and downgrading risk.

3 Risk factors of bond prices

3.1 1st risk source: changes in the reference rates

The government yield rates are used as the reference rates, called "risk-free rates", even if not so riskless anymore.
At issue date, for a standard bond, the coupon rate is determined such that price = face value, generally.

Later, for existing bonds:
how evolve the bonds prices if the rate for any term increase (translation of the curve)?
The new issued bonds are proposing a higher interest rate $\Rightarrow$ the prices for existing bonds go down.

The new bonds are now (interest rate is increased) issued with a better rate. Nobody accept to buy the old bonds unless their prices go down.
The interest rate risk is the most important for usual bonds (more impact than credit risk).

• **Sensitivity to a translation of the reference** (or risk-free) **yield curve**

For a general series of cash-flows $\{C_t,\ 1 \leq t \leq T\}$, due by a given issuer, we look for the sensitivity of the price to a small translation of the risk-free rate curve, assuming constant spreads for the credit risk premium.

The price can be written: $P(\rho) = \sum_{t=1}^{T} \frac{C_t}{(1+\rho)^t}$.

Each rate moves by Δr, thus the YTM moves roughly by $\Delta\rho = \Delta r$. Impact on the price?

We have:

$$P'(\rho) = -\sum_{t=1}^{T} \frac{tC_t}{(1+\rho)^{t+1}} = -\frac{1}{1+\rho}\sum_{t=1}^{T} \frac{C_t t}{(1+\rho)^t} = -\frac{DP(\rho)}{1+\rho}, \text{ since } D(\rho) = \frac{1}{P(\rho)}\sum_{t=1}^{T} \frac{tC_t}{(1+\rho)^t}.$$

From $\Delta P(\rho) \sim P'(\rho)\Delta\rho$ (linear approximation for a small variation $\Delta\rho$), we get: $\frac{\Delta P}{P} \sim -\frac{D(\rho)}{1+\rho}\Delta\rho$.

Therefore, the duration measures the bond price sensitivity to a translation of the rates curve.
Precisely, the instantaneous rate of change (or relative variation) of the price is measured by the sensitivity (or modified duration):

$S = \frac{D}{1+\rho}$, available in the quotations (as the modified duration). Note that $P'(\rho) = -S(\rho)P(\rho)$.

The rate of change in the price, $\frac{\Delta P}{P}$, implied by a small variation $\Delta\rho = \Delta r$ can therefore be approximated as: $\boxed{\frac{\Delta P}{P} \sim -S\Delta r}$ (i.e. the percent change in P is roughly equal to $-S\times$ change in rate).

When the interest rate drops by 10 basis points (1 bp = .01% = 0.0001), a bond with a sensitivity of 7.2 rises in price on the order of 0.72%.
(Traders are using the DV01, barely different than the sensitivity. It is just the dollar change given a one basis point yield change. DV01 = sensitivity * price * 0.0001).

$\Rightarrow$ long term bonds (ie with more weight on far maturities, thus higher duration) are more sensitive to such a rate move.

This notion of duration is fundamental in bond portfolio risk management and in asset-liability management (ALM) for any financial institution.
The duration gap is the difference between the duration of assets and liabilities. If this gap is positive (respectively negative), there will be losses when interest rates go up (respectively go down).

It is finally important to remember that, though bonds are perceived as low risk assets:
even though a bond has a substantial initial interest rate, its return can turn out to be negative if interest rates rise (ie the loss in price can exceed the current yield).

Since larger variations can affect the yield curve, a more precise approximation may be needed.

Second order term: to get a better approximation, we keep the second order term in the Taylor expansion: $\Delta P \sim -SP\Delta\rho + \frac{1}{2}P''(\rho)(\Delta\rho)^2$.

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+\rho)^t} \Rightarrow P'(\rho) = -\sum_{t=1}^{T} \frac{tC_t}{(1+\rho)^{t+1}} \Rightarrow P''(\rho) = \sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+\rho)^{t+2}} > 0 \text{ (for positive } C_t).$$

P is then a convex function of ρ. $\mathcal{C} = \frac{P''(\rho)}{P(\rho)}$ is called the convexity of the bond (the more convex P, the higher $\mathcal{C}$). The rate of change induced by a variation $\Delta\rho = \Delta r$ of the rate is therefore approximated by: $\boxed{\frac{\Delta P}{P} \sim -S\Delta r + \frac{1}{2}\mathcal{C}(\Delta r)^2}$

For large yield moves, regardless of rising or falling interest rates, the bonds with higher convexity are always preferable.

Note that callable bonds exhibit often negative convexity because when interest rates fall, the issuer can buy back the bonds, which limits the price appreciation that the bondholder would otherwise have.

• **Sensitivity to other deformations of the reference** (or risk-free) **yield curve**

The value of a bond is in fact affected by changes in interest rates of all possible maturities. There is more than one risk: beyond translation of the curve, slope changes or torsions can happen, whose effect should be taken into account when assessing a bond risk.

Nelson and (1987) proposed a model of the yield curve

$$y(0,t) = b_1 + b_2\frac{1-e^{-\lambda t}}{\lambda t} + b_3(\frac{1-e^{-\lambda t}}{\lambda t} - e^{-\lambda t}).$$

where $y(0,t)$ denotes the continuously-compounded zero-coupon nominal yield at time 0 for maturity t (i.e. such that $B(0,t) = e^{-y(0,t)t}$), and b_1, b_2, b_3, and λ are parameters (depending on the current date).

At each date, the parameter $b = (b_1, b_2, b_3, \lambda)$ will be estimated by minimising either the sum of squared price errors or the sum of squared yield errors, a described above (section 2.1.a).

Just from the choice of the 4 parameters, the model can generate a variety of yield curve shapes including upward sloping, downward sloping, humped, and inversely humped, and it is well suited to assess the sensitivity of the bond prices to the deformations of the reference yield curve.

Also, we can easily give economic interpretations to the parameters b_1, b_2, and b_3.

Indeed, we can interpret them as a **level factor**, a **slope factor**, and a **curvature factor**, respectively (which is particularly relevant as several studies show that these factors are sufficient to describe correctly the yield curve changes: in particular, Litterman and Scheinkman (1991) and Knez, Litterman, and Scheinkman (1994) use principal components and factor analysis to extract factors in bond returns. They conclude that a large portion (up to 98%) of bond return variation can be explained by the first three principal components or factors, and these 3 factors correspond exactly to the level, the slope, and the curvature, as above).

To see this, note that the loading on b_1 is 1, a constant that doesn't depend on the maturity. Thus b_1 affects yields at different maturities equally and hence can be regarded as a **level** factor.

The loading associated with b_2 is $\frac{1-e^{-\lambda t}}{\lambda t}$, which starts at 1 but decays monotonically to 0. Thus b_2

affects primarily short-term yields and hence changes the **slope** of the yield curve.
Finally, factor b_3 has loading $\frac{1-e^{-\lambda t}}{\lambda t} - e^{-\lambda t}$, which starts at 0, increases, and then decays. Thus b_3 has largest impact on medium-term yields and hence moves the **curvature** of the yield curve.

In short, the yield curve at any point of time can be expressed as a linear combination of the level, slope and curvature factors, the dynamics of which drive the dynamics of the entire yield curve.

Remark: In the original Nelson-Siegel formulation, the parameter λ may change with time. But it can be treated as fixed with little degradation of fit. Note that the parameter λ governs the exponential decay rate; small values of λ produce slow decay and can better fit the curve at long maturities, while large values of λ produce fast decay and can better fit the curve at short maturities. λ also governs where the loading on b_3 achieves its maximum.

An implication of this three-factor yield curve model is the natural movement that it suggests toward 'vector' sensitivity measures:
the usual duration measure works well only when the yield curve undergoes parallel shifts and when the shifts are small. It breaks down for more complicated yield curve movements. The three-factor term structure model suggests generalized duration components corresponding to the level, slope, and curvature risk factors.

For an arbitrary change of the yield curve, the price change can be decomposed into changes in the 3 risk factors: indeed, the price at time t of a given bond with cash-flow $\{C_t, t = 1, ..., T\}$ will be expressed as $P = \sum_{t=1}^{T} C_t e^{-ty(0,t)}$, and the yield curve $t \mapsto y(0,t)$ is linear in the 3 factors.
Hence we can compute a sensitivity component associated with each risk factor, the first component being the usual duration (see Diebold, F.X., Ji, L. and Li, C. (2006) *A Three-Factor Yield Curve Model: Non-Affine Structure, Systematic Risk Sources, and Generalized Duration*).

3.2 2nd risk source: credit risk

default risk $\Rightarrow$ part of coupons or nominal not reimbursed
+ (credit) downgrade risk or spread risk $\Rightarrow$ the spread increases and the bond value goes down (hence negative effect if you don't hold the bond until its maturity, also, anyway, the default risk increases).

YTM is computed under the assumption that the payments are sure. In fact, it is the maximum rate of return possible (ie if no default).

But the actual return ρ^* is random. For example for a 1-year investment:
ρ^* can be -100% (total loss) or between -100% and 0% if the recovery rate is positive.

Ex: 0-coupon 1-year ; price = 100, cash-flow = 106 in 1 year. YTM = 6%.

Assumed: full default with probability 1%. Then $\rho^* = \begin{cases} +6\% & \text{with proba } 99\% \\ -100\% & \text{with proba } 1\% \end{cases}$

YTM is the maximum of possible values for ρ^*, we have $\rho^* \leq YTM$ then the expected return satisfies $E(\rho^*) \leq$ YTM (equality iff ρ^* is deterministic, ie for a risk-free bond).

Note: The credit spread (given by: YTM = r+spread) involves 2 components:

$$\text{spread} = \underbrace{\text{YTM} - E(\rho^*)}_{induced\ by\ default\ risk} + \underbrace{E(\rho^*) - r}_{default\ risk\ premium}$$

YTM$-E(\rho^*)$ appears just from the default risk (ρ not always equal to YTM),
$E(\rho^*) - r$ is the excess return required by the investors to compensate for bearing the default risk (making ρ^* random).

It involves the risk-aversion of the market (and includes sometimes a liquidity premium).
The explanation for the spread is to compensate for this drop of the expected return YTM$-E(\rho^*)$ induced by the default risk (i.e. default risk $\Rightarrow$ randomness of ρ^*).

3.3 Other risk sources

The owner of a bond faces other kinds of risks:

1. Reinvestment risk (the uncertainty about the rate at which future cashflows can be reinvested, eg: at which rate will I be able to reinvest the received coupon?).

2. Inflation risk (depending on the inflation rate, a same amount does not guaranty the same purchasing power).
Real rate = nominal rate - inflation.
Real interest rates are adjusted for changes in the price level.
Nominal interest rates are not.

3. Risk of call (some bonds, called "callable bonds", include the right for the issuer, to buy back the issued bond, at a predetermined price, usually at par, or slightly above).
Security with a single maturity (ie who cannot be retired prior to maturity) are called term-security.

4. and like for any financial asset: potential currency risk, liquidity risk, operational risk...

4 Nonstandard bonds

Answers to some risks listed above.

· **Floating Rate Bonds**: the coupon paid on these bonds is indexed to some variable interest rate, e.g.: 3-month LIBOR (London Interbank Offered Rate), Euribor (described in exercise 19), French 10-year Constant Maturity Treasury bond yield, or Federal Funds Rate (FFR).
A spread, fixed at inception, can be added to the reference rate.
They are mainly issued by financial institutions and governments.

For 40 years, LIBOR was the primary benchmark, along with the Euribor, for short-term interest rates around the world.
The London Interbank Offered Rate was a daily reference rate based on the interest rates at which banks borrow unsecured funds from other banks in the London wholesale money market (or interbank market). It is an average of interbank deposit rates offered by designated contributor banks (panel banks), for maturities ranging from overnight to 1 year. Following reforms of 2013, Libor rates are calculated for 5 currencies (was 10 before).
This reference is 50-year old and was the most broadly used interest rate benchmark in the world with an estimated open notional exposure of $350-370$ trillion across derivatives, bonds, loans and other instruments.
Each day, the British Bankers' Association surveys a panel of banks (18 major global banks for the USD Libor), asking the question, "At what rate could you borrow funds, were you to do so by asking for and then accepting inter-bank offers in a reasonable market size just prior to 11 am?" The BBA throws out the highest 4 and lowest 4 responses, and averages the remaining middle 10.
But in 2012, significant fraud and collusion by member banks connected to the rate submissions have been detected, leading to the Libor scandal, one of the biggest scandals in banking history, and resulting in huge fines ($9bn penalties paid around the world).
Some traders were seeking to adjust the overall benchmark to flatter their open positions, while some banks were seeking to make themselves look stronger during the financial crisis when low borrowing rates were viewed as a sign of strength.

Similar problems occurred with Euribor.

In 2017 the UK's Financial Conduct Authority (FCA) announced that after 2021 it would no longer persuade or compel panel banks to submit the rates required to calculate LIBOR. But some instruments involving a reference to the LIBOR rate mature after this date (exposure of around $288 billion for USD LIBOR alone). These securities will likely require conversion to an alternative Risk Free Rate (RFR), that should be based on actual transactions, e.g. SONIA (UK), SOFR (US), ESTER (EU).

Such risk free rates have been now designated for each currency for which LIBOR was published. Risk free rates are benchmarks generally based on overnight deposit rates. They are considered to be more robust as they are based upon a larger volume of observable transactions.

Usually, reset frequency = coupon payment frequency. Then most FRBs have quarterly coupons.
Floating Rate Bond prices behave quite differently from straight bond prices, which adjust to fluctuations in the market interest rate. By contrast, floaters have coupons that adjust to interest rates. This means that Floating Rate Bonds exhibit great price stability when compared to straight bonds (see also next chapter).

Interest = bonds with a low sensitivity to rates moves (see Exercise 8):
a Floating Rate Bond (FRB) pays a series of variable coupons: at time $t \in \{1, ..., T\}$, it pays
$r(t-1,t) \times$ nominal where $r(t-1,t)$ is the rate prevailing at time $t-1$ for maturity t,
and the nominal is reimbursed at T.

The price of this FRB is N at each payment date. Indeed: the FRN is equivalent to:
invest N at time 0,
then at 1, receive $r(0,1)N$ and N, re-invest N, at 2, same... ... at T, receive $r(T,T-1)N$ and N.

· **Inflation-indexed bonds**: coupons and principal are indexed to the future inflation rates.

Interest = bonds with a low sensitivity to inflation variations.
Generally (except for above FRB), to price such indexed bonds, we need to know how to price "derivative products": indeed their cash flows are obtained as functions of another financial quantity (Underlying Asset): the variable interest rate or the inflation rate.

· **Strips** (separate trading of registered interest and principal)
= 0-coupon bonds mainly created by stripping government bonds of the G7 countries.

· A lot of instruments **intermediary between stocks and bonds**
Preferred stocks: a fix dividend paid by priority before usual dividend (ie preference in dividend payments).
In case of default: priority order = after bonds, before equities.
Most preferred stocks have no voting rights associated with them, but special voting rights may exist.

Some corporate bonds, called **convertible bonds**, have the additional feature of allowing the holder to convert them into a specified number of shares of stock at any time up to the maturity date.
This feature makes these convertible bonds more desirable to prospective purchasers than bonds without it, and allows the corporation to reduce its interest payments, because these bonds can increase in value if the price of the stock appreciates sufficiently.
Again these instruments require the derivative products pricing methods as they involve optional features (option to convert).

Chapter III Derivatives

Bibliography: Options, futures, and other derivative securities, J. Hull, Prentice-Hall.

1 Description and use

A derivative security is a security whose cash flows (hence value) depend on the values of other more basic underlying variables, which may be the prices of traded securities, prices of commodities, stock indices, exchange rates, or any observable variable (temperature, inflation rate...).
Derivative securities are also known as *contingent claims*.

1.1 Forward contracts

A **forward contract** is an agreement to buy or sell an asset (called the underlying asset, U.A.) at a certain future time T (called the maturity) for a certain price (called the delivery price).

Ex1: a wheat producer can specify in advance the price at which he will sell a given part of his production. This cancels his price risk (low price when the harvest is ready).

Ex2: A company using oil for its production can fix the price it will pay for it.
One of the parties to the contract assumes a long position and agrees to buy the underlying asset on T for the specified price.
The other party assumes a short position and agrees to sell the asset on T for the same price.
The contract must specify the quantity of the asset to be delivered, the quality, the place, and the way of delivery.

A very wide range of commodities and financial assets form the underlying assets, for the forward contracts, and for the other derivatives.

At the beginning, it was for seasonal products, products that are stored to satisfy the demand: agricultural products...
Then any storable product (coffee, sugar, wood, live cattle...).

Now there exists contracts on all sort of products, even non storable like electricity.
Financial products, interest rates, stock indices ($\Rightarrow$ cash delivery), currencies, bonds...
commodities: gold, silver, copper, aluminum...

Ex: FRA (Forward Rate Agreement) = Forward on an interest rate, eg between a bank and its client. It allows to fix now, the rate for a loan that the client will take in the future.

Contracts have been created on anything whose value fluctuates, to provide an hedge.

Examples:

-weather derivatives, on temperature, quantity of snow, rain (example: for a beer producer),...

- longevity swap: allows a pension scheme to remove the risk that members live longer than expected[1].

Payoff: Let S_T denote the price of the underlying asset on the date of expiration T and K the delivery price. The terminal payoff for the long position is $S_T - K$.

If the contract is exchanged on a market:
At the time the contract is entered into, the delivery price is chosen so that the value of the forward contract to both parties is zero: it costs nothing to take either a long or a short position.

[1]payments made to the pension scheme are based on the actual longevity experience of the pensioners or on an index (eg: changes in life expectancy for the national population)

Later the contract can have a positive or negative value depending on movements in the price of the underlying asset: if this price rises after the initiation of the contract, the value of a long position becomes positive (because it gives you the opportunity to buy at T at a better price than the current market forward price).

1.2 Futures contracts

Forward contracts are OTC = private contract between 2 counterparties, difficult to sell back + counterparty risk (default for delivery or for payment).
Futures contracts are the same products, but exchanged on a market.
⇒ standardization (to get liquidity): in amount, in term (ex: Mar, Jun, Sep, Dec), in quality.
1st future contract = on the CBOT for the wheat.
One of the key role of the exchange is to organize trading so that contract defaults are minimized.
When an investor enters a contract through a broker, the broker will require the investor to deposit funds in what is termed a margin account.
1. initial margin computed to cover the biggest loss possible for one day (depends on the underlying asset)
2. at the end of each trading day, the margin account is adjusted to reflect the investor's gain or loss.

Equivalent to terminate the contract every evening.
Note that the margin account is receiving interests on the amount posted.

$F(t, T)$ forward price at date t for the maturity T.

You enter a contract at time 0. At time 1, the new forward price is $F(1, T)$, if for example $F(1, T) < F(0, T)$, you could enter the same contract with a lower price $F(1, T)$

$\Rightarrow$ potential loss $= F(0, T) - F(1, T) =$ margin call.

If negative, money is put back in your account.

At maturity, you have paid $F(0, T) - F(T, T) = F(0, T) - S_T$ which is what you "owe" (can be negative) at time T, since you have to pay $F(0, T)$ to receive the underlying asset (in case of cash delivery, after the last margin call, you owe nothing).

The Clearinghouse acts as an intermediary between buyers and sellers. Members = brokers. It determines the close price depending on all the positions.
+ it organizes the exchange (daily price movement limits on the exchanged volumes..).