Université Paris 1 Panthéon Sorbonne
Exam M2 IRFA - MMMEF, 2022-2023

Exam: market risk measures

20th January 2023

- Documents and cell phone are prohibited. Calculator is allowed
- The duration of the exam is **2h00**.
- It is forbidden to leave without returning the copy of your exam sheet.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions.

Exercice 1 (Questions on the lectures (7 pts))

1. (2 pts) For a given loss L and confidence level $\alpha \in (0,1)$, recall the definitions of the Value-at-Risk at level α of L denoted by $\mathrm{VaR}_\alpha(L)$ and the Expected Shortfall, also known as Conditional Value-at-Risk, at level α of L denoted by $\mathrm{ES}_\alpha(L)$.
2. (1 pt) Under which sufficient assumption on L is the $\mathrm{VaR}_\alpha(L)$ unique?
3. (1 pt) Under the above assumption, provide the characterization (different from the very definition) of the $\mathrm{VaR}_\alpha(L)$ in terms of the cumulative distribution function of L.
4. (1 pt) Using the previous characterization on L, prove that for any $c \geqslant 0$ and $d \in \mathbb{R}$, one has
$$\mathrm{VaR}_\alpha(cL+d) = c\mathrm{VaR}_\alpha(L) + d.$$
5. (1 pt) We now assume that $L \overset{d}{=} \mathcal{N}(\mu, \sigma^2)$ for some parameters $\mu \in \mathbb{R}$ and $\sigma \geqslant 0$. Prove that for any $\alpha \in (0,1)$
$$\mathrm{VaR}_\alpha(L) = \mu + \sigma\Phi^{-1}(\alpha).$$
6. (1 pt) What is the main drawback of assessing the risk of L using $\mathrm{VaR}_\alpha(L)$?

Exercice 2 (Properties of a class of risk measures (7 pts))

Let $p \in [1,\infty)$. For $X \in L^p(\mathbb{P})$, we recall that the norm of X is defined by $\|X\|_p = \mathbb{E}[|X|^p]^{\frac{1}{p}}$. For x, we let $x_+ = \max(x,0)$. For $\tau \in [0,1]$, we define the map $\rho_\tau : L^p(\mathbb{P}) \to \mathbb{R}$ by
$$\rho_\tau(X) = \mathbb{E}[X] + \tau\|(X - \mathbb{E}[X])_+\|_p.$$

1. (2 pts) Prove that for any $c \geqslant 0$ and any $d \in \mathbb{R}$, it holds
$$\rho(cX+d) = c\rho(X) + d.$$
What is the name of the above property in the language of risk measures? What is its financial interpretation?
2. (1 pt) Prove that ρ is sub-additive.
3. (2 pts) Show that if $X \leqslant 0$ then $\rho(X) \leqslant 0$.
4. (1 pt) Deduce from the previous question that ρ is monotone.
5. (1 pt) Is ρ_τ a coherent risk measure for any $\tau \in [0,1]$?

Exercice 3 (Shortfall risk measures (6 pts)) Let $\ell : \mathbb{R} \to \mathbb{R}_+$ be a non-decreasing and convex function satisfying $\lim_{x\to+\infty} \ell(x) = +\infty$. Let x_0 be a fixed real number and assume that $\lim_{x\to-\infty} \ell(x) = \underline{\ell} < \ell(x_0)$. For any loss $L \in L^1(\mathbb{P})$ satisfying $\mathbb{E}[\ell(L-\xi)] < \infty$ for any $\xi \in \mathbb{R}$, we define the **shortfall risk measure** by
$$\rho(L) = \inf\{\xi \in \mathbb{R} : \mathbb{E}[\ell(L-\xi)] \leqslant \ell(x_0)\},$$
with the usual convention that $\inf \varnothing = +\infty$.

1. (2 pts) Our aim in this question is to prove that $\rho(L) \in \mathbb{R}$.
   (a) (1 pt) Using Jensen's inequality and the assumption on the function ℓ, prove that the set $\{\xi \in \mathbb{R} : \mathbb{E}[\ell(L-\xi)] \leqslant \ell(x_0)\}$ is bounded from below.

(b) (1 pt) Prove that $\{\xi \in \mathbb{R} : \mathbb{E}[\ell(L-\xi)] \leqslant \ell(x_0)\} \neq \varnothing$. Conclude.

2. (1 pt) Prove that $\rho(L)$ is the lowest solution to the equation

$$\mathbb{E}[\ell(L-\xi)] = \ell(x_0).$$

3. (1 pt) Prove that the risk measure ρ satisfies the cash invariance (also known as translation invariance) principle.

4. (1 pt) Propose a numerical algorithm to compute $\rho(L)$ based on a sequence $(L^{(n)})_{n\geqslant 1}$ of i.i.d. copies of the random variable L.

5. (1 pt) **Example:** We here set $\ell(x) = \exp(\lambda x)$, for some $\lambda > 0$. Provide an explicit expression for $\rho(L)$. This risk measure is called *the entropic risk measure of L.*