Micro 1.

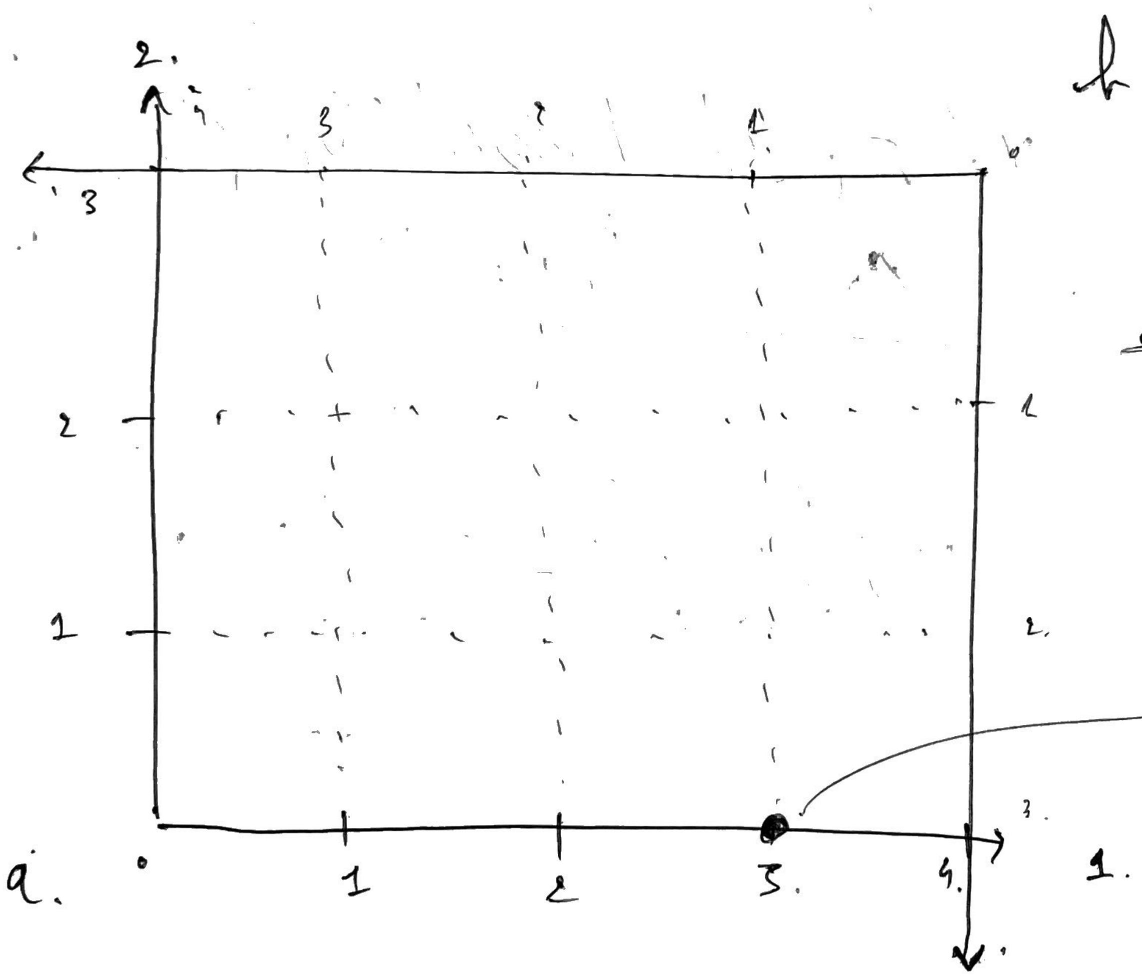
TD3

Exercice 1: Two goods. $L = \{1, 2\}$.
two consumers. $I = \{a, b\}$.

$\omega_a = (3, 0) \qquad \omega_b = (1, 3)$.

$u(x) = x_1^2 x_2^3$.

1) Find the Pareto optimal allocations
2) Determine the subset of Pareto optimal alloc. where ω initial endowment.
3) Competitive eq. are they Pareto?



$\underline{\omega = (4, 3)}$

ω_b
ω_a ..!!

the wealth of the consumer is NOT given exogenously! of not obtained!

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Def. An allocation x in the Edgeworth box is PARETO OPTIMAL if $\nexists$ x' with

$x_i' \succsim_i x_i$ for $i = a, b$ and $x_i' \succ x_i$ for some i

consumption bundle of consumer i

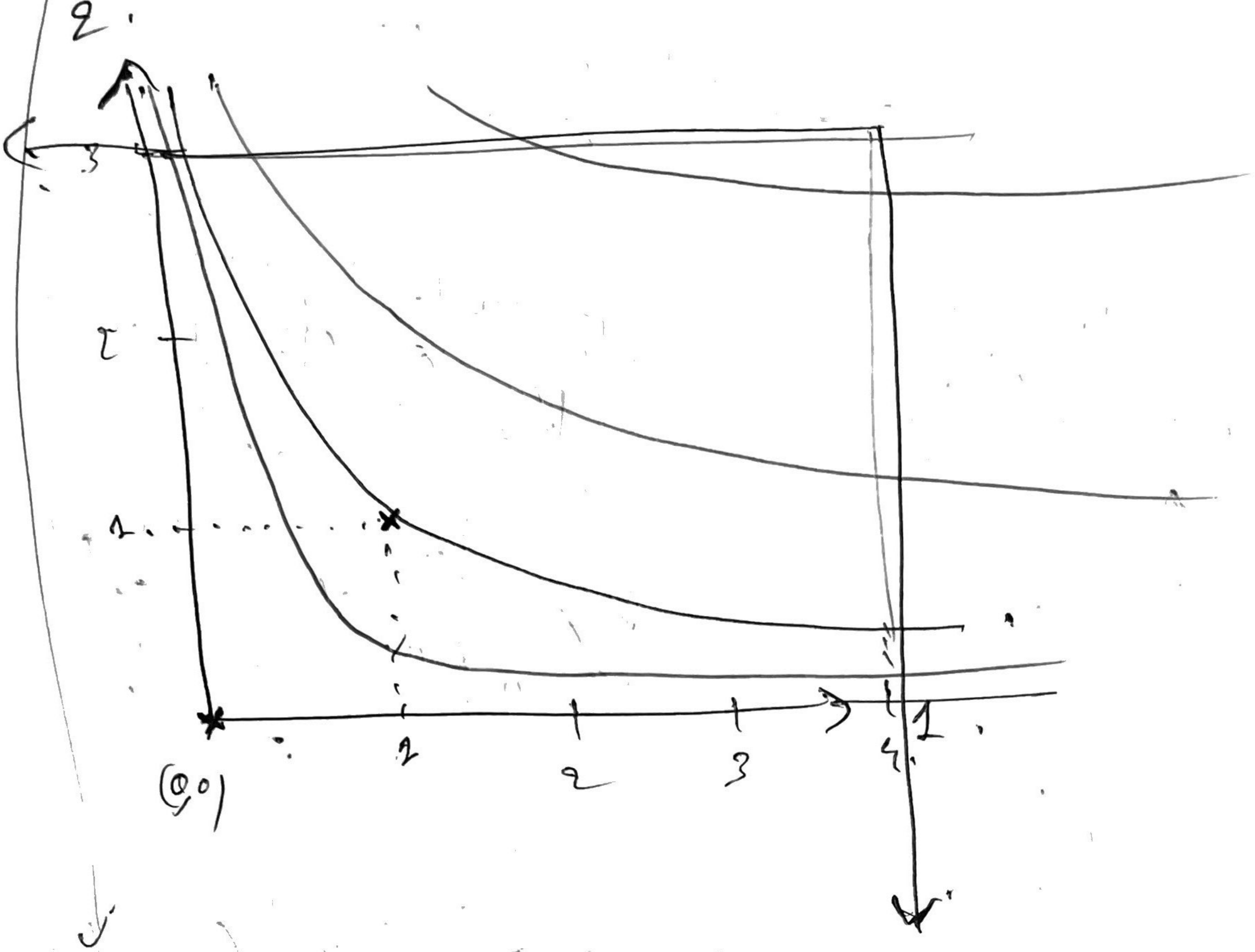
$\Downarrow$

NO WASTE!!

Note: Upper contour sets are independent of the price system!!

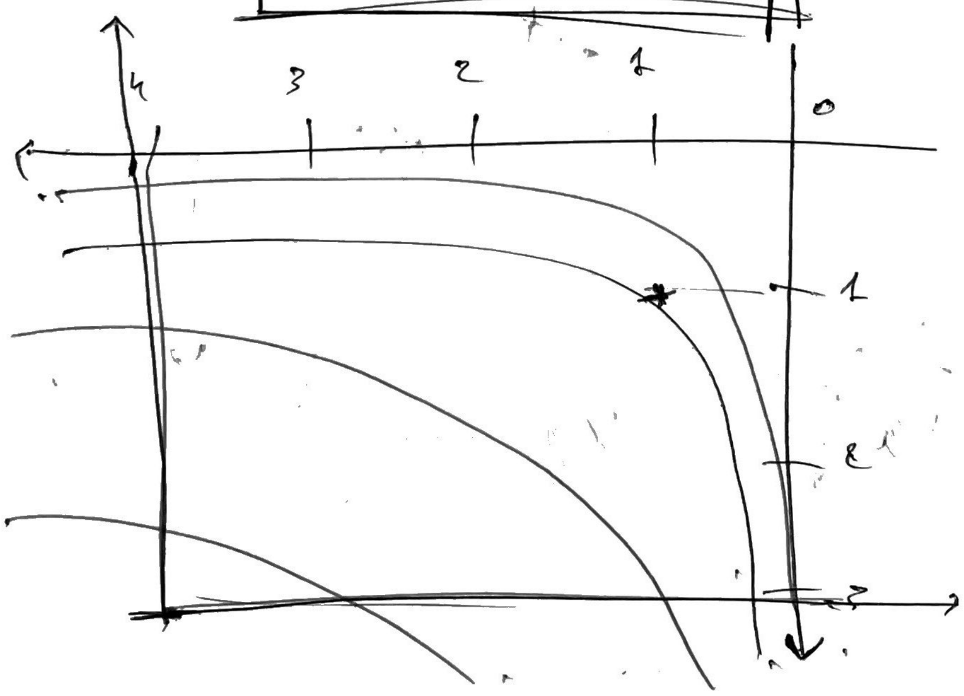
the price system gave the "budget" constraint only!

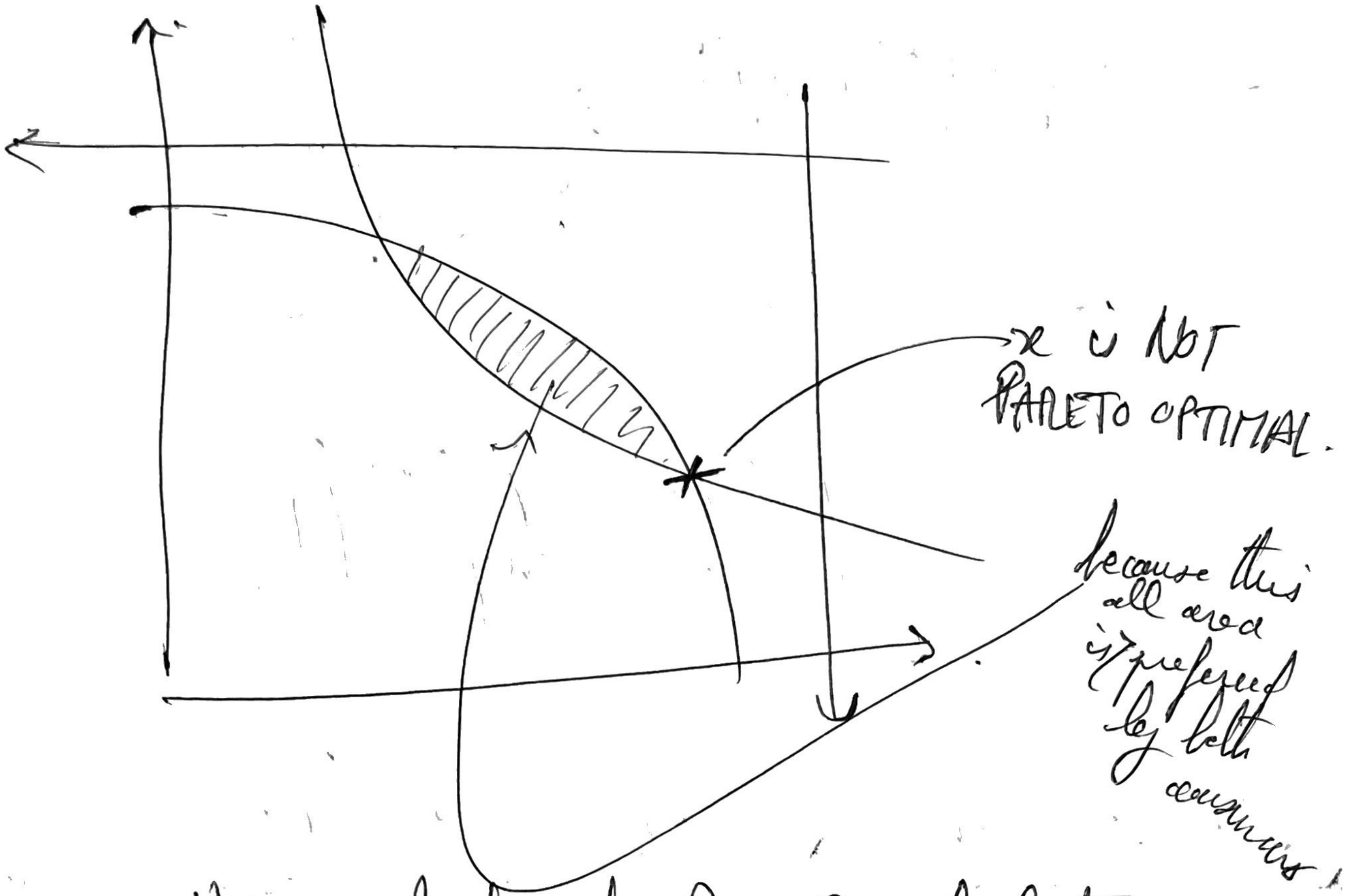
$u(x) = x_1^2 x_2^3$ for both consumers.



$u(x) = \text{cste} = u^a$

$\Longrightarrow \boxed{x_2 = \left(\frac{u^*}{x_1^2}\right)^{1/3}}$





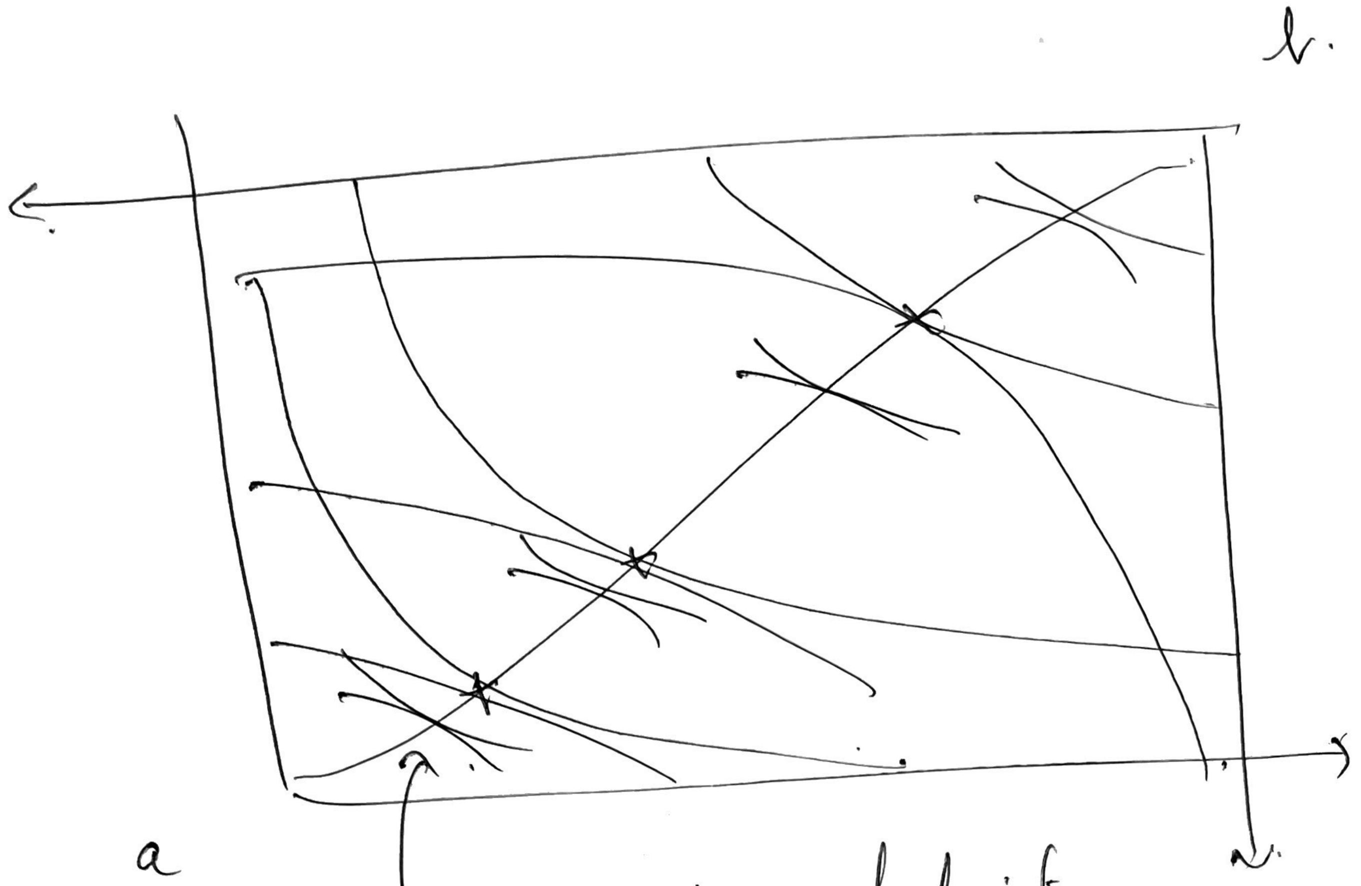
$\Longrightarrow$ Hence Pareto set $P = \{$ set of pts with indifference curves are tangent at that point. $\}$.

$$\Longrightarrow P = \{ x = \{ x_a, x_b \} \;;\; MRS^a_{12}(x_a) = MRS^b_{12}(x_b) \}.$$

$x_b = \omega - x_a$

total endowment.

slope of tangents are the MRS! recall en C12.



Pareto set = set of points with indifference curves of a and b are tangent!

Else, analytically, we can characterize the set of PARETO OPTIMA as the sol. of sols. to a UTILITY MAXIMIZATION PROBLEM.

$$\begin{cases} \max_{x_1} \; u_1(x_1). \\ \text{s.th.} \; u_2(x_2) \geq u^*. \\ \text{and} \; x_1 + x_2 = \omega. \end{cases} \longleftarrow \text{finding PARETO set!}$$

$$\Big\{ \quad u^* - u_2(\omega - x_1) \leq 0. \quad \|$$

$\Longrightarrow$ $\underset{>>0}{x_1^*}$ sol Θ. $\Longrightarrow$ KT cond Θ.

$$\begin{cases} \boxed{\nabla_{x_1} u_1(x_1^*) = -\lambda \, \nabla_{x_2} u_2(\omega - x_1^*) \qquad \lambda \geq 0.} \\ \text{and.} \; \lambda\left(u^* - u_2(\omega - x_1^*)\right) = 0 \qquad (*) \end{cases}$$

if $u_{1,2}$ monotone. $\Longrightarrow \lambda \neq 0.$

$\Longrightarrow u_2(x_2^*) = u^*$

By chain rule.

$$\nabla_{x_1} u_2(x_2) = \begin{pmatrix} \frac{\partial}{\partial x_{11}} u_2(x_2) \\ \\ \frac{\partial}{\partial x_{12}} u_2(x_2) \end{pmatrix} = \begin{pmatrix} \frac{\partial x_{21}}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}}(x_2). \\ \\ \frac{\partial x_{22}}{\partial x_{12}} \frac{\partial u_2(x_2)}{\partial x_{22}} \end{pmatrix}.$$

$$x_1 + x_2 = \omega \implies \frac{\partial x_{21}}{\partial x_{11}} = -1. \quad \text{and} \quad \frac{\partial x_{22}}{\partial x_{12}} = -1.$$

$$\implies \nabla_{\underline{x_1}} u_2(x_2) = -\nabla_{\underline{x_2}} u_2(x_2)$$

$$\implies (*) \longrightarrow \boxed{\nabla_{x_1} u_1(x_1^*) = \lambda \nabla_{\underline{x_2}} u_2(x_2^*).}$$

$$\implies \frac{\dfrac{\partial u_1}{\partial x_{11}}(x_1^*)}{\dfrac{\partial u_1}{\partial x_{12}}(x_1^*)} = \frac{\dfrac{\partial u_2}{\partial x_{21}}(x_2^*)}{\dfrac{\partial u_2}{\partial x_{22}}(x_2^*)}.$$

$$\Longleftrightarrow \boxed{MRS^1_{12}(x_1^*) = MRS^2_{12}(x_2^*).}$$

So the set of Pareto optimal allocations is given by:

$$P = \{ x = \{x_a, x_b\} ; \quad x_a + x_b = \omega \text{ and } MRS^a_{12}(x_a) = MRS^b_{12}(x_b) \}.$$

with $u_i(x_i) = x_{i1}^2 x_{i2}^3$ $\quad i = a, b$.

$$\rightarrow MRS^i_{12}(x_i) = \frac{\frac{\partial u_i(x_i)}{\partial x_{i1}}}{\frac{\partial u_i(x_i)}{\partial x_{i2}}} = \frac{2 x_{i1} x_{i2}^3}{3 x_{i1}^2 x_{i2}^2}$$

$$MRS^i_{12}(x_i) = \frac{2}{3} \frac{x_{i2}}{x_{i1}} \quad \forall i = a, b.$$

$$MRS^a_{12}(x_a) = MRS^b_{12}(x_b)$$

i.e. $\frac{\cancel{2}}{\cancel{3}} \frac{x_{a2}}{x_{a1}} = \frac{\cancel{2}}{\cancel{3}} \frac{x_{b2}}{x_{b1}}$.

$\Longrightarrow$ $\frac{x_{a2}}{x_{a1}} = \frac{\omega_2 - x_{a2}}{\omega_1 - x_{a1}}$.

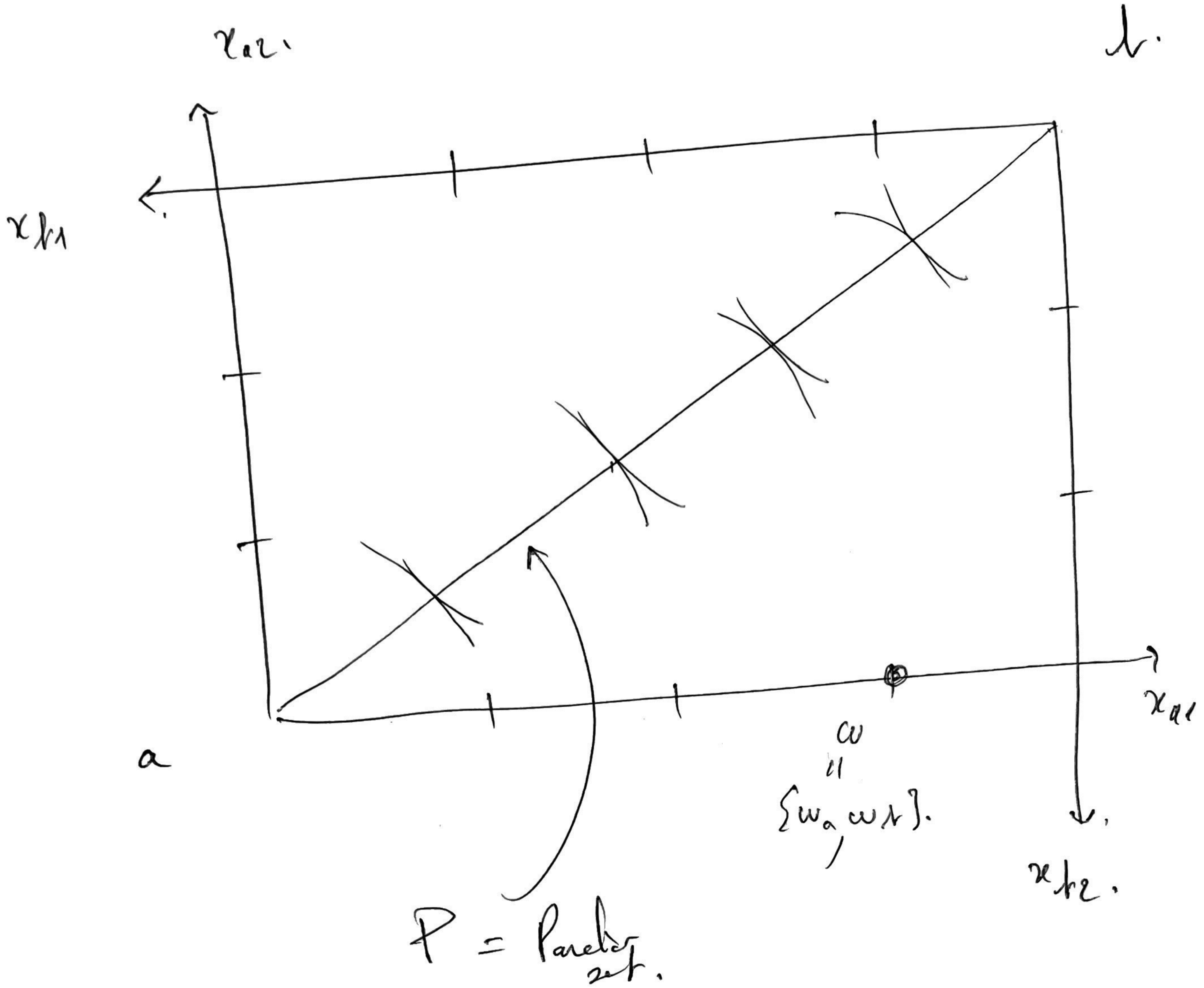
with $\omega = x_a + x_b$.

$\Longleftrightarrow$ $\frac{\omega_2}{x_{a2}} - \cancel{1} = \frac{\omega_1}{x_{a1}} - \cancel{1}$.

$\Longleftrightarrow$ $\boxed{x_{a2} = \frac{\omega_2}{\omega_1} x_{a1}}$, $= \frac{3}{4} x_{a1}$.

2) The contract curve = subset of Pareto set where both consumers are at least as good than at initial endowment.

i.e. $C = \{ x \in P,\ x_a \succsim_a \omega_a$ and $x_b \succsim_b \omega_b \}$.



$x \in C$ iff $x_{a2} = \frac{\omega_2}{\omega_1} x_{a1}$,

$x_b = \omega - x_a$.

and $u(x_a) \geq u(\omega_a) = 0$ true $\forall x_a \geq 0$.

and $u(x_b) \geq u(\omega_b) = 1^2 3^3 = 27$

$$u(x_b) = x_{b1}^{2}\, x_{b2}^{3} \geq 27.$$

$$\Big\Downarrow \quad x_b = \omega - x_a.$$

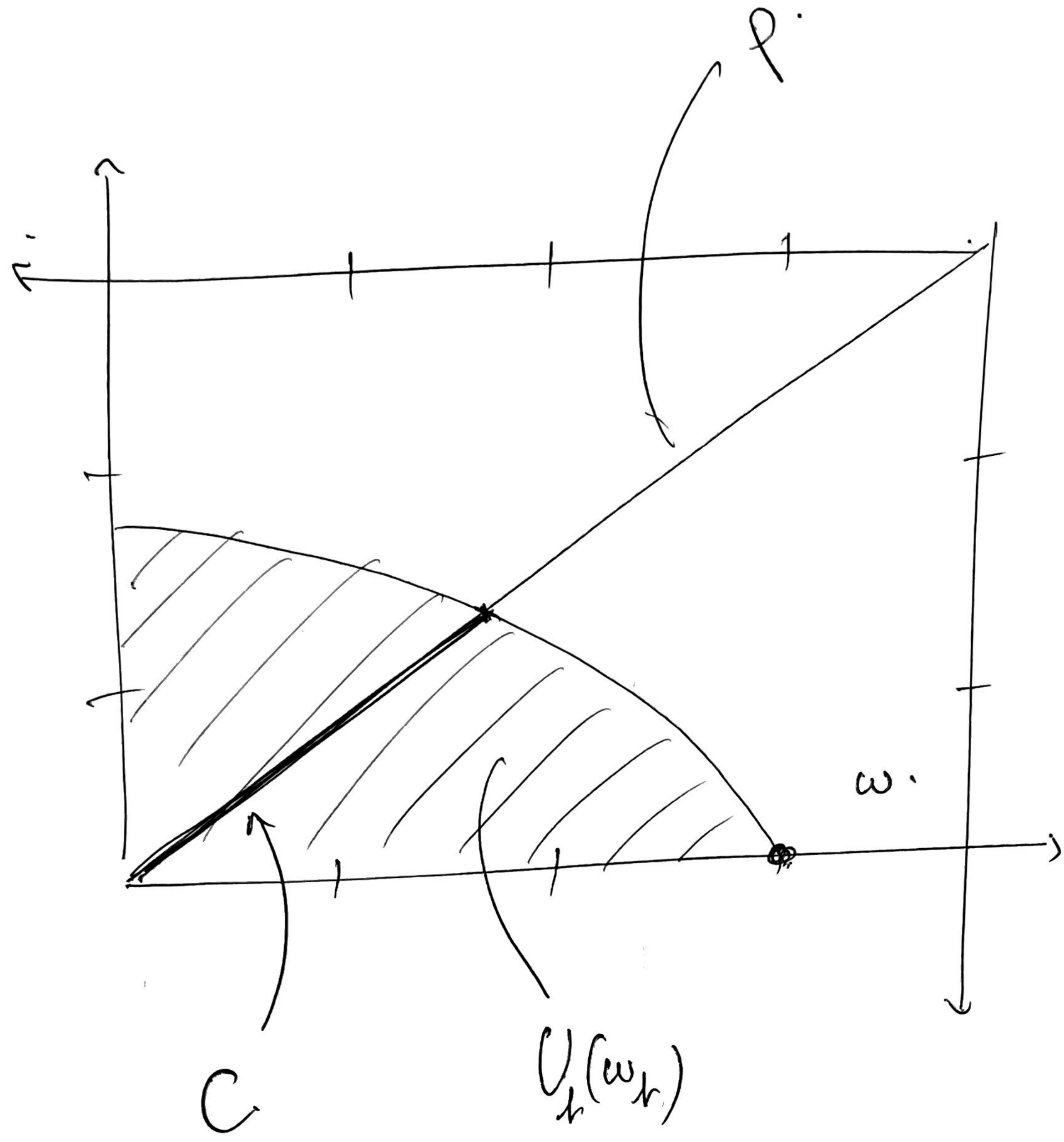
$$\underbrace{(x_{a1} - \omega_1)^{2}(\omega_2 - x_{a2})}^{3} \geq 27.$$

$$\underbrace{(\omega_1 - x_{a1})^{2} \cdot \left(\omega_2 - \frac{\omega_2}{\omega_1} x_{a1}\right)}^{3}$$

$$\underbrace{\left(\frac{\omega_2}{\omega_1}\right)^{3}} (\omega_1 - x_{a1})^{5} \geq 27.$$

$$\left(\frac{3}{4}\right)^{3} = \frac{27}{64} \qquad \text{i.e.} \quad \omega_1 - x_{a1} \geq 64^{1/5}$$

$$\text{i.e.} \quad x_{a1} \leq \underbrace{\omega_1 - 64^{1/5}}_{\approx 4 - 2{,}3.} = 1{,}7.$$



$= \{ x_r \succsim_r \omega_r \}.$

3) <u>Walrasian eq.</u> $x^* = (x_a^*, x_b^*)$ and p^*

eq. allocation

eq. price.

with. $x_i^* \succsim x_i' \quad \forall x_i' \in B_i(p^*)$.

(UMP). i.e. $p^* \cdot x_i' \leq p^* \cdot \omega_i$

$\forall i = a, b.$

and.

<u>market clearing</u>. $x_a^* + x_b^* = \omega$.

The constraints $p^* \cdot x_i \leq p^* \cdot \omega_i$ are unchanged by $p^* \to \lambda p^*$.

so we know only the ratio. $\frac{p_1^*}{p_2^*}$ matters.

$\Longrightarrow$ So we have only three unknowns x_{a1}^*, x_{a2}^* and $\left(\frac{p_1}{p_2}\right)^*$.

The KT condts. are thus.

$$\mathrm{UMP}_i \begin{cases} \max u_i(x_i) \\ p^* x_i \leq p^* \omega_i \end{cases} \qquad \forall i = a, b.$$

$$\Downarrow$$

$x_i^* \gg 0$ solts UMP_i

$$\longrightarrow \begin{cases} \boxed{\nabla_{x_i} u_i(x_i^*) = \lambda_i p^*} \quad (*) & \lambda_i \geq 0, \quad \forall i = a, b. \\ \text{and} \\ \lambda_i (p^* x_i^* - p^* \omega_i) = 0. \end{cases}$$

from $(*)$ we get.

$$MRS^i_{12}(x_i^*) = \frac{\frac{\partial u_i(x_i^*)}{\partial x_{i1}}}{\frac{\partial u_i(x_i^a)}{\partial x_{i2}}} = \frac{\cancel{\lambda_i} \; p_1^*}{\cancel{\lambda_i} \; p_2^*}.$$

indep of i.

$\forall i = a, b.$

so we have that. the Walrasian eq.

$$\left(x^* = \{x_a^*, x_b^*\}, \ p^*\right).$$

verifies. $MRS^a_{12}(x_a^*) = MRS^b_{12}(x_b^*)$.

and $x_a^* + x_b^* = \omega$.

$\Longrightarrow$ so the Walrasian eq. is a PARETO optimum!

(= 1st Welfare thm).

$x_b^* = \omega - x_a^*$.

We need to solve the system.

$$\begin{cases} MRS^a_{12}(x_a^*) = MRS^b_{12}(x_b^*) = \dfrac{p_1^*}{p_2^*}. \\ \text{and. } p^*\cdot x_a^* = p^*\cdot \omega_a. \end{cases}$$

3 equations for 3 unknowns

as we already computed.

$$MRS^a_{12} = MRS^b_{12}. \quad (1).$$

$$\longrightarrow \quad x_{a2} = \frac{\omega_2}{\omega_1} x_{a1}.$$

in addition.

$$MRS^a_{12} = \frac{2}{3}\frac{x_{a2}}{x_{a1}} = \frac{p_1}{p_2}. \quad (2).$$

$$\text{and.} \quad p_1 x_{a1} + p_2 x_{a2} = 3 p_1. \quad (3).$$

$= p \cdot \omega_a.$

$$(1) \text{ in } (2). \longrightarrow \frac{2}{3}\frac{\omega_2}{\omega_1} = \frac{\cancel{2}}{\cancel{3}}\frac{3}{4} = \boxed{\frac{1}{2} = \left(\frac{p_1}{p_2}\right)^*.}$$

$$\text{in } (3) \longrightarrow x_{a2} = \left(\frac{p_1}{p_2}\right)(3 - x_{a1}).$$

$$= \frac{1}{2}(3 - x_{a1})$$

$\longrightarrow$ in (1).

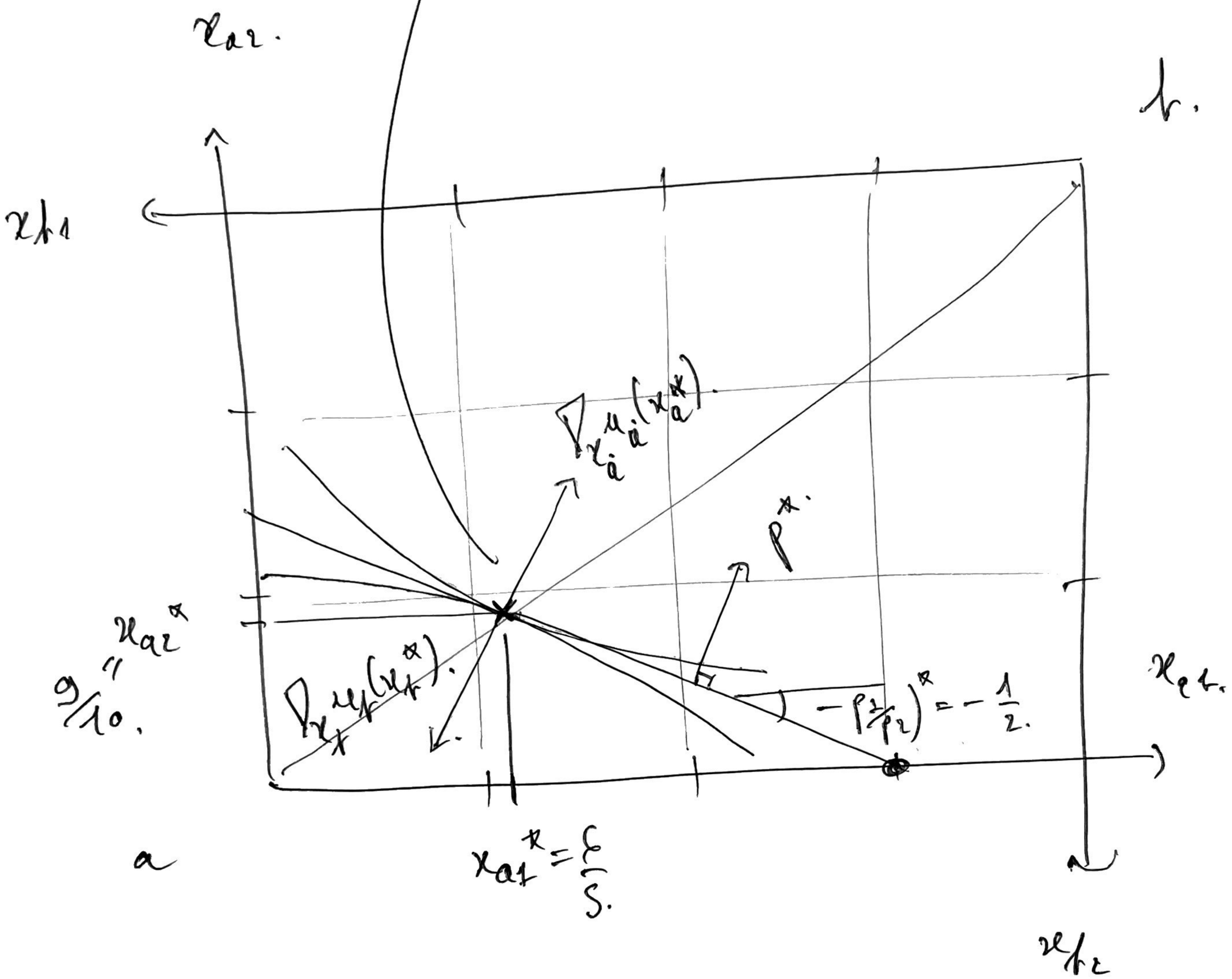
$$x_{a2} = \frac{1}{2}(3 - x_{a1}) = \frac{w_2}{w_1} x_{a1}. = \frac{3}{4} x_{a1}.$$

$$\longrightarrow. \quad \frac{3}{2} = \frac{5}{4} x_{a1}. \quad \longrightarrow \quad \boxed{x_{a1}^{*} = \frac{6}{5}.}$$

$$x_{a2} = \frac{3}{4} x_{a1} = \frac{3}{4} \frac{6}{5} = \boxed{\frac{9}{10} = x_{a2}^{*}.}$$

and $x_f^{*} = \omega - x_a^{*} \ldots$

$x^* = \{x_a^*, x_f^*\}$.



$x^* \in CCP$.

Exercise 4 $I = \{a, b\}$. $L = \{1, 2\}$.
ω_a ω_b.

From local nonsatiation, we know that both consumer will satisfy Walras' law, ie spending all their wealth, when maximizing.

thus both inequality constraints are saturated.

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$$p^* \cdot x_a^* = p^* \cdot \omega_a$$
and $p^* \cdot x_b^* = p^* \cdot \omega_b$.

$$\Longrightarrow \quad p^* \cdot (x_a^* + x_b^*) = p^* \cdot (\omega_a + \omega_b).$$

$$\cancel{p_1 (x_{a1}^* + x_{b1}^*)} + p_2 (x_{a2}^* + x_{b2}^*)$$
$$= \cancel{p_1 (\omega_{a1} + \omega_{b1})} + p_2 (\omega_{a2} + \omega_{b2}).$$

So if market for good 1 clears. ie. $x_{a1}^* + x_{b1}^* = \omega_{a1} + \omega_{b1}$. $\Longrightarrow$ then so does 2, ie $x_{a2}^* + x_{b2}^* = \omega_{a2} + \omega_{b2}$

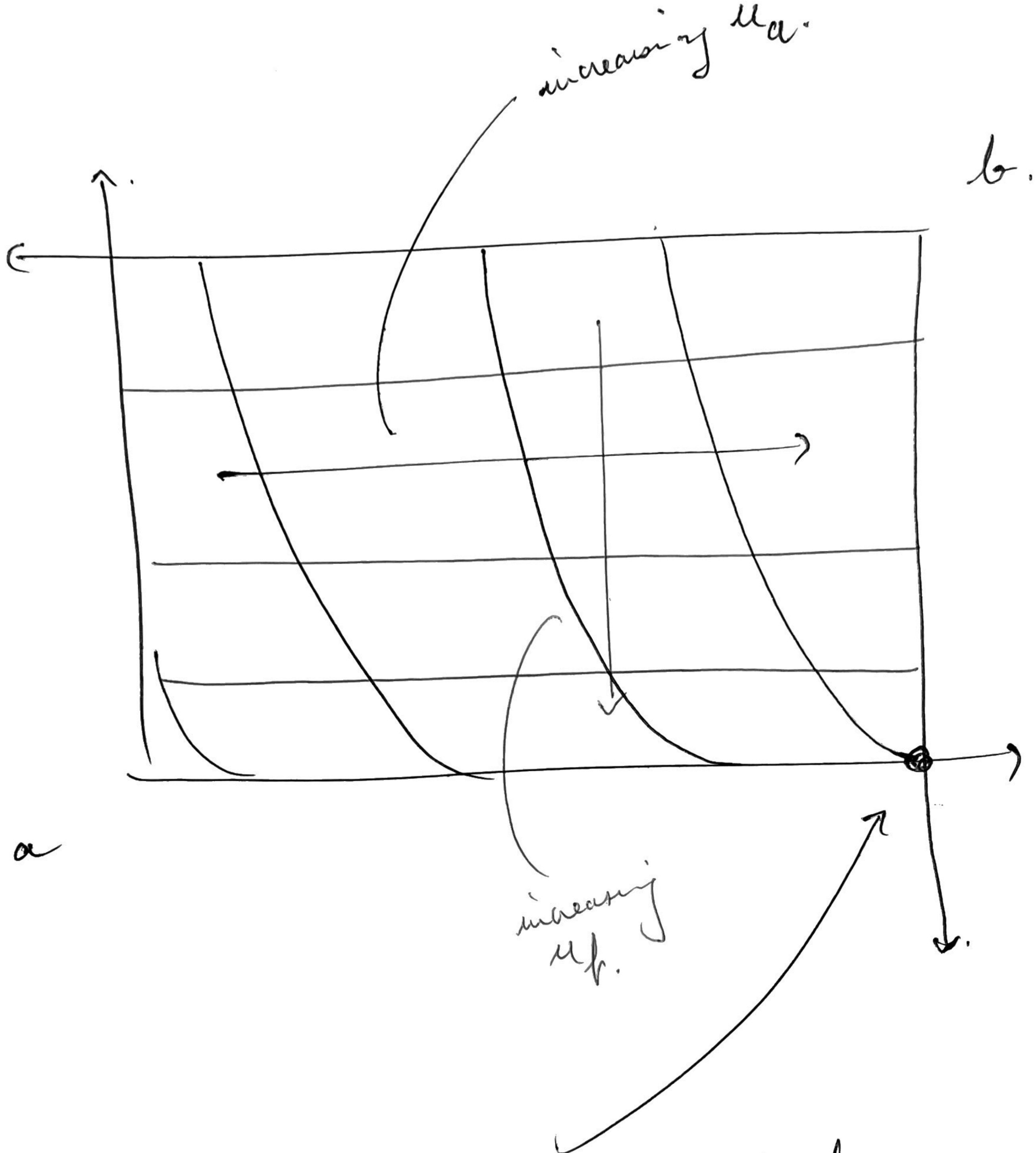
$\square$

② $u_a(x_a) = x_{a1} + \sqrt{x_{a2}}$.

$u_b(x_b) = x_{b2}$.

$$x_{a2} = (u^* - x_{a1})^2.$$

1).



$\Longrightarrow$ only Pareto optimal allocation...

$= \{\bar{x}_a = (\omega_1, 0), \bar{x}_b = (0, \omega_2)\}$.

A competitive equilibrium exists if that unique Pareto optimal allocation is attainable for a given equilibrium price system and initial endowment.

i.e. $\exists p^*$. $p^* \cdot \omega_a = p^* \cdot \bar{x}_a = p_1 \omega_1$.
and $p^* \cdot \omega_b = p^* \cdot \bar{x}_b = p_2 \omega_2$

2). if $\omega_a = (\omega_1, 0)$ and $\omega_b = (0, \omega_2)$, the eq. exists.

Note: when on the border of the Edgeworth box, i.e. such that at least one coordinate is binding $(= 0)$, equilibrium does not require $MRS = \frac{p_1^*}{p_2^*}$.

2) if $\omega_{a1} = 0$ it doesn't ...

~~[illegible]~~

5 1). $\bar{\omega} = (4,3)$.

$$\boxed{u_a(x_a) = \sqrt{x_{a1} x_{a2}} \quad , \quad u_b(x_b) = x_{b1}^{1/3} x_{b2}^{2/3}}.$$

$$P = \{ x = (x_a, x_b) \; ; \; MRS^a_{12}(x_a) = MRS^b_{12}(x_b). \; x_a + x_b = \bar{\omega} \}.$$

$$MRS^a_{12} = \frac{\cancel{\frac{1}{2}} \frac{\cancel{u_a(x_a)}}{x_{a1}}}{\cancel{\frac{1}{2}} \frac{\cancel{u_a(x_a)}}{x_{a2}}} = MRS^b_{12} = \frac{\cancel{\frac{1}{3}} \frac{u_b(x_b)}{x_{b1}}}{\frac{2}{\cancel{3}} \frac{u_b(x_b)}{x_{b2}}}.$$

$$\Rightarrow \quad \frac{x_{a2}}{x_{a1}} = \frac{1}{2} \frac{x_{b2}}{x_{b1}} \underset{x_a + x_b = \bar{\omega}}{\uparrow}= \frac{1}{2} \frac{3 - x_{a2}}{4 - x_{a1}}.$$

$$\Leftrightarrow \quad 2\left(\frac{4}{x_{a1}} - 1\right) = \frac{3}{x_{a2}} - 1$$

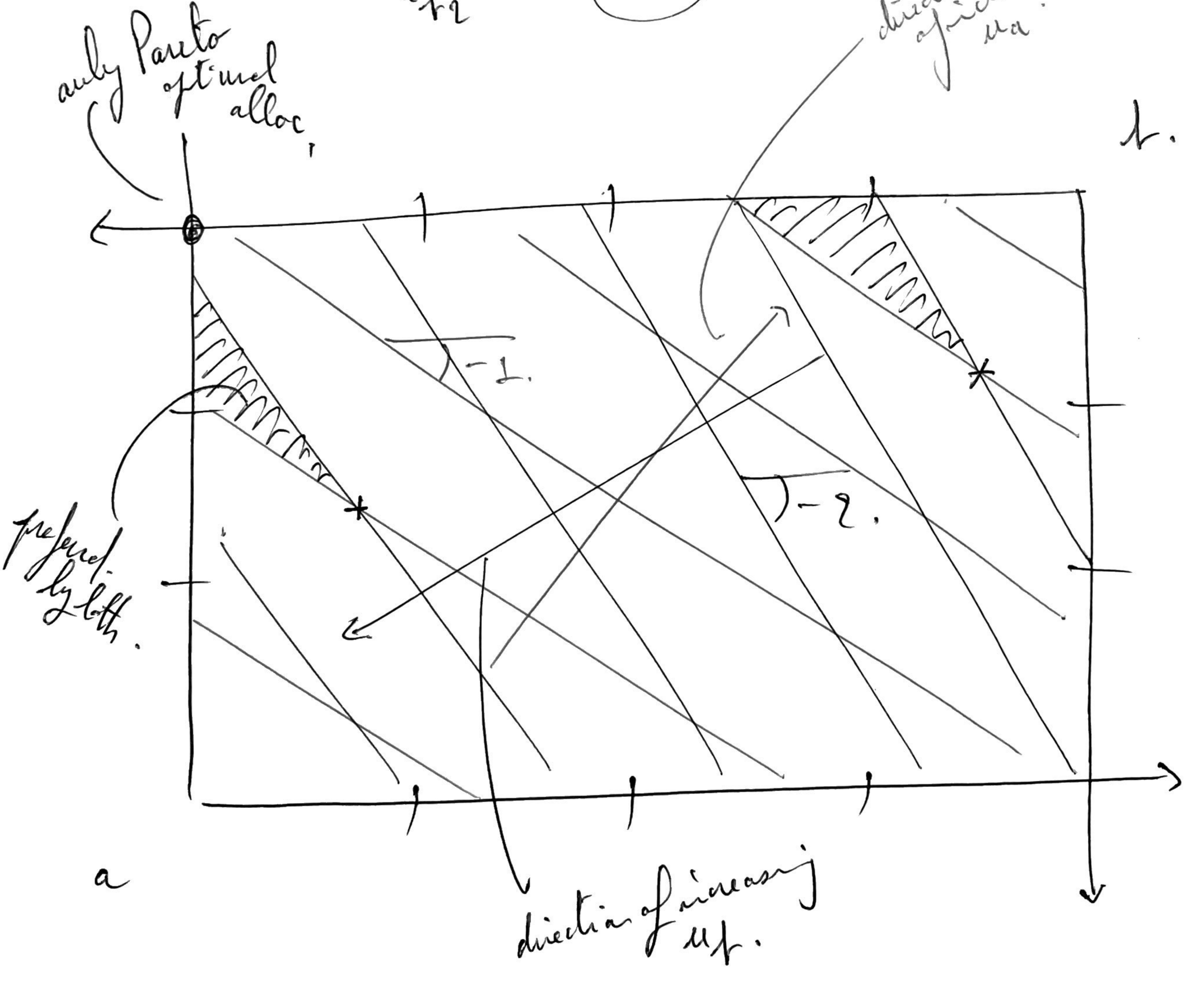
$$\Leftrightarrow \quad x_{a2} = \ldots$$

2).

$$u_a(x_a) = x_{a1} + x_{a2}.$$
$$u_b(x_b) = 2x_{b1} + x_{b2}.$$

the slope of the indifference curves is -2.

$$u^* = 2x_{b1} + x_{b2} \implies x_{b2} = u^* - 2\, x_{b1}.$$



3). $\boxed{\begin{aligned} u_a(x_a) &= \sqrt{x_{a1} x_{a2}} . \\ u_b(x_b) &= x_{b1} + 2x_{b2} . \end{aligned}}$

$$MRS^a_{12} = \frac{\cancel{\frac{1}{2}} \frac{\cancel{u_a(x_a)}}{x_{a1}}}{\cancel{\frac{1}{2}} \frac{\cancel{u_a(x_a)}}{x_{a2}}} = \frac{x_{a2}}{x_{a1}} = MRS^b_{12} = \frac{1}{2} .$$

$$\Longrightarrow . \quad P = \left\{ x = (x_a, x_b) ; \; x_{a2} = \frac{1}{2} x_{a1} . \text{ and. } x_b = \omega - x_a \right\} .$$