FINAL EXAM. QEM. 2023.

Exercise 3: Robinson Crusoe. 1c 1f.

$u(x) = x_2\sqrt{x_1}$. $\omega = (4, 1)$.

$f(z) = 2\sqrt{z}$.

a) Provide graphical rep., identify feasible allocations, and Pareto wasteful allocations.

b) Compute supply and profit fct. of the firm.

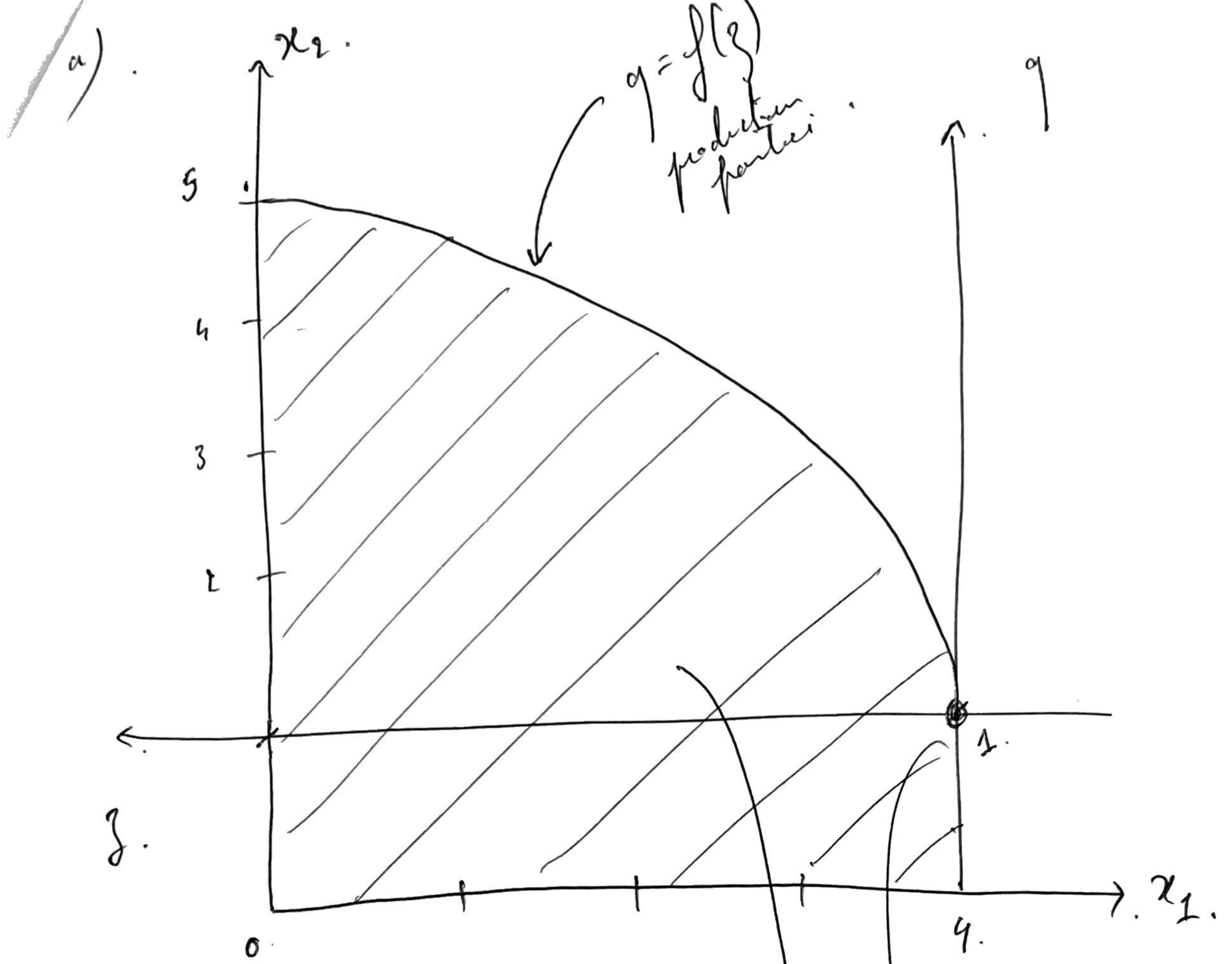
c) —— demand of the consumer.

d). Give def of. competitive eq., find them and rep. in graph. a).

e). Give def of Pareto opt. Can each be supported as a competitive eq? find supporting price ratios.

f) Modify the firm prod fct. s.th. the Pareto alloc. found in e) cannot be supported as competitive eq.
(graph answer is enough a)).

a)



at market clearing, we have

$$x = \omega + y$$

initial endowment $\omega = (4, 1)$

i.e. $\begin{cases} x_1 = 4 - z \\ x_2 = 1 + q \end{cases}$

feasible allocations i.e. with $q \leq f(z)$

$q = f(z)$

$\equiv$ non wasteful allocations

1). let $p \gg 0$. we want to solve the PMP.

$$\begin{cases} \max \; p \cdot y \\ f(y) \leq 0 \end{cases} \iff q = f(z) \text{ ie efficient production. and.} \begin{cases} \max\limits_{z \geq 0} \; p_q f(z) - p_z z \end{cases}$$

so $z > 0$ and 6. must solve.

$$\frac{\partial}{\partial z}\left(p_q f(z) - p_z z\right) = 0.$$

FOC is sufficient because f is concave.

ie. $\frac{p_z}{p_q} = \frac{1}{\sqrt{z}} \longrightarrow \boxed{z(p) = \left(\frac{p_q}{p_z}\right)^2}$.

$$\boxed{q(p) = f(z(p)) = 2\,\frac{p_q}{p_z}}.$$

The supply of the firm, as a fct of the price ratio, is:

$$y(p) = (-z(p),\, q(p)).$$

The profit fct is $\pi(p) = p \cdot y(p) = p_q \frac{2p_q}{p_z} - p_z \cdot \left(\frac{p_q}{p_z}\right)^2.$

$$\Longrightarrow \quad \boxed{\Pi(p) = \frac{p q^2}{p_3}}.$$

c) let $p \gg 0$, the wealth of the consumer, assuming they are the owner of the firm and hence receive $\Pi(p)$.

i.e. $\underline{w = p.\omega + \Pi(p)}$.

we want to solve the UMP.

$$\begin{cases} \max u(x) \\ \text{s.th. } p.x \leq w \end{cases}$$

KT condt.

$\downarrow$

$x \gg 0$ sol$^{\text{t}}$ if

$$\begin{cases} \nabla u(x) = \lambda p. \quad \lambda \geq 0. \\ \text{and.} \\ \lambda (p.x - w) = 0. \end{cases}$$

u mono.

$\Longrightarrow \lambda \neq 0 \Longrightarrow \boxed{p.x = w.}$

i.e. Walras' law.

$$\nabla u(x) = \begin{pmatrix} \dfrac{x_2}{2\sqrt{x_1}} \\ \sqrt{x_1} \end{pmatrix} = \lambda p$$

$$MRS_{12} = \frac{\frac{\partial u}{\partial x_1}(x)}{\frac{\partial u}{\partial x_2}(x)} = \boxed{\frac{1}{2}\frac{x_2}{x_1} = \frac{p_j}{p_g}}$$

$$p \cdot x = w = p_j x_1 + p_g x_2 = w.$$

$$\Longrightarrow x_2 = \frac{w}{p_g} - \frac{p_j}{p_g} x_1.$$

$$\Longrightarrow \frac{1}{2}\left(\frac{w}{p_g} - \frac{p_j}{p_g} x_1\right)\frac{1}{x_1} = \frac{p_j}{p_g}.$$

$$3\frac{p_j}{p_g} = \frac{w}{p_g}\frac{1}{x_1}.$$

and. $$\boxed{x_1 = \frac{1}{3}\frac{w}{p_j}, \quad x_2 = \frac{2}{3}\frac{w}{p_g}.}$$

So the demand of the consumer, as a fct. of the price system, is:

$$x(p) = \begin{pmatrix} x_1(p) \\ x_2(p) \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \frac{p.\omega + \Pi(p)}{p_1} \\ \frac{2}{3} \frac{p.\omega + \Pi(p)}{p_2} \end{pmatrix}.$$

---

d) i) A competitive equilibrium is an allocation, production plan and price system (x^*, y^*, p^*) s.th.

$$\begin{cases} x^* \text{ solves UMP } \begin{cases} \max u(x) \\ p^*.x \leq p^*.\omega + p^*.y^* \end{cases} \\ y^* \text{ solves PMP } \begin{cases} \max p^*.y \\ F(y) \leq 0 \end{cases} \\ x^* = \omega + y^* \quad \text{i.e. we have market clearing for both goods.} \end{cases}$$

---

ii) in b) we found. $y^*(p^*)$.

in c) we found $x^*(p^*)$.

now we just have to impose the market clearing condt°. (for 1 good is enough) to find the equilibrium price ratio p^* which reconcile both demands.

$$x^* = \omega + y^*.$$

$$\Rightarrow \begin{cases} x_1^* = 4 - z^* \\ \text{and.} \\ x_2^* = 1 + q^* \end{cases}$$

$$\Pi(p) = p_q^2 / p_3.$$

take the 1st eq., we must have:

$$x(p) = \frac{1}{3} \frac{p.\omega + \Pi(p)}{p_q} = 4 - z(p) = 4 - \left(\frac{p_q}{p_3}\right)^2.$$

$$= \frac{1}{3}\left(4 + \frac{p_q}{p_3} + \left(\frac{p_q}{p_3}\right)^2\right).$$

$$\longrightarrow \quad -8 + \frac{p_1}{p_2} + 4\left(\frac{p_1}{p_2}\right)^2 = 0.$$

$$\Delta = 1 + 128 = 129 > 0.$$

take only the positive root.

$$\longrightarrow \quad \left.\frac{p_1}{p_2}\right|_* = \frac{-1+\sqrt{129}}{8} \simeq 1{,}3$$

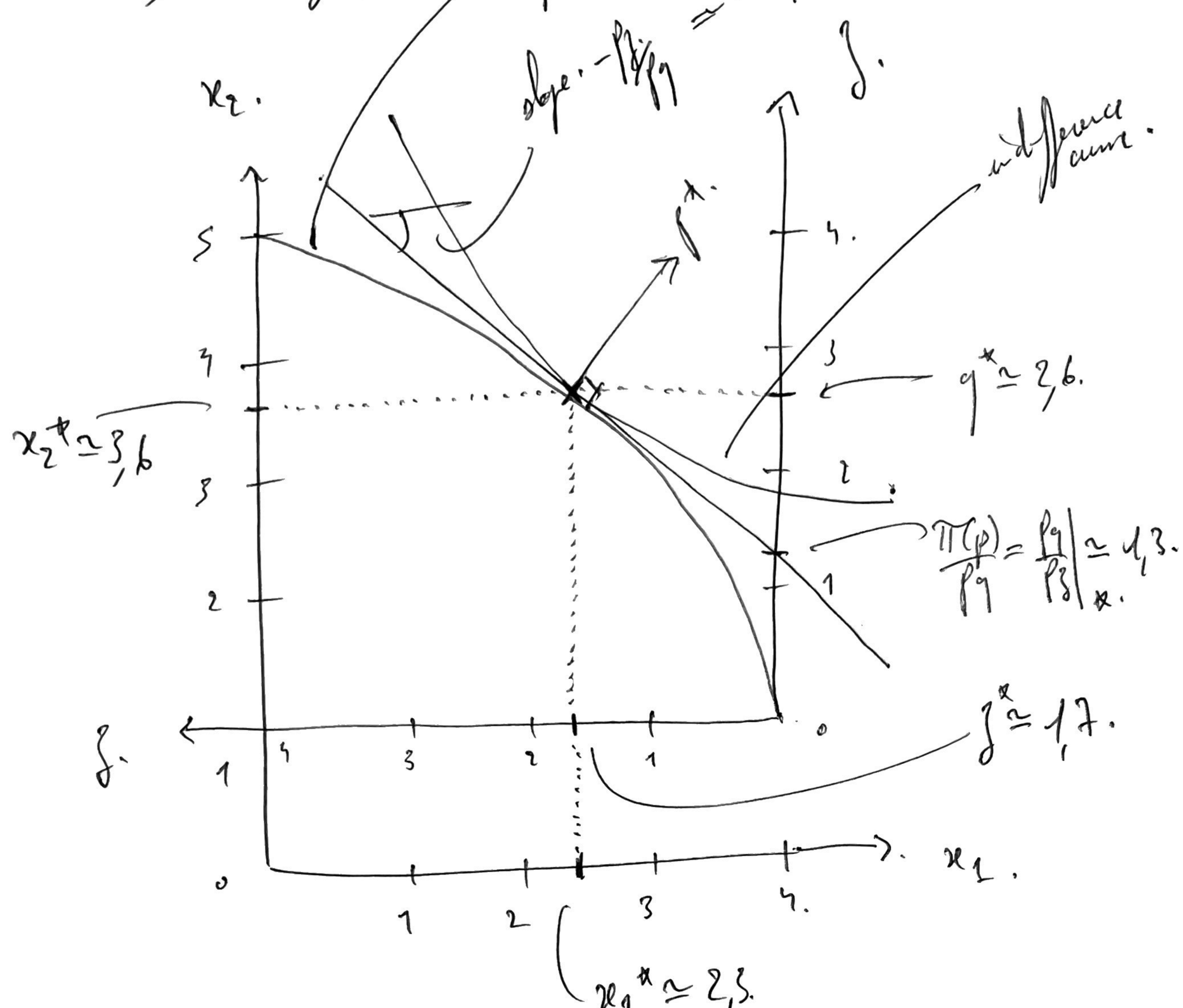
$$\longrightarrow \quad q^* = 2\left.\frac{p_1}{p_2}\right|_* \simeq 2{,}6$$

$$z^* = \left(\left.\frac{p_1}{p_2}\right|_*\right)^2 \simeq 1{,}7$$

$$x_1^* = 4 - z^* \simeq 2{,}3$$

$$x_2^* = 1 + q^* \simeq 3{,}6$$

iii) So, on the graph.



e) i) A Pareto optimal allocation (x^*, y^*) is: with it is impossible to make one consumer strictly better off without making another one strictly worse off.

ii) in the Robinson Crusoe case, when there is only one consumer, it just means that we are maximizing the utility of the said consumer under production constraints.

ie. $\{x^*, y^*\}$ must solve.

$$\begin{cases} \max\ u(x). \\ \text{s.th.}\quad x = \omega + y. \\ \quad \text{and.}\ f(y) \leq 0. \end{cases}$$

ie market clearing, = no waste.

ie production is feasible.

$x \gg 0$ sol^n. cf.

KT cond^n.
$$\begin{cases} \nabla_x u(x) \overset{(*)}{=} \lambda \nabla_x f(\underbrace{x-\omega}_{=y}) \\ \text{and.} \\ \lambda f(\underbrace{x-\omega}_{=y}) = 0. \end{cases} \qquad \lambda \geq 0.$$

u, f mono.
$\Longrightarrow \lambda \neq 0 \rightarrow f(y) = 0.$ ie production is efficient.

ie $q = f(z)$.

and. $(*)$.
$$\Longrightarrow \underset{=}{\nabla_x u(x)} = \lambda \nabla_x f(x-\omega) = \lambda \underset{=}{\nabla_y} f(y).$$

$x = \omega + y$.

dividing the i^{th} line by the second, we get:

$$MRS_{12}(u) = \frac{\frac{\partial u}{\partial x_1}(x)}{\frac{\partial u}{\partial x_2}(x)} = MRT_{12}(y) = \frac{\frac{\partial F}{\partial y_1}(y)}{\frac{\partial F}{\partial y_2}(y)}$$

I

So the Pareto set is :

$$P = \Big\{ (x,y) \;;\; \text{with } x = \omega + y \text{ (market clearing)}, \text{ and } F(y) = 0 \text{ (efficient production)}, \text{ and } MRS_{12}(u) = MRT_{12}(y) \Big\}.$$

II

$$MRS_{12}(u) = \frac{1}{2}\frac{x_2}{x_1}.$$

$$u(x) = x_2\sqrt{x_1}.$$

and $$MRT_{12}(y) = \frac{\frac{\partial}{\partial z} f(z)}{1} = \frac{1}{\sqrt{z}}.$$

$$F(y) = q - f(z) = y_2 - f(-y_1).$$

MC. $\quad x = \omega + y \implies \begin{cases} x_1 = 4 - z \\ x_2 = 1 + q \end{cases}$

$F(y) = 0 \implies q = f(z) = 2\sqrt{z}$

$MRS_{12}(x) = MRT_{12}(y) = \dfrac{1}{\sqrt{z}}$

4 equations for 4 unknowns! system is solvable.

4

$$\frac{1}{2}\frac{x_2}{x_1} = \frac{1}{2}\,\frac{1+q}{4-z} = \frac{1}{2}\,\frac{1+2\sqrt{z}}{4-z} = \frac{1}{\sqrt{z}}$$

i.e. $\quad \sqrt{z} + 2z = 8 - 2z$

i.e. $\quad 4z + \sqrt{z} - 8 = 0$

again. $\Delta = 1 + 128 = 129 > 0$

again take only positive root.

$$\implies \sqrt{z} = \frac{-1+\sqrt{129}}{8} \simeq 1{,}3$$

so.

$$z^* = \left(\frac{-1+\sqrt{129}}{8}\right)^2 \simeq \underline{1{,}7}$$

$$\left\{\begin{array}{l} q^* = 2\sqrt{z} \simeq \underline{2{,}6} \\ \\ x_1^* = 4 - z^* \simeq \underline{2{,}3} \\ x_2^* = 1 + q^* \simeq \underline{3{,}6} \end{array}\right.$$

is the unique Pareto optimal allocation ...

ii). It can be supported as a price equilibrium because Y is convex. (and $U(t)\ \forall t$).

requirement for the 2nd Welfare theorem to hold.

iii) The supporting price ratio must solve.

$$MRS_{12}(x^*) = MRT_{12}(y^*) = \left.\frac{p_z}{p_q}\right|_{a}.$$

FOC to satisfy PMP and UMP!

$$\longrightarrow \quad \left.\frac{p_2}{p_1}\right|_* = \frac{1}{\sqrt{3^*}} \simeq 0{,}77$$

f) φ convex. $\Longleftrightarrow$ f concave.

So it suffices to choose f not concave, e.g. convex, for the 2nd Welfare theorem not to hold anymore.

Graphically:

Take for instance.

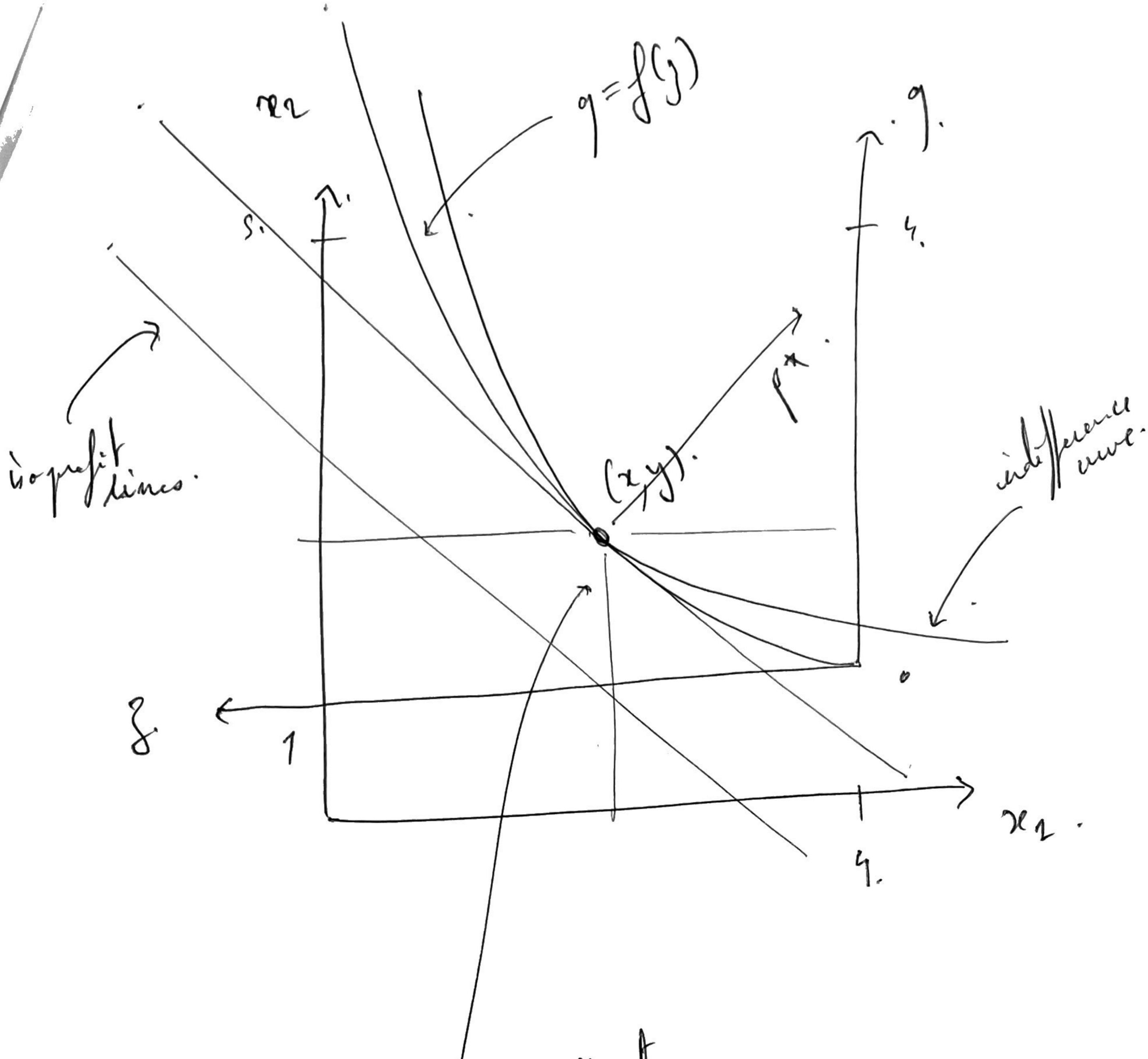
$$f(q) = \alpha q^2 \quad \text{for some } \alpha > 0.$$

we may still find a point s.th. it is a Pareto optimum.,

but because f is now convex, at p^* s.th.

$$\left.\frac{p_2}{p_1}\right|_* = MRT_{12}(y^*)$$

this point will now be a minimum, instead of a sol. to the PMP!



Pareto optimal allocation.

i.e. s.th. $MRS_{12}(x) = MRT_{12}(y)$.

$x = \omega + y$.

and. $f(y) = 0$.

(x, y) cannot be supported as a price equilibrium because y does not solve the PMP.

(FOC $\Longrightarrow$ y is an optimum; but f concave $\Longrightarrow$ y is a <u>minimum</u>)