Master MMMEF, 2025-2026
Lectures notes on:
General Equilibrium Theory:
Economic analysis of financial markets[1]

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Chapter 6

Existence of financial equilibria

The general equilibrium theory is a powerful tool to check whether or not a model is consistent. Indeed, it is quite easy to build a model but it is useful only if we are working on an object which exists in a quite large framework. Since the 50', the benchmark model is the Arrow-Debreu existence result and the associated assumptions. The purpose of this section is to present existence results for financial equilibria under assumptions which are at the same level of generality as the one for a competitive equilibrium.

Actually, in the first subsection, we show how to deduce a financial equilibrium from a competitive equilibrium of an auxiliary exchange economy in the one commodity case. We also show which assumptions are needed on the initial economy to get the necessary ones on the exchange economy. So we remark that we are really at the same level of generality as in the Arrow-Debreu existence result.

The second subsection is just a presentation of the most advanced existence results without proofs since they are far beyond the scope of this course.

6.1 Existence for the one commodity case

In this section, we provide an existence result for a two-period economy in the particular case where there is only one commodity per state, $\ell = 1$, which is a pure wealth model as it is commonly assumed in the literature in finance. The consumers take care only of their wealths in the different states.

In this framework, we use a correspondence due to Hart [13] between a financial equilibrium and a Walras equilibrium of an auxiliary economy where the commodities are the consumption at the initial node σ_0 and the assets.

We posit the following additional assumptions to complete Assumption C and S. First of all, $\ell = 1$, so $\mathbb{L} = \mathbb{D}$.

Assumption C1. For all $i \in \mathcal{I}$,

a) $X_i = \mathbb{R}^{\mathbb{D}}_+$;

b) u_i is strictly increasing on X_i.

Note that u_i strictly increasing is just the translation of Assumption NSS when there is only one commodity per state. The financial structure is composed by a finite set of $\mathcal{J}$ real assets and the payoff asset j in state σ is $p(\sigma)V_j(\sigma)$, where $V_j(\sigma)$ is an amount of the unique commodity. We denote by V the $\mathbb{D}_1 \times \mathcal{J}$ matrix whose entries are $V_j(\sigma)$. We assume that

Assumption F1.

a) For all $j \in \mathcal{J}$, $V_j \in \mathbb{R}_+^{\mathbb{D}_1} \setminus \{0\}$;

b) For all $\sigma \in \mathbb{D}_1$, there exists $j \in \mathcal{J}$ such that $V_j(\sigma) > 0$.

c) for all $i \in \mathcal{I}$, $Z_i = \mathbb{R}^{\mathcal{J}}$.

As already noticed in the previous section, assuming that the payoffs are non negative is not so restrictive since, if it is not satisfied, there exists an equivalent financial structure satisfying it under the mild sufficient condition that at least one portfolio has positive returns in all states.

Note that the assumptions a) and b) implies this condition that at least one portfolio has positive returns in all states.

Let us consider a financial equilibrium $((x_i^*, z_i^*), p^*, q^*)$. From the strict monotonicity of the utility functions, we deduces that $p^*(\sigma) > 0$ for all $\sigma \in \mathbb{D}$ and all the budget constraints are binding. So, for all $i \in \mathcal{I}$, for all $\sigma \in \mathbb{D}_1$,

$$p^*(\sigma)x_i^*(\sigma) = p^*(\sigma)e_i(\sigma) + \sum_{j \in \mathcal{J}} z_{ij}^* V_j(\sigma) p^*(\sigma).$$

Hence, $x_i^*(\sigma) = e_i(\sigma) + \sum_{j \in \mathcal{J}} z_{ij}^* V_j(\sigma)$. So, the consumption at date 1 is completely determined by the portfolio chosen on the financial market at date 0. Hence, we can reduce the choice of the consumer to her consumption at date 0 and her portfolio. Furthermore, the equilibrium portfolios must satisfy the constraints $e_i(\sigma) + \sum_{j \in \mathcal{J}} z_{ij}^* V_j(\sigma) \geq 0$ for all $\sigma \in \mathbb{D}_1$.

That is why we consider the following exchange economy $\tilde{\mathcal{E}}$ with the same set of consumers $\mathcal{I}$ than $\mathcal{E}$. The commodity space is $\mathbb{R} \times \mathbb{R}^{\mathcal{J}}$. For each $i \in \mathcal{I}$, the consumption set is $\Sigma_i = \{(x_i(\sigma_0), z_i) \in \mathbb{R} \times \mathbb{R}^{\mathcal{J}} \mid x_i(\sigma_0) \geq 0, e_i^1 + V z_i \geq 0\}$ where $e_i^1 \in \mathbb{D}_1$ is the restriction of e_i to the states in $\mathbb{D}_1$. The utility function is: $\tilde{u}_i(x_i(\sigma_0), z_i) = u_i(x_i(\sigma_0), e_i^1 + V z_i)$ and the endowments are $\tilde{e}_i = (e_i(\sigma_0), 0)$.

The following proposition shows the link between a Walras equilibrium of $\tilde{\mathcal{E}}$ and a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$.

Proposition 23 *Let $((x_i^*, z_i^*), p^*, q^*)$ be a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$. Then, $((x_i^*(\sigma_0), z_i^*), (p^*(\sigma_0), q^*))$ is a Walras equilibrium of $\tilde{\mathcal{E}}$.*

Conversely, let $((\tilde{x}_i(\sigma_0), \tilde{z}_i), (\tilde{p}(\sigma_0), \tilde{q}))$ be a Walras equilibrium of $\tilde{\mathcal{E}}$, then $((\tilde{x}_i, \tilde{z}_i), \tilde{p}, \tilde{q})$ is a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$ with for all $\sigma \in \mathbb{D}_1$

a) $\tilde{p}(\sigma) = 1$ *;*

b) $\tilde{x}_i(\sigma) = e_i(\sigma) + \sum_{j\in\mathcal{J}} \tilde{z}_{ij} v_j(\sigma)$.

Proof. Let $((x_i^*, z_i^*), p^*, q^*)$ be a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$. Then $\sum_{i\in\mathcal{I}} x_i^*(\sigma_0) = \sum_{i\in\mathcal{I}} e_i(\sigma_0)$ and $\sum_{i\in\mathcal{I}} z_i^* = 0$ so the market clearing conditions are satisfied for the exchange economy $\tilde{\mathcal{E}}$. Since x_i^* is affordable for z_i^*, the first budget constraint $p^*(\sigma_0)x_i^*(\sigma_0) + q^* \cdot z_i^* \leq p^*(\sigma_0)e_i(\sigma_0)$ holds true, so $(x_i^*(\sigma_0), z_i^*)$ belongs to the Walras budget set for the price $(p^*(\sigma_0), q^*)$ in the economy $\tilde{\mathcal{E}}$. If there exists $(x_i(\sigma_0), z_i)$ in the Walras budget set such that $\tilde{u}_i(x_i(\sigma_0), z_i) > \tilde{u}_i(x_i^*(\sigma_0), z_i^*)$, then, x_i, defined by $x_i(\sigma) = e_i(\sigma) + \sum_{j\in\mathcal{J}} z_{ij} V_j(\sigma)$ for all $\sigma \in \mathbb{D}_1$, satisfies $u_i(x_i) > u_i(x_i^*)$ and x_i is affordable by z_i for the prices (p^*, q^*) in $\mathcal{E}$. So, we get a contradiction with the fact that x_i^* is optimal as an equilibrium allocation. So, $(x_i^*(\sigma_0), z_i^*)$ is an optimal consumption in the Walras budget set. Consequently, $((x_i^*(\sigma_0), z_i^*), (p^*(\sigma_0), q^*))$ is a Walras equilibrium of $\tilde{\mathcal{E}}$.

Conversely, let $((\tilde{x}_i(\sigma_0), \tilde{z}_i), (\tilde{p}(\sigma_0), \tilde{q}))$ be a Walras equilibrium of $\tilde{\mathcal{E}}$. Since $(\tilde{x}_i(\sigma_0), \tilde{z}_i)$ belongs to σ, $\tilde{x}_i$ belongs to $\mathbb{R}_+^{\mathbb{D}}$. One easily check that $\tilde{x}_i$ is affordable for the portfolio $\tilde{z}_i$ for the price $(\tilde{p}, \tilde{q})$. From the market clearing conditions, $\sum_{i\in\mathcal{I}} x_i^*(\sigma_0) = \sum_{i\in\mathcal{I}} e_i(\sigma_0)$ and $\sum_{i\in\mathcal{I}} \tilde{z}_i = 0$, so one deduces that $\sum_{i\in\mathcal{I}} \tilde{x}_i(\sigma) = \sum_{i\in\mathcal{I}} e_i(\sigma)$ for all $\sigma \in \mathbb{D}_1$. Finally, if there exists x_i affordable for a portfolio z_i in the financial budget set and $u_i(x_i) > u_i(\tilde{x}_i)$, one has for all $\sigma \in \mathbb{D}_1$, $x_i(\sigma) \leq e_i(\sigma) + \sum_{j\in\mathcal{J}} z_i(\sigma) V_j(\sigma)$ since $\tilde{p}(\sigma) = 1$. So, since u_i is strictly increasing, the consumption x'_i define by $x'_i(\sigma_0) = x_i(\sigma_0)$ and $x'_i(\sigma) = e_i(\sigma) + \sum_{j\in\mathcal{J}} z_{ij} V_j(\sigma)$ is financially affordable for z_i and $u_i(x'_i) > u_i(\tilde{x}_i)$. So, from the definition of $\tilde{u}_i$, one deduces that $\tilde{u}_i(x'_i(\sigma_0), z_i) > \tilde{u}_i(\tilde{x}_i(\sigma_0), \tilde{z}_i)$ and $(x'_i(\sigma_0), z_i)$ belongs to the Walras budget set of $\tilde{\mathcal{E}}$. So, we get a contradiction with the optimality of $(\tilde{x}_i(\sigma_0), \tilde{z}_i)$ as an equilibrium allocation. Consequently, $((\tilde{x}_i, \tilde{z}_i), \tilde{p}, \tilde{q})$ is a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$. □

This proposition tells us that the existence of a financial equilibrium in $\mathcal{E}_{\mathcal{F}}$ is equivalent to the existence of a Walras equilibrium in the economy $\tilde{\mathcal{E}}$. We now check that the economy $\tilde{\mathcal{E}}$ satisfies the necessary conditions à la Arrow-Debreu for the existence of a Walras equilibrium but the boundedness of feasible allocations that we will discuss specifically.

Proposition 24 *If the economy* $\mathcal{E}_{\mathcal{F}}$ *satisfies Assumption C, C1, S and F1, then the economy* $\tilde{\mathcal{E}}$ *satisfies: for all* $i \in \mathcal{I}$

a) Σ_i *is nonempty, convex, closed;*

b) $\tilde{u}_i$ *is continuous, strictly increasing and quasi-concave on* σ_i.

c) $\tilde{e}_i \in \operatorname{int} \sigma_i$.

Proof.

a) Σ_i is nonempty, convex, closed since $(0, e_i^1)$ belongs to Σ_i and Σ_i is defined by a finite set of affine inequality constraints.

b) $\tilde{u}_i$ is continuous since u_i is so. It is strictly increasing with respect to $x_i(\sigma_0)$ since u_i is so. With respect to the portfolio component, if $z_i \geq z_i'$, $z_i \neq z_i'$, then there exists $j \in \mathcal{J}$ such that $z_{ij} > z_{ij}'$. Since V_j is non negative and not equal to 0, there exists $\sigma \in \mathbb{D}^1$ such that $V_j(\sigma) > 0$, so $e_i(\sigma) + \sum_{j\in\mathcal{J}} z_{ij}V_j(\sigma) > e_i(\sigma) + \sum_{j\in\mathcal{J}} z_{ij}'V_j(\sigma)$. Reccaling that V has only non negative entries and non zero entries, $e_i^1 + Vz_i \geq e_i^1 + Vz_i'$ and $e_i^1 + Vz_i \neq e_i^1 + Vz_i'$. So,

$$\tilde{u}_i(x_i(\sigma_0), z_i) = u_i(x_i(\sigma_0), e_i^1 + Vz_i) > u_i(x_i(\sigma_0), e_i^1 + V\tilde{z}_i) = \tilde{u}_i(x_i(\sigma_0), \tilde{z}_i).$$

Let $(x_i(\sigma_0), z_i)$, $(x_i'(\sigma_0), z_i\prime)$ and $(x_i''(\sigma_0), z_i'')$ three elements of Σ_i such that

$$\begin{cases} \tilde{u}_i(x_i(\sigma_0), z_i) \leq \tilde{u}_i(x_i'(\sigma_0), z_i') \\ \tilde{u}_i(x_i(\sigma_0), z_i) \leq \tilde{u}_i(x_i''(\sigma_0), z_i''). \end{cases}$$

Let $x_i = (x_i(\sigma_0), e_i^1 + Vz_i)$, $x_i' = (x_i'(\sigma_0), e_i^1 + Vz_i')$ and $x_i'' = (x_i(\sigma_0), e_i^1 + Vz_i'')$. Then, from the definition of $\tilde{u}_i$, $u_i(x_i) \leq u_i(x_i')$ and $u_i(x_i) \leq u_i(x_i'')$. Since u_i is quasi-concave, for all $t \in [0, 1]$, $u_i(x_i) \leq u_i(tx_i' + (1-t)x_i'')$ but

$$\begin{aligned} u_i(tx_i' + (1-t)x_i'') &= u_i(tx_i'(\sigma_0) + (1-t)x_i''(\sigma_0), t(e_i^1 + Vz_i') + (1-t)(e_i^1 + Vz_i'')) \\ &= u_i(tx_i'(\sigma_0) + (1-t)x_i''(\sigma_0), e_i^1 + V(tz_i' + (1-t)z_i'')) \\ &= \tilde{u}_i(t(x_i'(\sigma_0), z_i') + (1-t)(x_i''(\sigma_0), z_i'')). \end{aligned}$$

So $\tilde{u}_i$ is quasiconcave.

c) We remark that $\tilde{e}_i = (e_i(\sigma_0), 0)$ belongs to the interior of σ since $e_i(\sigma_0) > 0$ and $e_i^1 + V0 = e_i^1 \gg 0$ so, no constraint are binding in the definition of Σ_i. □

We now study the feasible set of the economy $\tilde{\mathcal{E}}$ which is

$$\mathcal{A} = \left\{ (\tilde{x}_i) \in \prod_{i\in\mathcal{I}} \Sigma_i \mid \sum_{i\in\mathcal{I}} \tilde{x}_i = \sum_{i\in\mathcal{I}} \tilde{e}_i \right\}$$

So $(\tilde{x}_i = (x_i(\sigma_0), z_i))$ belongs to $\mathcal{A}$ if $\sum_{i\in\mathcal{I}} x_i(\sigma_0) = \sum_{i\in\mathcal{I}} e_i(\sigma_0)$ and $\sum_{i\in\mathcal{I}} z_i = 0$. The key issue for the existence of an equilibrium is to prove that this set is bounded. Then the Arrow-Debreu Theorem implies the existence of an equilibrium for the exchange economy $\tilde{\mathcal{E}}$ and so, the existence of a financial equilibrium for the financial economy $\mathcal{E}_{\mathcal{F}}$.

For the boundedness of $\mathcal{A}$, we have no problem for the first component since $x_i(\sigma_0) \geq 0$ for all i. We focus on the portfolio component. Under which condition is the set

$$\mathcal{A}_Z = \{(z_i) \in \prod_{i\in\mathcal{I}} \mathbb{R}^{\mathcal{J}} \mid \sum_{i\in\mathcal{I}} z_i = 0, \forall i, e_i^1 + Vz_i \geq 0\}$$

bounded?

Indeed, contrary to the usual framework, the set Σ_i are not necessarily bounded from below. This is obvious when the matrix V is not one to one, which means that we have redundant assets. Indeed, in this case, for any non zero useless portfolio $\zeta \in \operatorname{Ker} V$, then $(0, \zeta) \in \Sigma_i$ for all consumers. So, Σ_i is not bounded from below.

But it may be also true with a one to one payoff matrix V. Let us consider an example. We study a date-event tree with three states of nature at date 1. A financial structure has two assets and the payoff matrix is:

$$V = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$$

Then,

$$\Sigma_i = \left\{ (x(\sigma_0), z_1, z_2) \in \mathbb{R} \times \mathbb{R}^2 \middle| x(\sigma_0) \geq 0, \left\{ \begin{array}{c} e_i(\sigma_1) + z_1 + z_2 \geq 0 \\ e_i(\sigma_2) + 2z_1 + z_2 \geq 0 \\ e_i(\sigma_3) + 3z_1 + z_2 \geq 0 \end{array} \right. \right\}$$

We remark that for all $t \geq 0$, $(0, t, -t)$ belongs to Σ_i so it is not bounded from below.

The next proposition shows that even if the consumption sets are not bounded from below, the asset attainable set $\mathcal{A}_Z$ is bounded when V is one to one. We know that this assumption is not really restrictive for the existence of an equilibrium in $\mathcal{E}_{\mathcal{F}}$. Indeed, if V is not one to one, we can consider the reduced financial structure obtained by suppressing the redundant assets. This reduced financial structure is free of useless portfolio or, equivalently, the reduced matrix is one to one. We have shown that we can build a financial equilibrium for the initial financial structure starting from a financial equilibrium for the reduced financial structure. Furthermore, we also check that the reduced financial structure satisfies Assumption F1. This is obvious for Assertions F1 (a) and (c) and is an exercise for Assertion F1 (b).

Proposition 25 *If* $\operatorname{card} I \geq 2$, *$\mathcal{A}_Z$ is bounded if and only if V is one to one.*

Proof. Let us assume that V is one to one. If $\mathcal{A}_Z$ is not bounded, there exists a sequence $(z^\nu)_{\nu \in \mathcal{N}}$ of $\mathcal{A}_Z$ such that $m^\nu = \max\{\|z_i^\nu\| \mid i \in \mathcal{I}\}$ tends to $+\infty$. Let us consider the "normalized" sequence $(\zeta^\nu = \frac{1}{m^\nu} z^\nu)$. From the definition of m^μ, one deduces that this sequence is bounded and $\mu^\nu = \max\{\|\zeta_i^\nu\| \mid i \in \mathcal{I}\} = 1$ for all ν. So, without any loss of generality, we can assume that this sequence converges to some $\bar{\zeta}$ satisfying $\max\{\|\bar{\zeta}_i\| \mid i \in \mathcal{I}\}) = 1$, so $\bar{\zeta} \neq 0$.

Since (z^ν) of $\mathcal{A}_Z$, for all $i \in \mathcal{I}$, $e_i^1 + V z_i^\nu \geq 0$, so $\frac{1}{m^\nu}(e_i^1 + V z_i^\nu) = \frac{1}{m^\nu} e_i^1 + V \zeta_i^\nu \geq 0$. At the limit, we get $V \bar{\zeta}_i \geq 0$ since (m^ν) tends to $+\infty$. Furthermore, since $\sum_{i \in \mathcal{I}} z_i^\nu = 0$, we also get $\sum_{i \in \mathcal{I}} \bar{\zeta}_i = 0$. So, $0 \leq \sum_{i \in \mathcal{I}} V \bar{\zeta}_i = V(\sum_{i \in \mathcal{I}} \bar{\zeta}_i) = V0 = 0$.

Hence, one deduces that $V\bar{\zeta}_i = 0$ for all i, and, since V is one to one, $\bar{\zeta}_i = 0$ for all i, which is in contradiction with $\bar{\zeta} \neq 0$. So $\mathcal{A}_Z$ is bounded.

Conversely, if $\operatorname{card} \mathcal{I} \geq 2$ and V is not one to one. Let $\zeta \in \operatorname{Ker} V \setminus \{0\}$. Let i and j two elements of $\mathcal{I}$. We check that for all $t \in \mathbb{R}$, z^t defined by $z_i^t = t\zeta$, $z_j^t = -t\zeta$, $z_{i'}^t = 0$ for i' different from i and j belongs to $\mathcal{A}_Z$ so $\mathcal{A}_Z$ is not bounded. □

So, summarising the previous discussion, we get the following existence result of a financial equilibrium.

Proposition 26 *If the unconstrained financial economy $\mathcal{E}_{\mathcal{F}}$ satisfies Assumption C, C1, S and* $\operatorname{Im} V \cap \mathbb{R}^{\mathbb{D}_1}_{++} \neq \emptyset$, *then a financial equilibrium exists.*

Proof. First, we have shown that there exists an equivalent financial structure V' such that for all $j \in \mathcal{J}$, $V'_j \in \mathbb{R}^{\mathbb{D}_1}_{+}$. Second, by eliminating the redundant assets, there exists an equivalent financial structure $\bar{V}'$ such that $\bar{V}'$ is one to one and for all $j \in \bar{\mathcal{J}}$, $\bar{V}'_j \in \mathbb{R}^{\mathbb{D}_1}_{+} \setminus \{0\}$. Since the range of the financial structure is the same than the one of V, $\operatorname{Im} \bar{V}' \cap \mathbb{R}^{\mathbb{D}_1}{++} \neq \emptyset$, so Assumption F1 (b) is satisfied. So, the financial economy with the financial structure $\bar{V}'$ satisfies all necessary conditions so that the associated exchange economy $\tilde{\mathcal{E}}$ satisfies the assumptions for the existence of a Walras equilibrium: Assumptions C and S, u_i locally non satiated for all i and the attainable set $\mathcal{A}$ is bounded. From a Walras equilibrium of $\tilde{E}$, one deduces the existence of a financial equilibrium with the financial structure $\bar{V}'$ and, by the equivalence of the financial structures, an equilibrium for $\mathcal{E}_{\mathcal{F}}$. □

6.2 Beyond the one commodity case

In this section, we consider a financial economy $\mathcal{E}_{\mathcal{F}}$ which satisfies the basic Assumptions C, S, NSS and F. We add an assumption on the portfolio sets:

Assumption Z: for all $i \in \mathcal{I}$, Z_i is closed convex and contains 0 in its interior.

We also define the consumption feasible set as follows:

$$\mathcal{A}_X = \left\{ (x_i) \in \prod_{i \in \mathcal{I}} X_i \mid \sum_{i \in \mathcal{I}} x_i = \sum_{i \in \mathcal{I}} e_i \right\}$$

This set is bounded since the individual consumption sets are bounded from below.

For a competitive equilibrium, we need a weaker non satiation assumption. We first provide an example of an economy without financial equilibrium since Assumption NSS is not satisfied whereas each utility function is locally non-satiated.

There are two states of nature at date 1, σ_1 and σ_2, only one commodity at each state and no financial asset, that is a pure spot market framework. They are two consumers $\mathcal{I} = \{i_1, i_2\}$, the consumption sets are $\mathbb{R}^3_+$ and the utility functions are:

$$u_1(x) = x(\sigma_0) - x(\sigma_1) + x(\sigma_2) \quad u_2(x) = x(\sigma_0) + x(\sigma_1) + x(\sigma_2)$$

The initial endowments are $e_1 = e_2 = (1,1,1)$. Assumption NSS is the only one which is not satisfied (for the first agent) since at the allocation $(1,0,2)$, it is impossible to increase the welfare of the first agent by moving only her consumption at state σ_1. There is no equilibrium since on the spot market at σ_1, the demand of the second consumer is infinite when the price is non positive and is equal to 1 when the price is positive. But, for a positive price, the demand of the first consumer is equal to 0. So the sum of the demand is strictly smaller than the endowments at this node, which is equal to 2. Nevertheless, note that a contingent commodity equilibrium exists, which is $x_1^* = \left(\frac{3}{2}, 0, \frac{3}{2}\right)$, $x_2^* = \left(\frac{1}{2}, 2, \frac{1}{2}\right)$ and $p^* = (1,1,1)$.

6.2.1 Bounded portfolio sets or nominal assets

We first state the Radner [19] existence results which assume that the set of attainable portfolio $\mathcal{A}_Z$ defined by

$$\mathcal{A}_Z = \left\{ (z_i) \in \prod_{i \in \mathcal{I}} Z_i \mid \sum_{i \in \mathcal{I}} z_i = 0 \right\}$$

is bounded. Note that the original result of Radner was considering bounded below portfolio sets for which $\mathcal{A}_Z$ is obviously bounded but the proof works under this more general condition.

Theorem 2 *The financial economy has a financial equilibrium under Assumptions C, S, NSS, F and Z and if $\mathcal{A}_Z$ is bounded.*

We can remark that the existence of a pure spot market equilibrium is a consequence of this theorem.

We now consider the case of a nominal asset structure, that is with a payoff matrix V independent of the spot price p. In this case, we can fix the present value vector $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$ and the asset price $q = V^t \lambda$.

Theorem 3 *Let $\mathcal{E}_{\mathcal{F}}$ be an unconstrained nominal financial economy satisfying Assumptions C, S, NSS. Then, for all $\lambda \in \mathbb{R}^{\mathbb{D}_1}_{++}$, there exists a financial equilibrium $((x_i^*, z_i^*), p^*, q^*)$ such that $q^* = V^t \lambda$.*

The proof of this theorem is very similar to the proof for bounded attainable portfolios but it uses the Cass trick [7] presented in Subsection 6.1.

6.2.2 Numéraire assets

We now consider a numéraire asset financial structure where $\nu \in \mathbb{R}^\ell \setminus \{0\}$ is the numéraire and R the matrix of payoffs stated in units of the numéraire. In this case, the payoff matrix $V(p)$ is equal to:

$$V(p) = \begin{pmatrix} p(\sigma_1)\cdot\nu & 0 & \dots & 0 \\ 0 & p(\sigma_2)\cdot\nu & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p(\sigma_{\mathrm{card}\,\mathbb{D}_1})\cdot\nu \end{pmatrix} R$$

and we check that the rank of the matrix $V(p)$ is the rank of R if $p(\sigma)\cdot\nu > 0$ for all $\sigma \in \mathbb{D}_1$. This properties is crucial to deduce the existence of a financial equilibrium for numéraire asset structures from the one for bounded attainable portfolios. Nevertheless, we need to assume that the numéraire commodity basket ν is strongly desirable at a feasible allocation in each state as precisely stated below in Assumption NNS.

Theorem 4 *Let $\mathcal{E}_\mathcal{F}$ be an unconstrained numéraire asset financial economy satisfying Assumptions C, S, NSS. As previously, ν denotes the numéraire basket of commodity in $\mathbb{R}^\ell$. We assume that:*
Assumption NNS: there exists $\rho > 0$ such that for every $x \in \mathcal{A}_X$, for every $\sigma \in \mathbb{D}_1$, there exists $i \in \mathcal{I}$ such that for all $x' \in \mathbb{R}^\mathbb{D}$ satisfying $x'(\sigma') = 0$ and $x'(\sigma) \in B_\ell(\nu,\rho)$, there exists $\tau > 0$ such that $u_i(x_i) < u_i(x_i + \tau x')$;

Then, there exists a financial equilibrium $((x_i^*, z_i^*), p^*, q^*)$ such that $p^*(\sigma)\cdot\nu > 0$ for all $\sigma \in \mathbb{D}$.

6.2.3 The real asset case

The real asset financial structure beyond the numéraire case exhibits a particular difficulty since the rank of the return matrix $V(p)$ may drop at some prices leading to a sharp reduction of the transfer possibilities offer by the financial structure, so a discontinuous demand for the consumers. This explain why the notion of pseudo-equilibrium was introduced as an intermediate concept. The main difficulty is to prove that a pseudo-equilibrium exists. Then a genericity argument shows that the pseudo-equilibrium is actually a financial equilibrium almost everywhere.

A counter example of existence

Hart [14] provides the first example of a real asset financial structure without financial equilibrium. We now present an example which is an adaptation by Cornet of an example of Magill and Shafer [16].

There are two states of nature at date 1, two commodities at each state and two consumers. The financial structure is composed of two real assets. The consumption sets are $\mathbb{R}^6_+$, the utility functions are:

$$u_i(x_i) = U_i(x_i(\sigma_0))\,[U_i(x_i(\sigma_1))]^{\rho_1}\,[U_i(x_i(\sigma_2))]^{\rho_2}$$

with $\rho_1 > 0$, $\rho_2 > 0$, $\rho_1 + \rho_2 = 1$. For $i = 1, 2$, $U_i(a, b) = a^{\alpha^i_1} b^{\alpha^i_2}$ with $\alpha^i_1 > 0$, $\alpha^i_2 > 0$, $\alpha^i_1 + \alpha^i_2 = 1$.

$e_1 = \left(\frac{1}{2}, \frac{1}{2}, 1-\epsilon, 1-\epsilon, \epsilon, \epsilon\right)$ and $e_2 = \left(\frac{1}{2}, \frac{1}{2}, \epsilon, \epsilon, 1-\epsilon, 1-\epsilon\right)$

The matrices representing the two real assets in the two states are identical equal to

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so, given the spot price p, the payoff matrix is

$$V(p) = \begin{pmatrix} p_1(\sigma_1) & p_2(\sigma_1) \\ p_1(\sigma_2) & p_2(\sigma_2) \end{pmatrix}$$

In other words, a unit of the first asset delivers the value of one unit of the first commodity and a a unit of the second asset delivers the value of one unit of the second commodity.

We remark that the rank of $V(p^*)$ at equilibrium is either 1 or 2 since the prices are positive due to the strict monotonicity of the utility function. Then, if the spot prices in the two states are colinear, the rank is 1 and if not, the rank is 2. We show that in both cases, we get a contradiction.

We now prove that it does not exists a financial equilibrium if $\alpha^1 \neq \alpha^2$ and $\varepsilon \neq \frac{1}{2}$.

Let us start by assuming that the rank $V(p^*)$ is 2. Then, in that case, the market is complete and the financial equilibrium is actually a competitive equilibrium for a price π^* which is obtained from p^* by discounting the spot prices at node σ_1 and σ_2 according to the unique present value vector.

Since the equilibrium allocation are strictly positive, the first order necessary condition for the demand of the consumers leads to the following equalities:

$$\frac{x^*_{1h}(\sigma_1)}{x^*_{1h}(\sigma_2)} = \frac{x^*_{2h}(\sigma_1)}{x^*_{2h}(\sigma_2)} = \frac{\pi^*_h(\sigma_2)\rho_1}{\pi^*_h(\sigma_1)\rho_2}$$

for all commodities h. Furthermore, from the market clearing condition, we get for all commodities h and all states $\sigma \in \mathbb{D}_1$, $x^*_{1h}(\sigma) + x^*_{2h}(\sigma) = 1$, so, one deduces from the previous equality that $\frac{\pi^*_h(\sigma_2)\rho_1}{\pi^*_h(\sigma_1)\rho_2} = 1$. Consequently, $\pi^*(\sigma_1)$ is collinear to $\pi^*(\sigma_2)$, which implies that $p^*(\sigma_1)$ is collinear to $p^*(\sigma_2)$ and the rank of the matrix $V(p^*)$ is then equal to 1. So, there is no equilibrium with the rank of $V(p^*)$ equal to 2.

If the rank $(V(p^*)$ is equal to 1, since we have a real asset structure, we can normalise the price vectors state by state and since they are collinear, we get that $p^*(\sigma_1) = p^*(\sigma_2)$. Using the first order necessary condition, one gets:

$$x^*_{ih}(\sigma) = \frac{\alpha^i_h(p^*(\sigma)\cdot e_i(\sigma) + p^*_1(\sigma)z^*_{i1} + p^*_2(\sigma)z^*_{i2})}{p^*_h(\sigma_1)}$$

for both consumers, both commodities and both states of nature. Since, at equilibrium the market clearing condition for both commodities and both states is $x^*_{1h}(\sigma) + x^*_{2h}(\sigma) = 1$, if we normalise the prices so that $p^*_1(\sigma) + p^*_2(\sigma) = 1$, we get

$$\begin{aligned} 1 &= \frac{\alpha^1_1(1-\varepsilon+p^*_1(\sigma_1)z^*_{11}+p^*_2(\sigma_1))z^*_{12})+\alpha^2_1(\varepsilon+p^*_1(\sigma_1)z^*_{21}+p^*_2(\sigma_1)+z^*_{22})}{p^*_1(\sigma_1)} \\ &= \frac{\alpha^1_2(1-\varepsilon+p^*_1(\sigma_1)z^*_{11}+p^*_2(\sigma_1))z^*_{12})+\alpha^2_2(\varepsilon+p^*_1(\sigma_1)z^*_{21}+p^*_2(\sigma_1)+z^*_{22})}{p^*_1(\sigma_1)} \\ &= \frac{\alpha^1_1(\varepsilon+p^*_1(\sigma_1)z^*_{11}+p^*_2(\sigma_1))z^*_{12})+\alpha^2_1(1-\varepsilon+p^*_1(\sigma_1)z^*_{21}+p^*_2(\sigma_1)+z^*_{22})}{p^*_1(\sigma_1)} \\ &= \frac{\alpha^1_2(\varepsilon+p^*_1(\sigma_1)z^*_{11}+p^*_2(\sigma_1))z^*_{12})+\alpha^2_2(1-\varepsilon+p^*_1(\sigma_1)z^*_{21}+p^*_2(\sigma_1)+z^*_{22})}{p^*_1(\sigma_1)} \end{aligned}$$

This implies that $(1-2\varepsilon)\alpha^1 + (2\varepsilon-1)\alpha^2 = 0$. Since $\varepsilon \neq \frac{1}{2}$, we get that α^1 and α^2 are not collinear. But since $\alpha^1_1 + \alpha^1_2 = 1 = \alpha^2_1 + \alpha^2_2 = 1$, this collinearity means that α^1 and α^2 are equal, which is a contradiction. So, it does not exist a financial equilibrium with the rank of $V(p^*)$ equal to 1.

Pseudo-equilibria

As already said, we now introduce the intermediary concept of pseudo-equilibria to get a generic existence result for financial equilibrium. Let $\mathcal{G}^r$ be the set of all linear subspaces of dimension r of $\mathbb{R}^{\mathbb{D}_1}$. This set is called the r-Grassmann manifold of $\mathbb{R}^{\mathbb{D}_1}$. In the definition of a pseudo-equilibrium, instead of considering the possible transfers of wealth among the two periods and the states of nature through a financial structure, we consider a transfer space $E \in \mathcal{G}^r$ and the marketable payoff are the vectors in E.

Définition 11 A r-pseudo-equilibrium of the economy $\mathcal{E}_{\mathcal{F}}$ is an element (x^*, p^*, q^*, E^*) in $\prod_{i\in\mathcal{I}} X_i \times \mathbb{R}^{\mathbb{L}} \times \mathbb{R}^{\mathcal{J}} \times \mathcal{G}^r$ such that:

(i) for every i, x^*_i is optimal for the utility function u_i in the budget set

$$B^G_i(p^*, E^*) = \{x_i \in X_i \mid \exists t_i \in E^*, p^* \,\square\, (x_i - e_i) \leq t_i\}$$

(ii) $\sum_{i\in\mathcal{I}} x^*_i = \sum_{i\in\mathcal{I}} e_i$

(iii) $\operatorname{Im} W(p^*, q^*) \subset E^*$.

We now present the link between a pseudo-equilibrium and a financial equilibrium.

Proposition 27 *Let (x^*, z^*p^*, q^*) be a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$, then, $(x^*, p^*, q^*; \operatorname{Im} W(p^*, q^*))$ is a r-pseudo-equilibrium where r is the rank of $W(p^*, q^*)$.*

Conversely, if (x^*, p^*, q^*, E^*) is a r-pseudo-equilibrium of $\mathcal{E}_{\mathcal{F}}$ and

$$E^* = \operatorname{Im} W(p^*, q^*)$$

then there exists some portfolios $z^* \in (\mathbb{R}^{\mathcal{J}})^{\mathcal{I}}$ such that (x^*, z^*, p^*, q^*) is a financial equilibrium of $\mathcal{E}_{\mathcal{F}}$.

The proof of this proposition is left as an exercise. The following existence result shows that a r-pseudo-equilibrium exists if $r \leq \operatorname{card} \mathbb{D}_1$.

Theorem 5 *Let $\mathcal{E}_{\mathcal{F}}$ be an unconstrained financial economy satisfying Assumptions C, S, NSS and F. Then for all $r \leq \operatorname{card} \mathbb{D}_1$, for all $\lambda \in \mathbb{R}^{\mathbb{D}}_{++}$, there exists a r-pseudo-equilibrium (x^*, p^*, q^*, E^*) such that $E^* \subset \lambda^{\perp}$.*

The proof of this existence result is the most technical one among all results presented in this course. Indeed, it involves a fixed point like theorem on the Grassmann manifold, which has a structure less tractable than the convex sets that are involved in the standard fixed-point theorem.

From this result, one deduces the following generic existence result for real asset financial structure. To do it, we need to strengthen the assumptions on the consumers by considering differentiable preferences as in the standard results of the general equilibrium theory from a differentiable viewpoint (See Balasko [4], Mas-Colell [17], Carosi et al. [20]).

Assumption SC: for every $i \in \mathcal{I}$,

a) $X_i = \mathbb{R}^{\mathbb{L}}_{++}$;

b) u_i is $\mathcal{C}^2$ on X_i and, for all $x_i \in X_i$, $\nabla u_i(x_i) \in \mathbb{R}^{\mathbb{L}}_{++}$ and

$$\text{for all } v \in \{\nabla u_i(x_i)\}^{\perp} \setminus \{0\},\ v \cdot H u_i(x_i)(v) < 0.$$

c) For all $x_i \in X_i$, the set $\{x_i' \in X_i \mid u_i(x_i') \geq u_i(x_i)\}$ is closed in $\mathbb{R}^{\mathbb{L}}$.

The real asset financial structure is represented by an element R in $(\mathcal{M}(\operatorname{card} \mathbb{D} \times \ell)^{\mathcal{J}}$.

Theorem 6 *Let $\mathcal{E}$ be an exchange economy satisfying Assumption SC. Then, there exists an open subset Ω of $(\mathbb{R}^{\mathbb{L}}_{++})^{\mathcal{I}} \times (\mathcal{M}(\operatorname{card} \mathbb{D} \times \ell))^{\mathcal{J}}$ of full Lebesgue measure such that for all $(e, R) \in \Omega$, the financial economy $\mathcal{E}_{\mathcal{F}}$ has a financial equilibrium.*

The proof of this theorem shows that generically on the pair of endowments and financial structure the matrix $W(p^*, q^*)$ associated for the pseudo-equilibrium price pair (p^*, q^*) has a maximal rank and then the existing pseudo-equilibrium is then a financial equilibrium.