# Master MMMEF, 2025-2026<br>Lectures notes on:<br>General Equilibrium Theory:<br>Economic analysis of financial markets[1]

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[1]I would like to thank Monique Jeanblanc, who have introduced me on this topic many years ago, Martine Quinzii, who was, with Michael, the author of the reference book on this matter [15], but also Bernard Cornet and Jean-Marc Bonnisseau, from whom I have borrowed a great part of the following notes and who is the author of decisive contributions and finally Cuong Le Van and Monique Florenzano.

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# Chapter 1

# Introduction

The purpose of this lecture notes is a presentation of the recent developments of the economic analysis of financial markets, also called incomplete market theory, with a special emphasis on the link with the financial literature studying the financial markets from a stochastic process point of view. Indeed, it seems very important that the students trained in finance had some knowledge on the economic background of the financial markets and some understanding on the market mechanisms from which the asset prices emerge.

The fundamental justification of the financial markets is the need of consumers, traders, economic agents to transfer money over time and over the events in the future to smooth their income all along the life or to insure them against bad circumstances. That is why we start by a presentation of the modelization of time and uncertainty with a finite date - event tree, which exhibits all features of the real cases and is easier to manipulate than the continuous time models with a continuum of states of nature.

We then define the commodity markets as in a standard microeconomic model and we show how the competitive equilibrium can be extended to this framework by considering a full set of contingent commodity markets as in Chapter 7 of Debreu [9]. Even if this approach is not realistic, it remains important as a benchmark for the other models. Indeed, the equilibrium allocations are then Pareto optimal.

The next step is the presentation of the Arrow Securities which shows that the need is not to promise the delivery of some quantities of commodities but just to transfer wealth which is then used to buy commodities on the spot markets. The main result of this part is the fact that an equilibrium with Arrow Securities is equivalent for the consumers to an equilibrium with contingent commodities with the suitable actualisation of the spot prices at the initial date. The price of the Arrow securities reveals the optimal present value factors for the consumers.

We then briefly present the pure spot market model as an example of a very inefficient market organisation.

Then, we address the main subject by presenting financial assets, financial structure and the associated equilibrium concept, which is the core of the theory. We are then closer to the contracts exchanged on the real financial markets. Nevertheless, to remain at a tractable level of complexity, we restrict our presentation to the two period model. We finally describe the three main categories of assets: real, numéraire and nominal.

The next section is devoted to the central concept of arbitrage opportunities, absence of arbitrage and the characterisation of arbitrage free asset prices. We state and prove the fundamental theorem about the existence of a positive present value vector associated to a no-arbitrage asset price. This is actually a simple direct consequence of the separation theorem between a simplex and a linear subspace. We explain the link between this present value vector, the interest rate and the so-called risk neutral probability measure when the consumers are maximising an expected returns for a subjective probability.

Then, we introduce the notion of redundant asset and we explain how we can eliminate the redundant assets without changing the transfer opportunities of the agents. The absence of redundant asset implies that for an affordable consumption, there is a unique portfolio supporting it or, there is a one to one correspondence between marketable payoffs and portfolios. This concept is important for the next section on pricing by arbitrage since this notion is based on the redundancy.

The pricing by arbitrage is a major topic in finance since it is at the basis of the pricing of option and other derivatives. We present the principle of this method for a redundant asset of an existing financial structure. We explain it from an economic point of view, in particular the fact that this new asset does not modify the market outcome only if it is redundant with the existing ones. We show how the price of this asset is computed from the present value vector, which means that it is the discounted expectation of the payoffs computed with a risk-neutral probability. We also show that the price is well defined even if they are several risk-neutral probabilities.

In the next subsection, we briefly present the over hedging pricing which is sometimes called the cost of the financial structure. This pricing is relevant for the assets which are not redundant with the existing ones. We provide the basic property of the cost function and we give an interpretation of this price as the highest price for which the new asset would enlarge the transfer possibilities of the consumers if a market is open for it.

Then we extend the characterisation of absence of arbitrage opportunities when the consumers face short selling constraints, which is the framework of the seminal paper of Radner [19]. This is the only parts where we consider market restrictions represented by the portfolio sets of the consumers. In this case, the asset price is no more equal but higher than the discounted expected value of the payoffs. It comes from the fact that the consumers cannot get advantage of some arbitrage opportunities if it is necessary to sell larger and larger quantities to get an higher payoff.

The next subsection is devoted to the notion of complete financial structures, defined as those which offer the same possibilities of transfer than a complete set of contingent commodities. Then the equilibrium allocations at the financial equilibrium are the same than the one at the contingent commodity equilibrium and, thus, are Pareto optimal. We show that, when the spot prices are all not vanishing, the financial structure is complete if the rank of the payoff matrix is equal to the number of states of nature. Then we deduce that the present value vector associated to an arbitrage free asset price is unique. In that case, any new asset is redundant with the existing one and can be priced by arbitrage, or, equivalently, the cost function of the financial structure is equal to the pricing by arbitrage.

The final subsection is devoted to the notion of equivalent financial structures. Indeed, what really matters for the consumers is not the assets themselves but the marketable payoffs that they generate. So, we say that two financial structures are equivalent if they offer the same transfer possibilities to the consumers. We show that, when the spot prices are all not vanishing, two financial structures are equivalent if and only if the ranges of the payoff matrices coincide. We reinterpret the elimination of redundant assets by noticing that a nominal financial structure is equivalent to a financial structure without redundant asset.

The last section is a fist approach of the existence of a financial equilibrium. This deserves very long developments and we just present the simplest result and state the sufficient condition for the other case. Indeed, when we have a two-period pure wealth economy, that is only one commodity per state, then, following Hart [13], we show that we can build on auxiliary exchange economy and that we can easily deduce a financial equilibrium of the original economy from a competitive equilibrium of the auxiliary economy. So, in this particular framework, we are able to provide a complete existence proof based on the well-known Arrow-Debreu existence result.

When we switch to the general case with several commodities per states, the issue is much more complex. In particular, the payoff matrix may exhibit a drop of its rank for some spot prices, leading to discontinuities in the demands of the consumers and then to the non-existence of an equilibrium. We provide an example to illustrate this phenomenon due to Hart [14]. But, the question of the boundedness of the attainable portfolios is also at stake since, in the unconstrained case, the portfolio sets are not bounded from below as often assumed for the consumption sets, to get the boundedness of the attainable consumptions. Nevertheless, using several elaborated arguments, including the so-called Cass trick for nominal assets, it is possible to overcome these difficulties to get a general existence result for nominal assets, for numéraire asset under the strong desirability of the numéraire, and a generic existence result for real assets under the usual differentiable assumptions on the preferences.

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# Chapter 2

# Exchange economies with time and uncertainty

We present the model and the notations, which are borrowed from Angeloni-Cornet[1] and are essentially the same as those of Magill-Quinzii[15].

## 2.1 Time and uncertainty

We[1] consider a multi-period exchange economy with $(T+1)$ dates, $t \in \mathcal{T} := \{0, ..., T\}$. The uncertainty is described by a date-event tree $\mathbb{D}$ of length $T+1$. The set $\mathbb{D}_t$ is the set of nodes (also called date-events) that could occur at date $t$ and the family $(\mathbb{D}_t)_{t\in\mathcal{T}}$ defines a partition of the set $\mathbb{D}$; for each $\sigma \in \mathbb{D}$, we denote by $t(\sigma)$ the unique date $t \in \mathcal{T}$ such that $\sigma \in \mathbb{D}_t$.

At date $t = 0$, there is a unique node $\sigma_0$, that is $\mathbb{D}_0 = \{\sigma_0\}$. As $\mathbb{D}$ is a tree, each node $\sigma$ in $\mathbb{D}\setminus\{\sigma_0\}$ has a unique immediate predecessor denoted $pr(\sigma)$ or $\sigma^-$. At each period $t \geq 1$, the mapping $pr$ maps $\mathbb{D}_t$ to $\mathbb{D}_{t-1}$. Each node $\sigma \in \mathbb{D} \setminus \mathbb{D}_T$ has a set of immediate successors defined by $\sigma^+ = \{\xi \in \mathbb{D} : \sigma = \xi^-\}$.

At each period $t \geq 1$, only one node prevails among the immediate successors of the node, which prevailed at the period $t-1$. So, the sequential revelation of uncertainty is represented by a path joining the initial node $\sigma_0$ to a terminal node $\sigma_T$. This path will be denoted by $(\sigma(0) = \sigma_0, \sigma(1), \ldots, \sigma(T) = \sigma_T)$ in such a way that for $t \geq 1$, $\sigma(t-1) = (\sigma(t))^-$. Hence, at a given period $t$, the possible

[1]We use the following notations. A $(\mathbb{D}\times\mathcal{J})$-matrix $A$ is an element of $\mathbb{R}^{\mathbb{D}\times\mathcal{J}}$, with entries $(a^j_\sigma)_{(\sigma\in\mathbb{D}, j\in\mathcal{J})}$; we denote by $A_\sigma \in \mathbb{R}^{\mathcal{J}}$ the $\sigma$-th row of $A$ and by $A^j \in \mathbb{R}^{\mathbb{D}}$ the $j$-th column of $A$. We recall that the transpose of $A$ is the unique $(\mathcal{J}\times\mathbb{D})$-matrix ${}^tA$ satisfying $(Ax)\cdot_{\mathbb{D}} y = x\cdot_{\mathcal{J}} ({}^tAy)$ for every $x \in \mathbb{R}^{\mathcal{J}}$, $y \in \mathbb{R}^{\mathbb{D}}$, where $\cdot_{\mathbb{D}}$ [resp. $\cdot_{\mathcal{J}}$] denotes the usual inner product in $\mathbb{R}^{\mathbb{D}}$ [resp. $\mathbb{R}^{\mathcal{J}}$]. We denote by $\operatorname{rank} A$ the rank of the matrix $A$ and by $\mathrm{Vect}(A)$ the range of $A$, that is the linear sub-space spanned by the column vectors of $A$. For every subset $\tilde{\mathbb{D}} \subset \mathbb{D}$ and $\tilde{\mathcal{J}} \subset \mathcal{J}$, the matrix $A^{\tilde{\mathcal{J}}}_{\tilde{\mathbb{D}}}$ is the $(\tilde{\mathbb{D}}\times\tilde{\mathcal{J}})$-sub-matrix of $A$ with entries $a^j_\sigma$ for every $(\sigma, j) \in (\tilde{\mathbb{D}}\times\tilde{\mathcal{J}})$. Let $x$, $y$ be in $\mathbb{R}^n$; $x \geq y$ (resp. $x \gg y$ ) means $x_h \geq y_h$ (resp. $x_h > y_h$) for every $h = 1, \ldots, n$ and we let $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x \gg 0\}$. We also use the notation $x > y$ if $x \geq y$ and $x \neq y$. The Euclidean norm in the Euclidean different spaces is denoted $\|.\|$ and the closed ball centered at $x$ and of radius $r > 0$ is denoted $\overline{B}(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$.

future states are only the one of the sub-tree with initial node $\sigma(t)$.

For $\tau \in \mathcal{T} \setminus \{0\}$ and $\sigma \in \mathbb{D} \setminus \cup_{t=0}^{\tau-1} \mathbb{D}_t$, we define $pr^{\tau}(\sigma)$ by the recursive formula: $pr^{\tau}(\sigma) = pr(pr^{\tau-1}(\sigma))$. We then define the set of successors and the set of predecessors of $\sigma$ as follows:

$$\mathbb{D}^{+}(\sigma) = \{\sigma' \in \mathbb{D} : \ \exists \tau \in \mathcal{T} \setminus \{0\} \mid \sigma = pr^{\tau}(\sigma')\}$$

$$\mathbb{D}^{-}(\sigma) = \{\sigma' \in \mathbb{D} : \ \exists \tau \in \mathcal{T} \setminus \{0\} \mid \sigma' = pr^{\tau}(\sigma)\}$$

If $\sigma' \in \mathbb{D}^{+}(\sigma)$ [resp. $\sigma' \in \mathbb{D}(\sigma) := \mathbb{D}^{+}(\sigma) \cup \{\sigma\}$], we shall use the notation $\sigma' > \sigma$ [resp. $\sigma' \geq \sigma$]. Note that $\sigma' \in \mathbb{D}^{+}(\sigma)$ if and only if $\sigma \in \mathbb{D}^{-}(\sigma')$ and similarly $\sigma' \in \sigma^{+}$ if and only if $\sigma = (\sigma')^{-}$.

In the example of the event tree of Figure 1,

$$\mathbb{D} = \{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\},$$

$T = 2$, the length of $\mathbb{D}$ is 3, $\mathbb{D}_2 = \{\sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$, $\sigma_1^{+} = \{\sigma_3, \sigma_3, \sigma_5\}$, $\mathbb{D}^{+}(\sigma_2) = \{\sigma_6, \sigma_7\}$, $t(\sigma_3) = t(\sigma_4) = t(\sigma_5) = t(\sigma_6) = t(\sigma_7) = 2$, $\mathbb{D}^{-}(\sigma_3) = \{\sigma_0, \sigma_1\}$, $\mathbb{D}(\sigma_2) = \{\sigma_2, \sigma_6, \sigma_7\}$.

$T = 2$, the length of $\mathbb{D}$ is 3,

| | |
|---|---|
| $\mathbb{D}_0 = \{\sigma_0\}$ | $t(\sigma_0) = 0$ |
| $\mathbb{D}_1 = \{\sigma_1, \sigma_2\}$ | $t(\sigma_1) = t(\sigma_2) = 1$ |
| $\mathbb{D}_2 = \{\sigma_3, \sigma_4, \sigma_5, \sigma_6, \sigma_7\}$ | $t(\sigma_3) = t(\sigma_4) = t(\sigma_5) = t(\sigma_6) = t(\sigma_7) = 2$ |

$\sigma_1^{+} = \{\sigma_3, \sigma_4, \sigma_5\}$, $\mathbb{D}^{+}(\sigma_2) = \{\sigma_6, \sigma_7\}$, $\mathbb{D}^{-}(\sigma_3) = \{\sigma_0, \sigma_1\}$, $\mathbb{D}(\sigma_2) = \{\sigma_2, \sigma_6, \sigma_7\}$.

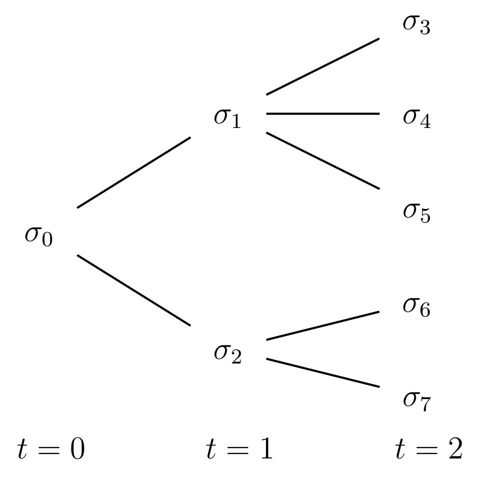


Figure 2.1: The tree $\mathbb{D}$

## 2.2 Commodities and prices

At each node $\sigma \in \mathbb{D}$, there is a finite set of $\ell$ divisible physical goods available. We assume that each good does not last for more than one period. So a commodity is a couple $(h, \sigma)$) of a physical good $h$ and a node $\sigma \in \mathbb{D}$ at which it will be available. Hence the commodity space is $\mathbb{R}^{\mathbb{L}}$, where $\mathbb{L} = \ell \times \text{card}(\mathbb{D})$.

At the initial date 0, a commodity $(h, \sigma)$ for the state $\sigma \in \mathbb{D}_t$ for $t \geq 1$ is a contingent commodity since it will be available only if $\sigma$ prevails at date $t$.

A consumption or consumption plan is an element $x = (x(\sigma))_{\sigma \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$, where the components of $x(\sigma)$ are the quantities of the $\ell$ contingent commodities in the state $\sigma$. For example, to be sure to have one unit of commodity $h$ tomorrow, you need to have one unit of the contingent commodities $(h, \sigma)_{\sigma \in \mathbb{D}_1}$. So, the consumption plan is such that $x_h(\sigma) = 1$ for all $\sigma \in \mathbb{D}_1$.

A price vector $p = (p(\sigma))_{\sigma \in \mathbb{D}} \in \mathbb{R}^{\mathbb{L}}$ specifies the prices of the contingent commodities.

## 2.3 Consumers

The economic agents, called consumers or traders, are in finite number and they are represented by the index $i$, $i \in \mathcal{I}$. Each agent has a consumption set $X_i \subset \mathbb{R}^{\mathbb{L}}$ and her preferences are represented by a utility function $u_i : X_i \to \mathbb{R}$.

Each agent has also an initial endowments $e_i \in \mathbb{R}^{\mathbb{L}}$, which is a basket of contingent commodities. So the endowments $e_{ih}(\sigma)$ for the contingent commodity $h$ at the node $\sigma$ is received by the consumer $i$ only if the date-event $\sigma$ prevails.

**Basic assumptions:**
Assumption C. For all $i \in \mathcal{I}$,

a) $X_i$ is nonempty, convex, closed and bounded from below;

b) $u_i$ is continuous and quasi-concave on $X_i$.

Assumption S. *(Survival Assumption)* For all $i \in \mathcal{I}$, $e_i \in \text{int}\, X_i$.

In the standard general equilibrium model, we usually assume that the preferences are locally non satiated, which means that, for each consumer, each consumption in the consumption set is in the closure of the strictly preferred set or, equivalently, there exists a sequence of strictly preferred consumptions which converges to it. We now state a stronger version of this assumption, entitled Nonsatiation at every state, which plays a key role in many result and, in particular, in the characterisation of the absence of arbitrage opportunity.

Assumption NSS. For all $i \in \mathcal{I}$, for all $x_i \in X_i$, for every $\sigma \in \mathbb{D}$, there exists a sequence $(x_i^\nu)$ of $X_i$ such that $x_i(\sigma') = x_i^\nu(\sigma')$ for all $\sigma' \neq \sigma$, $u_i(x_i^\nu) > u_i(x_i)$ and $\lim_{\nu \to \infty} x_i^\nu = x_i$.

This assumption means that the consumption at each state, node of the tree $\mathbb{D}$, has a real influence on the welfare of the consumers. In other words, there is no negligible state.

**Remark 1** Note that the above assumptions are satisfied when $X_i = \mathbb{R}^{\mathbb{L}}_+$, $e_i \gg 0$ and the preferences are strictly increasing.

To relate this presentation with the one in Finance where we have a probability space as a primary concept, we can assume that there is a probability $\pi_\sigma$ at each node $\sigma'$ on the set $\sigma^+$ of successors of $\sigma$.

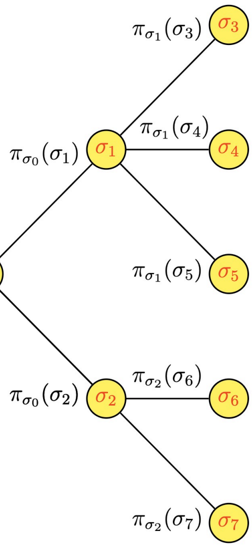


Figure 2.2: Probabilities on the tree $\mathbb{D}$

From these transition probability, one can define a probability $\pi_t$ on the states at period $t$ by the following recursive formula:

$$\pi_t(\sigma) = \pi_{pr(\sigma)}(\sigma)\pi_{t-1}(pr(\sigma))$$

and $\pi_0(\sigma_0) = 1$.

We have in particular a probability $\pi_T$ on the terminal node. We can reconstruct the probabilities $(\pi_t)$ backward, starting from $\pi_T$, as follows: if the probability $\pi_{t+1}$ on $\mathbb{D}_{t+1}$ is known, for all $\sigma \in \mathbb{D}_t$, we define $\pi_t(\sigma) = \sum_{\sigma' \in \sigma^+} \pi_{t+1}(\sigma')$. Note that we can show by induction that these probability is the conditional probability of $\pi_T$ on the set $\mathbb{D}_T$ on the partition $\mathcal{P}_t = \{S_\sigma \mid \sigma \in \mathbb{D}_t\}$ with $S_\sigma = \mathbb{D}^+(\sigma) \cap \mathbb{D}_T$, the set of terminal nodes which are successors of $\sigma$. It is also true that this probability is the conditional probability of $\pi_{t+1}$ on the set $\mathbb{D}_{t+1}$ on the partition $\mathcal{P}_t^{t+1} = \{\sigma^+(\sigma) \mid \sigma \in \mathbb{D}_t\}$, the set of immediate successors of $\sigma$.

With this probability, we can define a discounted expected utility $u$ starting from a Bernouilli utility function $v$ defined on a subset $X$ of $\mathbb{R}^\ell$ as follows:

$$u(x) = \sum_{t=0}^{T} \beta^t \sum_{\sigma \in \mathbb{D}_t} \pi_t(\sigma) v(x(\sigma))$$

where $\beta \in ]0,1[$ is the discount factor or $\beta = \frac{1}{1+r}$ with $r$ being the interest rate. The global welfare of the agent is measured as the discounted sum of the expected welfare at each period. Note that we can slightly generalize this type of utility functions by assuming that the interest rate depends on the period $t$.

# Chapter 3

# Contingent commodity equilibrium and Arrow securities

## 3.1 Contingent commodity equilibrium

The first equilibrium concept presented now is the Contingent commodity equilibrium borrowed from Chapter 7 of Theory of Value [9]. Actually, the interest of this equilibrium comes from the fact that it is a benchmark model for the other concepts studied in this course. Indeed, it is not very realistic but its outcome is Pareto optimal, which is a desirable property.

In this concept, we assume that there is a unique market at state $\sigma_0$ on which all current commodities available at date 0 are exchanged, which is a spot market but also all contingent commodities $(h, \sigma)$ for all commodities and all nodes of the tree $\mathbb{D}$. So, there is $\mathbb{L}$ commodities traded on this market according to a price vector $p \in \mathbb{R}^{\mathbb{L}}$.

Each consumer comes to the market with her current endowments $e_i(\sigma_0)$ and their contingent endowments $(e_i(\sigma))_{\sigma \in \mathbb{D}_1^+}$. The exchanges take place for the current commodity and contracts are signed for the contingent commodity promising the delivery of a given quantity of a given commodity at node $\sigma$ if this node prevails in the future and nothing if this node does not prevail. So, after the market, each consumer has an allocation $x_i^* \in \mathbb{R}^{\mathbb{L}}$. The exchanges take place according to the market price $p^*$. The price for the initial node $\sigma_0$ are ordinary prices whereas the prices for the future nodes are future prices with an irrevocable paiement now for a contingent delivery of one unit of the given commodity at the given node in the future. The consumers have access only to the financially affordable consumption, which means those which are in the budget set:

$$B_i^W(p, p \cdot e_i) = \{x_i \in X_i \mid p \cdot x_i \leq p \cdot e_i\}$$

Finally, the market clearing condition imposes that the sum of the final allocations are equal to the sum of the endowments. Note that the markets are not reopen afterwards so, each consumer expects the consumption in the future according to her final allocation in contingent commodities. So, we get the following definition:

**Définition 1**

A Contingent Commodity equilibrium of the private ownership exchange economy $\mathcal{E} = (\mathbb{D}, \mathbb{R}^{\mathbb{L}}, (X_i, u_i, e_i)_{i \in \mathcal{I}})$ is an element $((x_i^*), p^*)$ of $(\mathbb{R}^{\mathbb{L}})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}}$ such that

(a) [Preference maximization] for every $i \in \mathcal{I}$,

$x_i^*$ is a "maximal" element of $u_i$ in the budget set $B_i^W(p^*, p^* \cdot e_i)$ in the sense $x_i^* \in B_i^W(p^*, p^* \cdot e_i)$ and $B_i^W(p^*, p^* \cdot e_i) \cap \{x_i' \in X_i \mid u_i(x_i') > u_i(x_i^*)\} = \emptyset$;

(b) [attainability]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i.$$

Actually, a Contingent Commodity equilibrium is a standard Walras equilibrium of an exchange economy. We just remark that Assumption (NSS), non satiation state by state, implies that the price $p^*(\sigma)$ in $\mathbb{R}^\ell$ is not vanishing for all $\sigma \in \mathbb{D}$. Usually, the non satiation assumption leads to a non zero price $p^*$. So the existence of a Contingent Commodity equilibrium is obtained under our basic assumptions as a consequence of the Arrow-Debreu existence result for a competitive equilibrium, see [9].

**Theorem 1** *Under Assumptions C, S and NSS, there exists a Contingent Commodity Equilibrium.*

Applying the First and Second Theorems of Welfare Economics (see, [9]), we get the two following results on the optimality of a Contingent Commodity equilibrium allocation and the decentralisation of a Pareto optimal economy.

**Proposition 1** *(Optimality of the equilibrium allocations) Under Assumptions C and NSS, if $((x_i^*), p^*)$ is a Contingent Commodity Equilibrium, then $(x_i^*)$ is a Pareto optimal allocation, which means that it does not exist an allocation $(x_i') \in \prod_{i \in \mathcal{I}} X_i$ such that $\sum_{i \in \mathcal{I}} x_i' = \sum_{i \in \mathcal{I}} e_i$, $u_i(x_i') \geq u_i(x_i^*)$ for all $i \in \mathcal{I}$ with a strict inequality for at least one consumer.*

**Proposition 2** *(Decentralisation of optimal allocations) Under Assumptions C and NSS, let $(\bar{x}_i)$ be a Pareto optimal allocation. Then, there exists a non zero price $\bar{p}$ such that for all $i \in \mathcal{I}$, for all $x_i \in X_i$ such that $u_i(x_i) \geq u_i(\bar{x}_i)$, $\bar{p} \cdot \bar{x}_i \leq \bar{p} \cdot x_i$.*

*Furthermore, if there exists a consumption $\underline{x}_i \in X_i$ such that $\bar{p} \cdot \underline{x}_i < \bar{p} \cdot \bar{x}_i$, then $\bar{x}_i$ is a maximal element for $u_i$ in the budget set $B_i^W(\bar{p}, \bar{p} \cdot \bar{x}_i)$.*

Let us assume that $\bar{x}_i$ is not minimising the cost $\bar{p} \cdot x_i$ over $X_i$ for all $i$. Then, $((\bar{x}_i), \bar{p})$ is a Contingent Commodity Equilibrium of the economy where the initial endowments is $\bar{e}_i = \bar{x}_i$. In other words, if the initial endowments are Pareto optimal, the associated equilibrium is a no trade equilibrium since the equilibrium allocation is equal to the initial endowments.

Let us now come back to the reopening of the markets after the initial state $\sigma_0$. For this, we assume that the preferences of the consumers are represented by a discounted average welfare function, that is:

$$u_i(x_i) = \sum_{t=0}^{T} \beta^t \sum_{\sigma \in \mathbb{D}_t} \pi_t(\sigma) v_i(x_i(\sigma))$$

Let us assume that $((x_i^*), p^*)$ is a contingent commodity equilibrium. At date 1, one state $\sigma_1 \in \mathbb{D}_1$ prevails. If we reopen the market at this date, the consumers will only consider the subtree $\mathbb{D}(\sigma_1)$ of the successors of $\sigma_1$ and the preferences on this new commodity space are represented by:

$$u_i^{\sigma_1}(x_i) = \sum_{t=1}^{T} \beta^{t-1} \sum_{\sigma \in \mathbb{D}_t \cap \mathbb{D}(\sigma_1)} \pi_t(\sigma) v_i(x_i(\sigma))$$

The reopening of the markets does not cancel previous contracts. So, the new initial endowments of the consumers are given by $(e_i'(\sigma))_{\sigma \in \mathbb{D}(\sigma_1)} = (x_i^*(\sigma))_{\sigma \in \mathbb{D}(\sigma_1)}$, which is the sum of the initial endowments and the net trades during the market at date 0. So, we check that the truncated allocation $(x_i^{*\sigma_1})$ where we only keep the consumption for the sub-tree $\mathbb{D}(\sigma_1)$ and the corresponding truncated price $p^{*\sigma_1}$ are a contingent equilibrium. Indeed, otherwise there exists a consumer $i$ and a consumption $\bar{x}_i^{\sigma_1}$ such that $p^{*\sigma_1} \cdot \bar{x}_i^{\sigma_1} \leq p^{*\sigma_1} \cdot e_i$ and $u_i^{\sigma_1}(\bar{x}_i^{\sigma_1}) > u_i^{\sigma_1}(x_i^{*\sigma_1})$. If we extend the consumption $\bar{x}_i$ on $\mathbb{D}$ by defining $\bar{x}_i(\bar{\sigma}) = x_i^*(\sigma)$ for the node $\sigma \notin \mathbb{D}(\sigma_1)$ and $\bar{x}_i(\bar{\sigma}) = \bar{x}_i(\sigma)$ for the node $\sigma \in \mathbb{D}(\sigma_1)$ , we check that $p^* \cdot \bar{x}_i \leq p^* \cdot x_i^* \leq p^* \cdot e_i$ and $u_i(\bar{x}_i) > u_i(x_i^*)$, which is in contradiction with the optimality of $x_i^*$ as an equilibrium consumption for the initial market.

So, the reopening of the markets is useless since the agents have no gain to exchange and the new equilibrium is a no trade equilibrium. Nevertheless this result is valid only if the forecast of the future price and of the future endowments are realised at the node $\sigma_1$ which prevails at date 1. In other words, the anticipations are perfect foresight.

## 3.2 Equilibrium with Arrow securities

The model with a complete set of contingent commodity markets is clearly not realistic even if few markets are really functioning in the world for this kind of future delivery for example for grain or oil at the Chicago Mercantile Exchange. Nevertheless, note that these markets are not pure contingent commodity markets since the delivery at date $t$ is irrevocable and does not depend on the state at this date. So the contract exchanged on this market is a future contract with one unit of each contingent commodity contract for the states at this period.

Kenneth Arrow, in a pioneering work [3], presents another way of describing the possible exchanges in the above framework with time and uncertainty. First of all, there are spot markets at each node $\sigma$ where the current commodities are traded at a spot price $p(\sigma) \in \mathbb{R}^\ell$. This means that the paiements are made at the node $\sigma$ and not at date 0 as for the contingent commodities.

Then, at the initial node $\sigma_0$, a financial market is open where so-called Arrow securities are traded. An Arrow security $j^\sigma$ is associated to a node $\sigma \in \mathbb{D}^+(\sigma_0) = \mathbb{D} \setminus \{\sigma_0\}$. It is a contract signed at date $\sigma_0$ with a paiement at this date which promised to deliver one unit of unit of account at the node $\sigma$ if this node prevails and nothing otherwise. So, it can be understood as an insurance contract again losses at this particular node. The price of this asset on the financial market at $\sigma_0$ is $q_{j^\sigma}$. We assume that all Arrow securities are traded on the financial market. We denote by $q \in \mathbb{R}^{\mathbb{D}^+(\sigma_0)}$ the vector of prices of the Arrow securities.

In this model, the agent $i$ chooses a consumption $x_i \in X_i$ as previously but also a portfolio $z_i \in \mathbb{R}^{\mathbb{D}^+(\sigma_0)}$. So $|z_i(\sigma)|$ is the quantity of the asset $j^\sigma$ sold ($z_i(\sigma) < 0$) or bought ($z_i(\sigma) > 0$) on the financial market at date 0. She is now facing card $\mathbb{D}$ budget constraints, one at each node. At node $\sigma_0$, the budget constraint is:

$$p(\sigma_0) \cdot x_i(\sigma_0) + q \cdot z_i \leq p(\sigma_0) \cdot e_i(\sigma_0)$$

At each node $\sigma \in \mathbb{D}^+(\sigma_0)$, the budget constraint is

$$p(\sigma) \cdot x_i(\sigma) \leq p(\sigma) \cdot e_i(\sigma) + z_i(\sigma)$$

So, at the global level, the budget set of the agent is $B_i^A(p, q, e_i)$ defined by:

$$\left\{ x_i \in X_i \middle| \exists z_i \in \mathbb{R}^{\mathbb{D}^+(\sigma_0)} \begin{array}{l} p(\sigma_0) \cdot x_i(\sigma_0) + q \cdot z_i \leq p(\sigma_0) \cdot e_i(\sigma_0) \\ p(\sigma) \cdot x_i(\sigma) \leq p(\sigma) \cdot e_i(\sigma) + z_i(\sigma), \quad \forall \sigma \in \mathbb{D}^+(\sigma_0) \end{array} \right\}$$

An equilibrium is obtained when all market clearing conditions are satisfied for all markets, spot markets and financial markets. Formally:

**Définition 2**

A Arrow financial equilibrium of the private ownership exchange economy $\mathcal{E} = (\mathbb{D}, \mathbb{R}^{\mathbb{L}}, (X_i, u_i, e_i)_{i \in \mathcal{I}})$ is an element

$$((x_i^*, z_i^*), p^*, q^*) \in (\mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathbb{D}^+(\sigma_0)})^{\mathcal{I}} \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathbb{D}^+(\sigma_0)}$$

such that

(a) [Preference maximization] for every $i \in \mathcal{I}$,

$(x_i^*, z_i^*)$ is a "maximal" element of $u_i$ in the budget set $B_i^A(p^*, q^*, e_i)$ in the sense that

$$\left\{ \begin{array}{l} p^*(\sigma_0) \cdot x_i^*(\sigma_0) + q^* \cdot z_i^* \leq p^*(\sigma_0) \cdot e_i(\sigma_0) \\ p^*(\sigma) \cdot x_i^*(\sigma) \leq p^*(\sigma) \cdot e_i(\sigma) + z_i^*(\sigma), \quad \forall \sigma \in \mathbb{D}^+(\sigma_0) \end{array} \right.$$

and

$$B_i^A(p^*, q^*, e_i) \cap \{x_i \in X_i \mid u_i(x_i) > u_i(x_i^*)\} = \emptyset;$$

(b) [Market clearing conditions on the spot markets]

$$\sum_{i \in \mathcal{I}} x_i^* = \sum_{i \in \mathcal{I}} e_i;$$

(c) [Market clearing conditions on the financial market]

$$\sum_{i \in \mathcal{I}} z_i^* = 0.$$

**Exercise 1** *Show that, under Assumptions C and NSS, the last condition in the above definition, Market clearing conditions on the financial market, is redundant in the sense that if we remove it, an element* $((x_i^*, z_i^*), p^*, q^*)$ *satisfying all of other conditions also satisfies the market clearing condition on the financial market.*

We now compare the two notions of equilibrium and we show that they are equivalent from the point of view of the consumers in the sense that the equilibrium consumption allocations are the same leading to the same welfare levels.

We first remark that a basic application of the absence of arbitrage implies that the prices of the Arrow securities are positive.

**Proposition 3** *Under Assumptions C and NSS, if* $((x_i^*, z_i^*), p^*, q^*)$ *is a Arrow financial equilibrium, then* $q_{j^\sigma}^* > 0$ *for all* $\sigma \in \mathbb{D}^+(\sigma_0)$.

**Proof.** If $q_{j^\sigma}^* \leq 0$ for some $\sigma \in \mathbb{D}^+(\sigma_0)$, then the consumer can increase its quantity $z_i(\sigma)$ without violating the budget constraint at node $\sigma_0$ and enlarge the consumption possibility at node $\sigma$ in such way that she can improve her welfare thanks to Assumption NSS. This is in contradiction with the optimality of $(x_i^*, z_i^*)$ at equilibrium. □

**Proposition 4** *We consider an exchange economy satisfying Assumptions C and NSS.*

*Let* $((x_i^*, z_i^*), p^*, q^*)$ *be a Arrow financial equilibrium. Let* $\tilde{p}^*$ *defined by* $\tilde{p}^*(\sigma_0) = p^*(\sigma_0)$ *and for all* $\sigma \in \mathbb{D}^+(\sigma_0)$, $\tilde{p}^*(\sigma) = q_{j^\sigma}^* p^*(\sigma)$. *Then, for all* $i \in \mathcal{I}$,

$$B_i^A(p^*, q^*, e_i) \subset B_i^W(\tilde{p}^*, \tilde{p}^* \cdot e_i)$$

*Consequently,* $((x_i^*),\ \tilde{p}^*)$ *is a contingent commodity equilibrium.*

*Conversely, let* $((x_i^*),\ \bar{p}^*)$ *be a contingent commodity equilibrium, let* $\bar{q}^*$ *be the asset price such that* $\bar{q}^*(\sigma) = 1$ *for all* $\sigma \in \mathbb{D}^+(\sigma_0)$ *and for all* $i \in \mathcal{I}$, $z_i^*$ *be the portfolio defined by* $z_i^*(\sigma) = \bar{p}^*(\sigma) \cdot (x_i^*(\sigma) - e_i(\sigma))$ *for all* $\sigma \in \mathbb{D}^+(\sigma_0)$. *Then, for all* $i \in \mathcal{I}$,

$$B_i^W(\tilde{p}^*, \tilde{p}^* \cdot e_i) \subset B_i^A(p^*, q^*, e_i)$$

*Consequently,* $((x_i^*, z_i^*), \bar{p}^*, \bar{q}^*)$ *is a Arrow financial equilibrium.*

**Proof.** For the first part, it suffices to multiply the budget constraints at all nodes $\sigma \in \mathbb{D}^+(\sigma_0)$ by $q_{j^\sigma}^*$ and to add them to check that $x_i$ belongs to $B_i^W(\tilde{p}^*, \tilde{p}^* \cdot e_i)$. The market clearing condition for the commodities are the same and the optimality condition for the consumption is obtained thanks to the above relation between the two budget sets.

For the converse part, it suffices to define $z_i(\sigma) = \bar{p}^*(\sigma) \cdot (x_i(\sigma) - e_i(\sigma))$ for all $i \in \mathcal{I}$ and $\sigma \in \mathbb{D}^+(\sigma_0)$ and to check that we obtain the budget constraint at node $\sigma_0$ thanks to the fact that the Arrow-Debreu budget constraint can be written as follows:

$$\bar{p}^*(\sigma_0) \cdot (x_i(\sigma_0) - e_i(\sigma_0)) = - \sum_{\sigma \in \mathbb{D}^+(\sigma_0)} \bar{p}^*(\sigma) \cdot (x_i(\sigma) - e_i(\sigma)) = - \sum_{\sigma \in \mathbb{D}^+(\sigma_0)} z_i(\sigma)$$

For the equivalence of the equilibrium, we check that the market clearing condition for the financial market comes from the fact that for all $\sigma \in \mathbb{D}^+(\sigma_0)$

$$\sum_{i \in \mathcal{I}} z_i^*(\sigma) = \sum_{i \in \mathcal{I}} \bar{p}^*(\sigma) \cdot (x_i^*(\sigma) - e_i(\sigma)) = \bar{p}^*(\sigma) \cdot \sum_{i \in \mathcal{I}} (x_i^*(\sigma) - e_i(\sigma)) = \bar{p}^*(\sigma) \cdot 0 = 0$$

since the market clearing condition for the commodities implies that $\sum_{i \in \mathcal{I}} (x_i^*(\sigma) - e_i(\sigma)) = 0$. □

The above result shows that the two market organisations, with contingent commodities and with Arrow securities, leads to the same outcome even if the markets are not similar. In the next part, we will generalise this conclusion with more general financial structures.

**Remark 2** Note that the equivalence between Contingent commodity equilibrium and Arrow financial equilibrium holds true since we have a complete set of Arrow Securities, that is an Arrow security for each future node of the tree $\mathbb{D}$. Let us show that this is no more the case if some Arrow security are missing.

For example, let us consider the simplest tree $\mathbb{D}$ with $T = 1$ and just one node $\sigma_0$ at date 0 and one, $\sigma_1$ at date 1. We also assume that there is just one commodity per date, $\ell = 1$. Then, we have two agents $\mathcal{I} = \{1, 2\}$ having the identical preferences on $\mathbb{R}^2_+$ defined by $u(x_0, x_1) = x_0 x_1$ and initial endowments $e_1 = (2, 1)$ and $e_2 = (1, 2)$. Then if the unique Arrow security is missing, we have only two spot markets. The budget constraints are $p_0 x_0 \leq 2p_0$ and $p_1 x_1 \leq p_1$ for the first agent and $p_0 x_0 \leq p_0$ and $p_1 x_1 \leq 2p_1$ for the second agent. So, actually, for a positive price vector $p$, the agents are constrained to consume less than their initial endowments. We check that an equilibrium price vector must be positive, otherwise there is no optimal consumption in the budget set. Then, the only possible equilibrium is $x^1 = e_1$ and $x^2 = e_2$, which is not optimal since the utility levels are equal to 2, whereas the utility levels of the attainable allocation $(3/2, 3/2)$, $(3/2, 3/2)$ is $\frac{9}{4} > 2$. Hence the missing Arrow security prevents the market mechanism to reach an optimal solution as the contingent commodity equilibrium does.

**Remark 3** The return of the Arrow securities can be expressed in real terms of the value of a numéraire commodity or a numéraire commodity basket. For example, if a commodity $h$ is chosen as numéraire or if a numéraire commodity basket $\nu \in \mathbb{R}^\ell_{++}$ is chosen, then the return of one unit of the Arrow security

$j^\sigma$ at node $\sigma$ is equal to $p_h(\sigma)$ or $p(\sigma)\cdot\nu$. Then the budget constraints become $p(\sigma)\cdot x_i(\sigma)\le p(\sigma)\cdot e_i(\sigma)+p_h(\sigma)z_i(\sigma)$ or $p(\sigma)\cdot x_i(\sigma)\le p(\sigma)\cdot e_i(\sigma)+(p(\sigma)\cdot\nu)z_i(\sigma)$.

Then the above equivalence results holds true if the equilibrium prices $p_h^*(\sigma)$ or the value of the numéraire basket $p^*(\sigma)\cdot\nu$ are positive at every node $\sigma\in\mathbb{D}^+(\sigma_0)$. For example, it holds true if the preferences of one agent are strictly increasing with respect to the commodities $(h,\sigma)$ for all $\sigma$ or with respect to the consumption $\nu^\sigma$ defined by $\nu^\sigma(\sigma')=0$ for all $\sigma'\neq\sigma$ and $\nu^\sigma(\sigma)=\nu$.

To conclude this section, we explore the link with the standard literature in finance. For this, we assume that we have just one commodity ($\ell=1$) at each node, that is the consumers take care only on their wealths. We normalise the spot price $p(\sigma)$ to 1 on each spot market. Let us consider a risk neutral consumer $i$ with a discounted expected utility $u$ defined by:

$$u_i(x_i)=\sum_{t=0}^{T}\beta^t\sum_{\sigma\in\mathbb{D}_t}\pi_t(\sigma)x_i(\sigma)$$

Assume that the Arrow financial equilibrium allocation $x_i^*$ of this consumer is an interior point. Then the first order optimality conditions tell us that there exists multipliers such that $\lambda_\sigma=\beta^t\pi_t(\sigma)$ and $\lambda_\sigma=\lambda_{\sigma_0}q^*_{j^\sigma}$ for all $\sigma\in\mathbb{D}_t$. So $\beta^t\pi_t(\sigma)=\lambda_{\sigma_0}q^*_{j^\sigma}=q^*_{j^\sigma}$ since $\lambda_{\sigma_0}=p(\sigma_0)=1$. Note that $q^*_{j^\sigma}$ is the cost paid at $\sigma_0$ to have one additional unit of wealth at node $\sigma$ or in other words is the price at date 0 of a unit of wealth at node $\sigma$.

To have one additional unit of wealth at all nodes of date $t$, the cost is $\sum_{\sigma\in\mathbb{D}_t}q^*_{j^\sigma}$. Since $(\pi_t(\sigma))$ is a probability on $\mathbb{D}_t$, the total price is $\beta^t$ at date 0. So in terms of interest rate $r$, we note that the return at date $t$ of a paiement of $\beta^t$ at date 0 is $\beta^t(1+r)^t=1$, or, in other words, $\beta=\frac{1}{1+r}$.

Now we remark that the discounted price process on the final states $\mathbb{D}_T$, $(\frac{1}{\beta^T}q^*_{j^\sigma}=(1+r)^Tq^*_{j^\sigma})_{\sigma\in\mathbb{D}_T}$ defined a "risk-neutral" probability measure on the final states and $(\frac{1}{\beta^t}q^*_{j^\sigma}=(1+r)^tq^*_{j^\sigma})_{\sigma\in\mathbb{D}_t}$ is the conditional probability on the states at date $t$. These are the usual assumptions on a price process in a standard financial model.

## 3.3 Pure spot market economy

In the two previous equivalent organisation of the markets, the consumers have access to enough financial instruments, either contingent commodities or Arrow Securities, to be able to transfer incomes over time and among the different states of the world at each period. So, they reach an optimal allocation of commodities at the end of the market process.

We briefly sketch the other extreme case where there is no financial instruments and only pure spot markets at each node. So the consumers face the following $\operatorname{card}\mathbb{D}$ budget constraints:

$$p(\sigma)\cdot x_i(\sigma)\le p(\sigma)\cdot e_i(\sigma),\quad\forall\sigma\in\mathbb{D}$$

The affordable consumption of a consumer at a node $\sigma$ are those which are less expensive than the endowments at this node computing with the current spot price. As already illustrated in the previous example, the lack of transfer of wealth leads to a non optimal allocation. Clearly, the budget set of an agent is strictly included in the one obtained with Arrow securities.

This observation is the reason which justifies the introduction of a financial market to extend the possibilities of the agents for the transfer of wealth among dates and states of nature. Clearly, the existence of a full set of Arrow securities is far from what we observe on the financial markets. So, we present in the next section a more realistic point of view through the concept of financial structure.

**Remark 4** Note that the above assumptions C, S and NSS are sufficient to guarantee the existence of a pure spot market equilibrium. It suffices to adapt the proof of a standard Competitive equilibrium checking that the budget sets have a closed graph and are lower semicontinuous which implies that the quasi-demands are upper semicontinuous if we truncate in a suitable way the consumption sets. Then, the step from a quasi-equilibrium to an equilibrium is obtained thanks to the survival assumption and the non satiation at each state.

**Exercise 2** *We consider a new organisation of the markets for Arrow securities. At each node $\sigma \in \mathbb{D}$, as usual, we have a spot market where the price for the commodities is denoted $p(\sigma) \in \mathbb{R}^\ell$. Then, at each node $\sigma \in \mathbb{D} \setminus \mathbb{D}_T$, there exists a market for Arrow securities for the nodes, which are immediate successors of $\sigma$, that is the node $\sigma' \in \sigma^+$. The Arrow security $j_{\sigma'}$ has a payoff equal to 1 at node $\sigma'$ and nothing at the other nodes. The price of this Arrow security is denoted $q_{\sigma'}$ and the exchange and the paiement of the Arrow security $j_{\sigma'}$ take place at note $\sigma$.*

*1) Write the budget set of a consumer with this new market organization.*

*2) Show that, under Assumptions C and NSS, all Arrow security prices $q_{\sigma'}$ are positive.*

*3) Show that, under Assumptions C and NSS, the equilibrium allocation for this market organisation is also a contingent commodity market allocation and explain the link between the prices of the contingent commodities on one hand and the spot prices and the Arrow security in the other hand.*