2020. Exercises 7 and 25 from problem sheet.

November 27, 2023.

7. Edgeworth box. $l = 2$ $I = 2$.

$e_1 = (1,1)$ $u_1(x_1) = x_{11}^{1/3} x_{12}^{2/3}$.
$e_2 = (1,3)$ $u_2(x_2) = x_{21}^{1/2} x_{22}^{1/2}$.

1) Total endowment. $\bar{e} = (2,4)$.

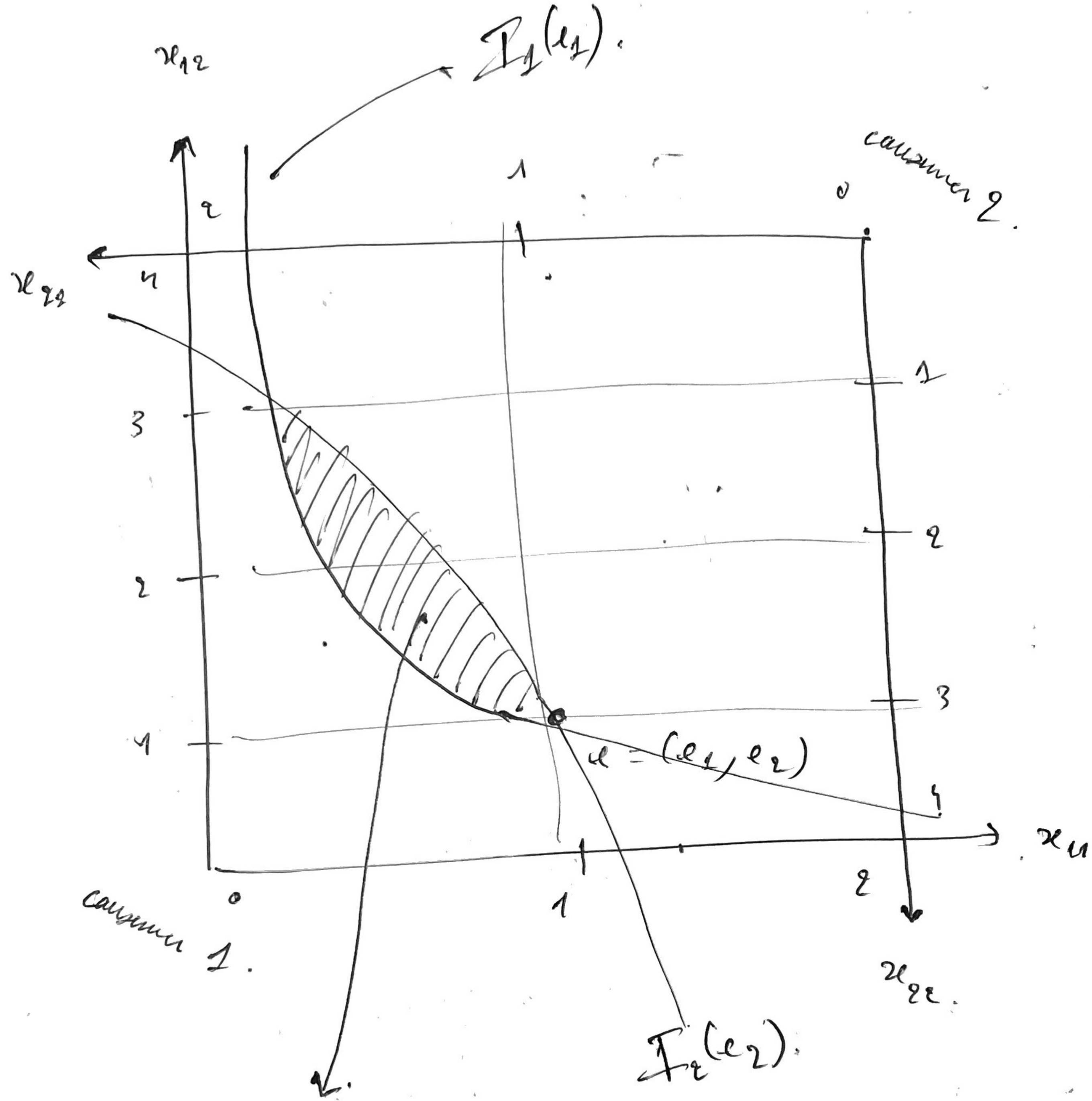
Indifference curves:

* $I_1(e_1) = \{ x_1 ; \underbrace{u_1(x_1)}_{x_{11}^{1/3} x_{12}^{2/3}} = \underbrace{u_1(e_1)}_{1} \}$.

$$\Rightarrow \boxed{I_1(e_1) = \left\{ x_1 = (x_{11}, x_{12}) ; x_{12} = \frac{1}{\sqrt{x_{11}}} \right\}}$$

* $I_2(e_2) = \{ x_2 ; \underbrace{u_2(x_2)}_{x_{21}^{1/2} x_{22}^{1/2}} = \underbrace{u_2(e_2)}_{\sqrt{3}} \}$.

$$\Rightarrow \boxed{I_2(e_2) = \left\{ x_2 = (x_{21}, x_{22}) , x_{22} = \frac{3}{x_{21}} \right\}}$$



2.) $B_e = \{ x = (x_1, x_2) \; ; \; x_1 \succsim_1 e_1 \, , \, x_2 \succsim_2 e_2$

upper contour sets and $x_1 + x_2 = \bar{e} \}$.

$$\boxed{B_e = \{ x = (x_1, x_2) \; ; \; x_1 \in U_1(e_1) \, , \, x_2 \in U_2(e_2) \text{ and } x_1 + x_2 = \bar{e} \}.}$$

B_e is the set of Allocations which Pareto dominate $e = (e_1, e_2)$.

↳ ie x in the Edgeworth box.

3) x^* is a Pareto optimum iff.
(x_1^*, x_2^*) $\nexists x' = (x_1', x_2')$ s.th.

$$\boxed{\left(x_i' \succsim x_i^* \quad \forall i = 1,2 \text{ and } \exists i \in \{1,2\} \text{ s.th. } x_i' \succ x_i^* \right)}$$

→ ie it is impossible to make one of the two consumer better off without making the other one worst off.

→ we should add that both x^* and x' need to be in the edgeworth box, ie.

$$x_1^* + x_2^* = \bar{e} \text{ and } x_1' + x_2' = \bar{e}.$$

We can translate the above definition to state that

$$\boxed{x^* \text{ is a Pareto optimum iff. } B_{x^*} = \{x^*\}.}$$

ie. x^* is only pareto dominated by itself.

reduced to a single tone.

4) See notes on Pareto sets.

The set C of Pareto optima for this economy verifies:

$$C = \left\{ x = (x_1, x_2) \; ; \; x_1 + x_2 = \bar{e} \text{ and } MRS^1_{12}(x_1) = MRS^2_{12}(x_2) \right\}$$

Proof: it comes from the fact that x^* is a Pareto optimal allocation if it solves the maximization problem:

$$\begin{cases} \max\limits_{x_1} \; u_1(x_1) \\ \text{s.th } \; x_1 + x_2 = \bar{e} \\ \text{and } \; u_2(x_2) \geq u^* \end{cases}$$

i.e the allocation is in the Edgeworth box.

i.e the utility of consumer 2 cannot go below a certain level.

meaning, indeed, that it would not be possible to make consumer 1 better off without making consumer 2 worse off.

Now, the condition $x_1 + x_2 = \bar{e}$
makes x_2 dependent on x_1,
so the KT condition, for an interior solution,
yields:

$$\nabla_{x_1} u_1(x_1^*) = \lambda \; \underline{\nabla_{x_2}} u_2(x_2^*)$$

gradient of the objective fct.

"proportional"

gradient of the constraint

from the chain rule. (see other notes)

$$\nabla_{x_1} u_1(x_1^*) = \lambda \nabla_{x_2} u_2(x_2^*) = \lambda \begin{pmatrix} \frac{\partial x_{21}}{\partial x_{11}} \frac{\partial u_2}{\partial x_{21}}(x_2^*) \\ \frac{\partial x_{22}}{\partial x_{12}} \frac{\partial u_2}{\partial x_{22}}(x_2^*) \end{pmatrix}$$

$-1.$

$$= -\lambda . \underline{\nabla_{x_2}} u_2(x_2^*) .$$

from which follows:

$$\boxed{MRS^1_{12}(x_1^*) = MRS^2_{12}(x_2^*).}$$

and we also know from the monotonicity of u_1 and u_2 that $u_2(x_2^*) = u^*$

i.e., as always, the solution is found on the boundary of the constraint. $\square$

So, going back to solving.

$$C = \left\{ x = (x_1, x_2) \; ; \; x_1 + x_2 = \bar{e} = (2,4) \text{ and } MRS^1_{12}(x_1) = MRS^2_{12}(x_2) \right\}.$$

We have:

$$MRS^1_{12}(x_1) = \frac{\cancel{\frac{1}{3}} \frac{1}{x_{11}} \cancel{u_1(x_1)}}{\cancel{\frac{2}{3}} \frac{1}{x_{12}} \cancel{u_1(x_1)}} = \frac{1}{2} \frac{x_{12}}{x_{11}}$$

and

$$MRS^2_{12}(x_2) = \frac{\cancel{\frac{1}{2}} \frac{1}{x_{21}} \cancel{u_2(x_2)}}{\cancel{\frac{1}{2}} \frac{1}{x_{22}} \cancel{u_2(x_2)}} = \frac{x_{22}}{x_{21}}.$$

So

$$x \in C \iff \frac{1}{2}\frac{x_{12}}{x_{11}} = \frac{x_{22}}{x_{21}} \underset{x_1 + x_2 = \bar{e}}{=} \frac{4 - x_{12}}{2 - x_{11}}$$

$$\iff \frac{1}{x_{11}} - \frac{1}{2} = \frac{4}{x_{12}} - 1.$$

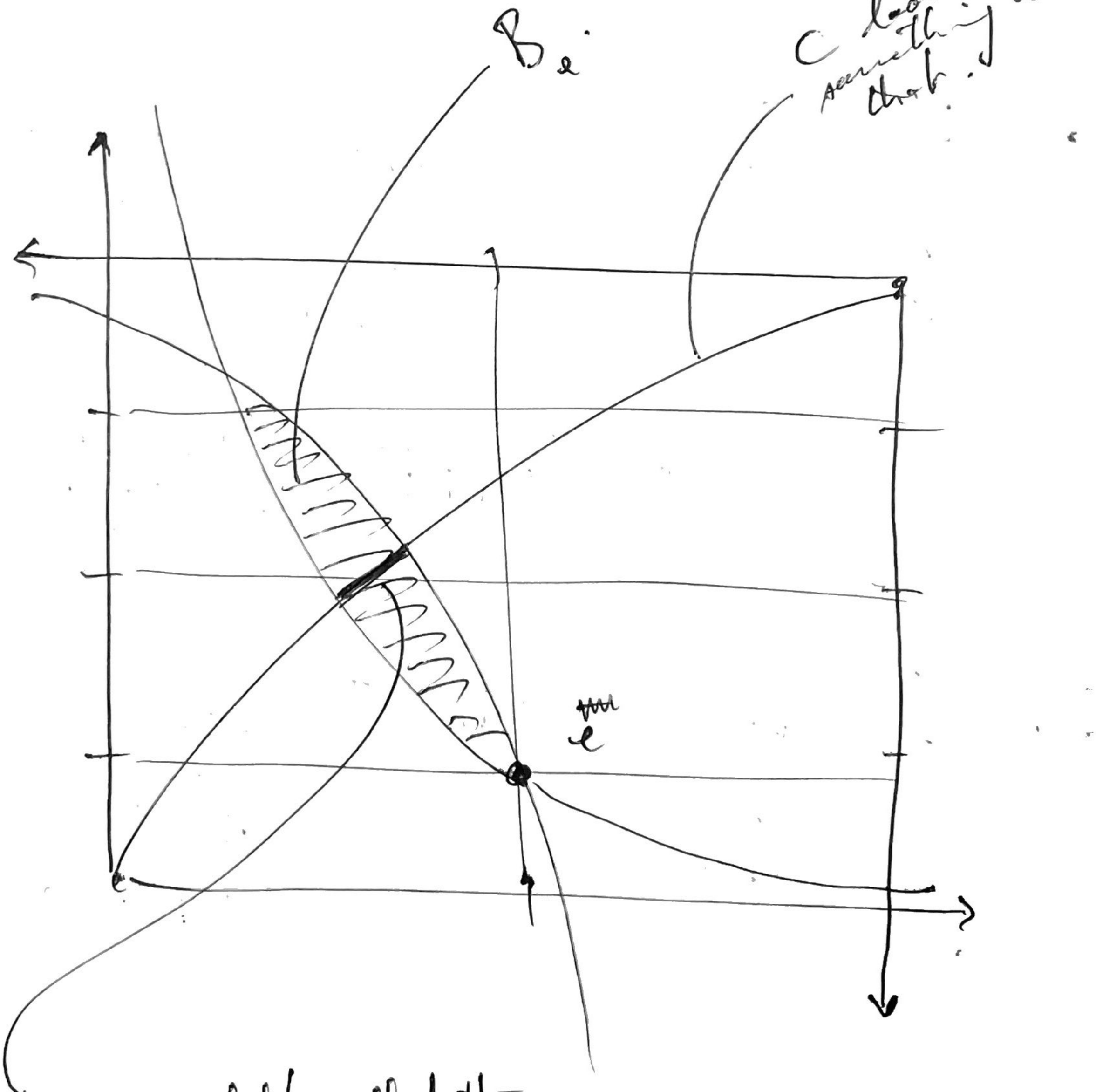
$$\iff \boxed{x_{12} = \frac{4}{1/x_{11} + 1/2}.}$$

→ So...

$$\boxed{C = \left\{ x = (x_1, \bar{e} - x_1)\ ;\ \text{s.th.}\ x_{12} = \frac{4}{1/x_{11} + 1/2} \right\}.}$$

PARETO SET.

5)



$C \cap B$ is what's called the "CONTRACT CURVE", i.e. the set of Pareto optimal allocations (i.e. $x \in C$) s.th. they are better than the initial endowment e for both agents (i.e. $x \in B$).

... beyond that remark however there is no reason why B and C would "coincide" at all.

6). p^*, x^* is a general / Walrasian (competitive) price equilibrium for this economy. iff it satisfies.

* both agents maximize their utility.

 i.e. x_i^* is a sol° to.

$$UMP_i \begin{cases} \max u_i(x_i) \\ p^* \cdot x_i \le p^* \cdot e_i \end{cases} \quad \forall i = 1, 2.$$

and.

* market clearing condition.

 i.e. $x_1^* + x_2^* = \bar{e}$.

7) So. p^*, x^* solves

$$MRS_{12}^{i}(x_i^*) = \frac{p_1^*}{p_2^*}$$

$$\nabla u_i(x_i^*) = \lambda_i p^* \leftarrow \text{KT conds. for } x_i^* \gg 0.$$

$$p^* \cdot x_i^* = p^* \cdot e_i \leftarrow \text{Walras' law.}$$

$\forall i = 1, 2.$

and. $x_1^* + x_2^* = \bar{e}$

$\longrightarrow$ p^*, x^* solves

$$\begin{cases} MRS_{12}^1(x_1^*) = \boxed{\frac{1}{2}\frac{x_{12}^*}{x_{11}^*} = \frac{p_1^*}{p_2^*}} = \frac{x_{22}^*}{x_{21}^*} = MRS_{12}^2(x_2^*) & (a)\ (b) \\ \boxed{\frac{p_1^*}{p_2^*}x_{11}^* + x_{12}^* = \frac{p_1^*}{p_2^*} + 1.} & (a') \\ \boxed{\frac{p_1^*}{p_2^*}x_{21}^* + x_{22}^* = \frac{p_1^*}{p_2^*} + 3} & (b') \\ x_{11}^* + x_{21}^* = 2. \\ x_{12}^* + x_{22}^* = 4. \end{cases}$$

The technique is always the same; we first obtain the demand of the consumers as a fct of the ratio of the prices. (i.e. their OFFER CURVE) and then we impose market clearing (only one of the two markets need to clear and the other clears automatically) to find the equilibrium price (and this corresponds to the two offer curves crossing each other in the Edgeworth box).

So using (a) and (a'), we have.

$$x_{12}^* = 2x_{11}^*\frac{p_1^*}{p_2^*}.$$

$\Longrightarrow$ injecting in (a'). yields.

$$3\,x_{11}^{*}\,\frac{p_1^*}{p_2^*} = \frac{p_1^*}{p_2^*} + 1.$$

$\longrightarrow$ $\boxed{x_{11}^{*} = \frac{1}{3} + \frac{1}{3}\frac{p_2^*}{p_1^*}}$ and $\boxed{x_{12}^{*} \overset{(a)}{=} \frac{2p_1^*}{p_2^*}x_{11}^{*} = \frac{2}{3} + \frac{2}{3}\frac{p_1^*}{p_2^*}}$.

for the 2nd consumer, we do the same,

(b) yields. $x_{22}^{*} = x_{21}^{*}\,\frac{p_1^*}{p_2^*}$,

that we inject into (b') to get.

$$2\,x_{22}^{*}\,\frac{p_1^*}{p_2^*} = \frac{p_1^*}{p_2^*} + 3.$$

$\longrightarrow$ $\boxed{x_{21}^{*} = \frac{1}{2} + \frac{3}{2}\frac{p_2^*}{p_1^*}}$ and $\boxed{x_{22}^{*} = \frac{3}{2} + \frac{1}{2}\frac{p_1^*}{p_2^*}}$

Finally we obtain the equilibrium price ratio by. imposing that the market for good 1 clears,

i.e. $x_{11}^{*} + x_{21}^{*} = 2$.

i.e $\frac{1}{3} + \frac{1}{3}\frac{p_2^*}{p_1^*} + \frac{1}{2} + \frac{3}{2}\frac{p_2^*}{p_1^*} = 2$.

i.e. $\frac{5}{6} + \frac{11}{6}\frac{p_2^*}{p_1^*} = 2.$

i.e. $$\boxed{\frac{p_2^*}{p_1^*} = \frac{7}{11}.}$$

$\longrightarrow$ $x_{11}^* = \frac{1}{3} + \frac{7}{33} = \frac{18}{33} = \frac{6}{11}$

$x_{12}^* = \frac{2}{3} + \frac{2}{3}\frac{11}{7} = \frac{36}{21} = \frac{12}{7}$

$x_{21}^* = \frac{1}{2} + \frac{3}{2}\frac{7}{11} = \frac{32}{22} = \frac{16}{11}.$

$x_{22}^* = \frac{3}{2} + \frac{1}{2}\frac{11}{7} = \frac{32}{14} = \frac{16}{7}.$

$\longrightarrow$ So the competitive equilibrium is:

$$\boxed{CE = \left\{ \frac{p_1^*}{p_2^*} = \frac{11}{7} ;\ x_1^* = \left(\frac{6}{11}, \frac{12}{7}\right) ;\ x_2^* = \left(\frac{16}{11}, \frac{16}{7}\right). \right\}.}$$

To ease the calculi, we could have use the fact that, since the 1ST WELFARE THM applies, p^* & x^* verify $x^* \in C$ and $MRS_{12}(x_1^*) = \frac{p_1^*}{p_2^*}$.

We could also have normalized one of the prices, as suggested, because as you see only the ratio of the prices matters.

(25)

$$\boxed{\begin{array}{lll} L=2 & I=2 & J=1. \\ \omega_1=(6,0) & \theta_1=\frac{1}{2} & u_1(x_1)=x_{11}x_{12}. \\ \omega_2=(4,2) & \theta_2=\frac{1}{2} & u_2(x_2)=x_{21}x_{22}. \\ f(z)=z \;;\; z\geq 0. \end{array}}$$

1). Derive the supply and profit of the firm.

$Y=\{(-z,q)\,;\, z\geq 0 \text{ and } q-f(z)\leq 0\}.$

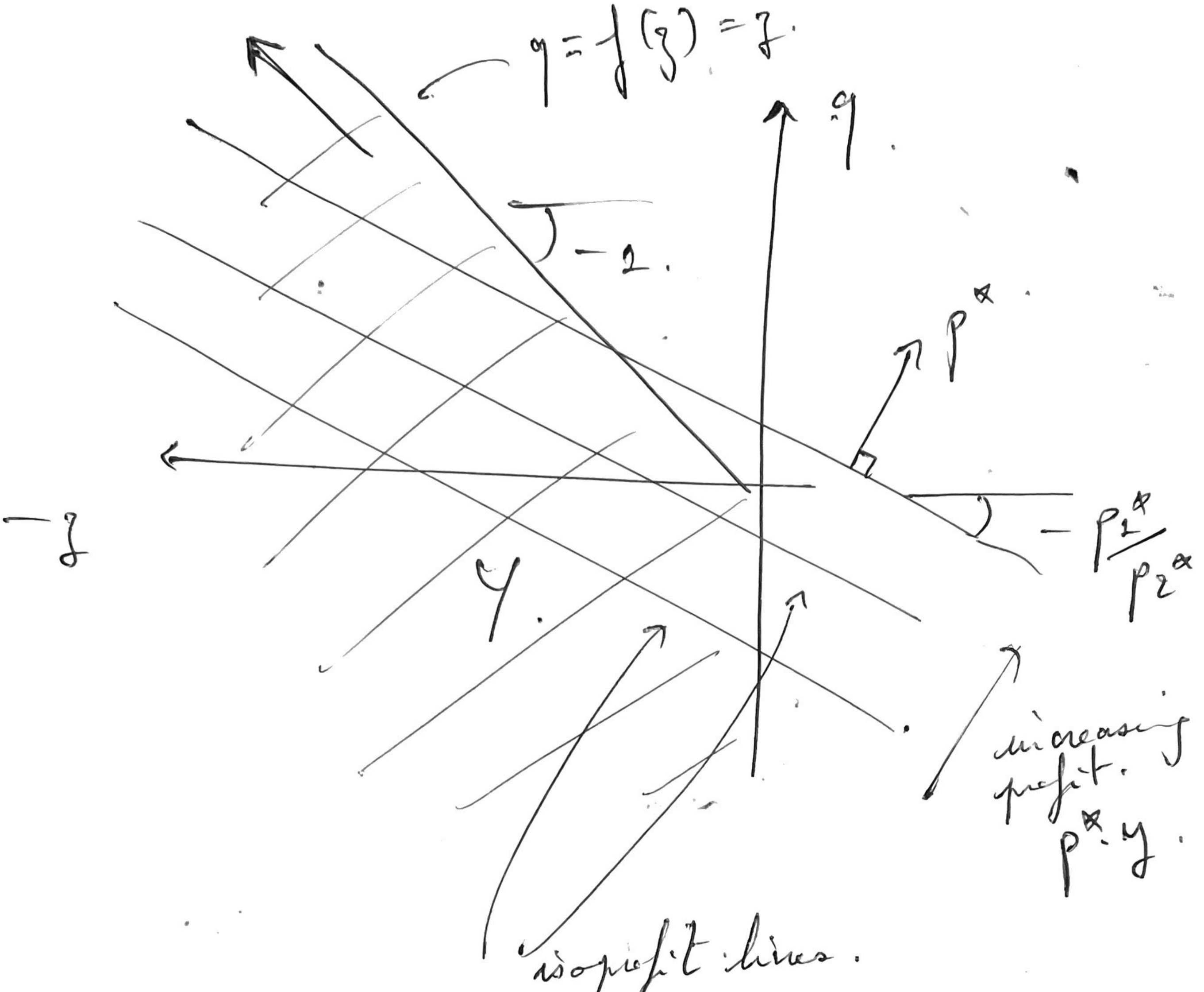
↑ production set.

$f(z)=z$ is homogenous of deg 1.

$\Longrightarrow$ Y exhibits CONSTANT RETURNS TO SCALE.

$\Longrightarrow$ so it exhibits increasing returns to scale, so we know that the profit fct $\pi(p^*)$ will either be ≤ 0 or $=+\infty$.

Graphically (see next page), we deduce that depending on the value of $\frac{p_1^*}{p_2^*}$ we have:



* if $\frac{p_1^*}{p_2^*} < 1 \Longrightarrow$ $\boxed{\pi(p^*) = +\infty \text{ and } y^* = (-\infty, +\infty)}$

with ratio 1 between z and q

* if $\frac{p_1^*}{p_2^*} > 1$ $\Longrightarrow$ $\boxed{\pi(p^*) = 0 \text{ and } y^* = (0, 0)}$

* if $\frac{p_1^*}{p_2^*} = 1$ $\boxed{\pi(p^*) = 0 \text{ and } y^* = \{(-z, z) ; z \geq 0\}}$

2). We exclude the case $\frac{p_1^*}{p_2^*} < 1$.

because of unbounded. so Θ $\Longrightarrow$ $\nexists$ no competitive equilibrium. so Θ.

Let x_i^* sol to.

$$UMP_i \quad \begin{cases} \max u_i(x_i) & i = 1, 2. \\ p^* \cdot x_i \leq p^* \cdot \omega_i + \theta_i \pi(p^*) \end{cases}$$

$\Longrightarrow$

Walras' law $\quad p^* \cdot x_i^* = p^* \cdot \omega_i + \theta_i \pi(p^*)$.

and KT condition for $x_i^* \gg 0$. $\quad \nabla u_i(x_i^*) = \lambda_i \frac{p_1^*}{p_2^*}$.

$$\Downarrow$$

$$MRS^i_{12}(x_i^*) = \frac{p_1^*}{p_2^*} \qquad \forall i = 1, 2.$$

---

for both cases of interest, $p_1/p_2 = 1$ and $p_1/p_2 > 0$ we have. $\pi(p^*) = 0$.

So. x_1^* solves.
$$\begin{cases} \frac{x_{12}^*}{x_{11}^*} = \frac{p_2^*}{p_2^*} \\ \text{and. } \frac{p_1^*}{p_2^*} x_{11}^* + x_{12}^* = 6 \frac{p_1^*}{p_2^*}. \end{cases}$$

$$\Longrightarrow 2\frac{p_1^*}{p_2^*}x_{11}^* = 6\frac{p_1^*}{p_2^*}.$$

$$\Longrightarrow \boxed{x_{11}^* = 3 \quad \text{and} \quad x_{12}^* = 3\frac{p_1^*}{p_2^*}.}$$

and x_2^* solves. $\begin{cases} \dfrac{x_{22}^*}{x_{21}^*} = \dfrac{p_1^*}{p_2^*}. \\ \text{and } \dfrac{p_1^*}{p_2^*}x_{21}^* + x_{22}^* = 4\dfrac{p_1^*}{p_2^*} + 2. \end{cases}$

$$\Longrightarrow 2\frac{p_1^*}{p_2^*}x_{21}^* = 4\frac{p_1^*}{p_2^*} + 2.$$

$$\Longrightarrow \boxed{x_{21}^* = 2 + \frac{p_2^*}{p_1^*} \quad \text{and.} \quad x_{22} = 1 + 2\frac{p_1^*}{p_2^*}}.$$

3) Competitive eq. are.

p^*, x^*, y^* s.th. both consumers maximize utility,

the firm maximize profits.

and all markets clear.

4) Can we have a competitive equilibrium with $\frac{p_1^*}{p_2^*} > 1$?

market clearing i.e. $x_1^* + x_2^* = \bar{\omega} + \underbrace{y^*}_{\substack{= \\ (0,0) \\ \text{in this case}}}$

i.e., for good 1. $\boxed{\frac{p_1^*}{p_2^*} = \frac{1}{5} < 1.}$

$\Downarrow$

$x_{11}^* + x_{21}^* = \bar{\omega}_1$

i.e. $5 + \frac{p_2^*}{p_1^*} = 10$

CONTRADICTION, so NOT POSSIBLE.

So let's consider the case $\underline{\frac{p_1^*}{p_2^*} = 1.}$)

$\Longrightarrow$ in which case

$y(p^*) = \{(-z, z) ; z \geq 0\}$.

$\uparrow$ the supply corr.

hence the market clearing condition writes

$$\begin{cases} x_{11}^* + x_{21}^* = \bar{\omega}_1 - z^* & \text{for good 1.} \\ \text{and} & \\ x_{12}^* + x_{22}^* = \bar{\omega}_2 + z^* & \text{for good 2.} \end{cases}$$

$$\Longrightarrow \begin{cases} 3 + 2 + \dfrac{p_2^*}{p_1^*} = 10 - z^* \\ \text{and} \\ 3\dfrac{p_1^*}{p_2^*} + 1 + 2\dfrac{p_1^*}{p_2^*} = 2 + z^* \end{cases}$$

summing up the two, we get:

$$6 + 6\frac{p_1^*}{p_2^*} = 12.$$

$$\Longrightarrow \frac{p_1^*}{p_2^*} = 1.$$

NO CONTRADICTION. with

!

and $z^* = 10 - 6 = 4.$

$$\boxed{y^* = (-z^*, z^*) = (-4, 4)}$$

$\Longrightarrow$ The competitive equilibrium is:

$$\boxed{CE = \left\{ \frac{p_1^*}{p_2^*} = 1 \; ; \; x_1^* = (3,3) \; ; \; x_2^* = (3,3) \; ; \; y^* = (-4,4) \right\}}$$

5) Yes, that CE is PARETO OPTIMAL.

because $u_{i=1,2}$ strongly monotone $\Longrightarrow$ LNS.

$\Longrightarrow$ 1ST WELFARE THM applies.

6) See notes on PARETO SETS, pages 8–12 and like in exercise 20.

$\longrightarrow$

x^* is PARETO OPTIMAL.

iff $\nexists x'$ s.th. ($x_i' \succsim x_i^* \ \forall i = 1, 2$

and $\exists i \in \{1, 2\}$, $x_i' \succ x_i^*$).

i.e. you cannot make one consumer better off without making the other one worse off.

it means that both consumers are maximizing there utilities under the constraint that market clears. $x_1^* + x_2^* = \bar{\omega} + y$

and under the constraint of feasibility $F(y) \leq 0$.

7) This translate into the maximization problem.

$x^* = (x_1^*, \bar{\omega} + y^* - x_1^*)$ is sol. to.

$$\begin{cases} \max\limits_{x_1} u_1(x_1) \\ \text{s.th. } x_1 + x_2 = \bar{\omega} + y \\ \text{and } u_2(x_2) \geq u^* \\ \text{and } F(y) \leq 0 \end{cases}$$

$x_1 + x_2 = \bar{\omega} + y \leftarrow$ market clearing links x_2 to x_1 and x_1 to y.

$u_2(x_2) \geq u^* \leftarrow$ minimum utility level for 2.

$F(y) \leq 0 \leftarrow$ y is a feasible production plan i.e. $y \in Y$.

$\Longrightarrow$ due to monotonicity of u_1, u_2, F.
$\Longrightarrow$ LNS,

the solution verifies

$$\boxed{\begin{array}{l} u_2(x_2^*) = u^* \\ \text{and } F(y^*) = 0 \quad \text{ie } y^* = f(z^*) \end{array}}$$

sol found on the border of the constraints.

$$z^*$$

and KT conditions for an interior solution

$$\boxed{\nabla_{x_1} u_1(x_1^*) = \lambda \nabla_{x_1} u_2(x_2^*) = -\lambda \nabla_{x_2} u_2(x_2^*)}$$

chain rule and $x_2 = \bar{w} + y - x_1$.

$\Longrightarrow \nabla_{x_1} x_2 = -1$.

$\Downarrow$

$$\boxed{MRS^1_{12}(x_1^*) = MRS^2_{12}(x_2^*)}$$

ie $\boxed{\dfrac{x_{12}^*}{x_{11}^*} = \dfrac{x_{22}^*}{x_{21}^*}}$.

and.

another Lagrange multiplier (there's one per constraint).

$$\boxed{\nabla_{x_1} u_1(x_1^*)} = \lambda' \nabla_{x_1} F(y^*)$$

$$= \boxed{\lambda' \nabla_y F(y^*).}$$

using chain rule and $y = x_1 + x_2 - \overline{\omega}$

$$\Longrightarrow \nabla_{x_1} y = 1.$$

$$\boxed{MRS^1_{12}(x_1^*) = MRT_{12}(y^*).} \quad (*).$$

"marginal rate of transformation"

"

$$\frac{\frac{\partial F}{\partial y_1}(y^*)}{\frac{\partial F}{\partial y_2}(y^*)}$$

transformation fct.   production fct.

with $F : (-z, q) \longmapsto q - f(z)$

$= q - z$.

$\updownarrow$

$F : (y_1, y_2) = y \longmapsto y_2 + y_1$.

we have

$$\boxed{\frac{\partial F}{\partial y_1}(y) = 1 = \frac{\partial F}{\partial y_2}(y) \quad \forall y \in Y.}$$

$\Downarrow$

$$\boxed{MRT_{12}(y) = 1 \quad \forall y \in Y}$$

So (a) $\Longrightarrow$

$$\boxed{\frac{x_{12}^*}{x_{11}^*} = 1.}$$

So the Pareto set writes

$$\begin{cases} \dfrac{x_{12}^*}{x_{11}^*} = 1 = \dfrac{x_{22}^*}{x_{21}^*}. \\ x_{21}^{a} x_{22}^{a} = u^* \longleftarrow \text{which can span.} \\ q^* = z^*. \\ x_{11}^* + x_{21}^* = 10 - z^*. \\ x_{12}^* + x_{22}^* = 2 + q^* \end{cases}$$

$$\longrightarrow \quad x_{11}^* + x_{21}^* = 10 - z^*$$
$$\ominus \quad x_{12}^* + x_{22}^* = 2 + z^*$$

$$0 + 0 = 8 - 2z^*$$

many

$x_{11}^* = x_{12}^*$
and $x_{22}^* = x_{21}^*$ $\Longrightarrow \boxed{z^* = 4}$

$\boxed{x_{12}^* = x_{11}^*}$ $\longrightarrow \boxed{y^* = (-4, 4)}$

$$\Longrightarrow \quad x_{21}^* = 10 - 4 - x_{11}^*$$
$$\Longrightarrow \boxed{x_{21}^* = 6 - x_{11}^*}$$

and $x_{22}^* = 2 + 4 - x_{12}^* \Longrightarrow \boxed{x_{22}^* = 6 - x_{11}^*}$

So the PARETO SET writes.

$$\boxed{P = \left\{ x_1 = (x_{11}, x_{11}),\ x_2 = (6 - x_{11}, 6 - x_{11}),\ y = (-4, 4)\ ;\ \text{with } x_{11} \in [0, 6] \right\}.}$$

maximal value it can get to keep $x_2 \geq 0$

$\square$