Microeconomics 1

Antoine Mandel[1]

2025-2026

September 8, 2025

[1] antoine.mandel@univ-paris1.fr

Logistics

- Midterm +Final Exam + Attendance.
- Problem set and recommended readings at the end of the class.
- Main reference for the class: Mas-Colell, Whinston and Green (MWG) Micro-economic Theory, Oxford University Press.
- timetable
- online learning resources

Outline of the course

Introduce the basics of neoclassical economic models:

- A mathematical model of the economic environment.
- Two actors (at most): consumers and producers.
- Three types of actions/interactions: consumption, production, exchange.
- Rational individual choices represented as solutions to optimization problems.
- Interactions between individuals through market institutions: general equilibrium

Lecture 1: commodities and preferences

- Objective: introduce the building blocks for the representation of consumers and producers' choices.
- References: MWG 1A, 1B, 2A, 2B, 2C, 3B.

Commodities

- A finite number of goods and services $\ell = 1, \cdots L$ is considered given.
- These goods and services are referred to as commodities.
- Commodities are characterized by their nature/essence but possibly also by their date, place and conditions (state of nature) of availability.
- Space, time and uncertainty are mostly implicit in the first part of the class (uncertainty discussed at the end of the class).
- However time and uncertainty are key in financial economics.

Consumption bundles and consumption set

- Commodities can be consumed/produced in fractional/real quantities, i.e a quantity of good ℓ consumed/produced is represented by a real number $x_\ell \in \mathbb{R}$.
- A commodity vector, used e.g. to represent the consumption bundle of a consumer, is represented by a vector of the form $x = (x_1, \cdots, x_L) \in \mathbb{R}^L$.
- There are constraints (physical, cultural, legal,...) on the consumption bundles a consumer can actually consume .
- The consumptions set $X \subset \mathbb{R}^L$ of a consumer is the set of consumption bundles, the consumer can actually consume.
- In general, consumptions sets are assumed to be convex, i.e $x, y \in X \Rightarrow \forall \lambda \in [0, 1]\ \lambda x + (1 - \lambda)y \in X$.
- In the following, we assume for sake of simplicity $X = \mathbb{R}^L_+$ (Most results hold in a more general setting, provided the consumption set is convex).

Preference Relations

- Preferences/tastes of consumers are represented by binary "preference" relations on the consumption set X.
- Namely, given consumption bundles $x, y \in X$ and a preference relation $\succsim$, one has $x \succsim y$ if x is at least as good as y (preferred or indifferent).

Example of preference relations

The following are rational preference relations over a consumption set $X = \mathbb{R}_+^L$.

- Linear preferences assume that each commodity ℓ is assigned a non-negative weight a_ℓ and commodity bundles are compared through weighted sums. Namely, $x \succsim_a y$ if and only if $\sum_{\ell=1}^L a_\ell x_\ell \geq \sum_{\ell=1}^L a_\ell y_\ell$.
- Leontieff preferences are determined by the commodity that is consumed in minimal amount, i.e. $x \succsim_{\text{leon}} y$ if and only if $\min(x_1, \cdots, x_L) \geq \min(y_1, \cdots, y_L)$.
- Lexicographic preferences assume that commodities are ordered by decreasing importance and thus commodity bundles are ordered in a dictionary-like manner: given $x \neq y \in X$, let $\ell* = \min\{\ell \in \{1, \cdots, L\} \mid x_\ell \neq y_\ell\}$ then one has $x \succ_{\text{lex}} y$ if $x_{\ell*} > y_{\ell*}$ and $y \succ_{\text{lex}} x$ otherwise.

Graphical Representations of preferences

Given a preference relation $\succsim$ on a consumption set X, and a consumption bundle $x \in X$

- The indifference set (curve) of x is the set of bundles that are indifferent to x, i.e.

$$\mathcal{I}_x := \{y \in x \mid y \sim x\}$$

- The upper contour set of x is the set of bundles that are preferred to x, i.e.

$$\mathcal{U}_x := \{y \in x \mid y \succsim x\}$$

- The lower contour set of x is the set of bundles that x is preferred to, i.e.

$$\mathcal{L}_x := \{y \in x \mid x \succsim y\}$$

Auxiliary relations

Given a preference relation $\succsim$ on a consumption set X, and a consumption bundle $x \in X$

- To a preference relation $\succsim$, one can associate:
  - The strict preference relation $\succ$ defined by $x \succ y$ if $x \succsim y$ and $y \not\succsim x$.
  - The indifference relation $\sim$ defined by $x \sim y$ if $x \succsim y$ and $y \succsim x$.

Reminder: properties of relations

A binary relation $\mathcal{R}$ on a set E is said to be:

- Complete if for all $x, y \in E$, either $x\mathcal{R}y$ or $y\mathcal{R}x$.
- Transitive if for all $x, y, z \in X$, if $x \succsim y$ and $y \succsim z$ then $x \succsim z$.
- Reflexive if for all $x \in E$, $x\mathcal{R}x$.
- Irreflexive if for all $x \in E$, $\neg(x\mathcal{R}x)$.
- Symmetric if for all $x, y \in E$, one has $x\mathcal{R}y \Leftrightarrow y\mathcal{R}x$.
- An equivalence relation if it is reflexive, transitive and symmetric.

Rational preferences

- Throughout, we shall assume that preferences are transitive and complete.
- Such preferences are called rational

Proposition

If $\succsim$ is a rational preference relation, then

1. *$\succsim$ is reflexive and thus a complete preorder*
2. *$\sim$ is reflexive, transitive and symmetric, i.e. an equivalence relation*
3. *$\succ$ is irreflexive and transitive.*
4. *If $x \succ y$ and $y \succsim z$ then $x \succ z$.*

Discussion of transitivity

- Transitivity:
  - Just perceptible differences.
  - Condorcet paradox (social preferences, see next slide)
  - Framing (Khaneman and Tversky 1984, see next slide)

Framing effects

Another potential problem arises when the manner in which alternatives are presented matters for choice. This is known as the *framing* problem. Consider the following example, paraphrased from Kahneman and Tversky (1984):

> Imagine that you are about to purchase a stereo for 125 dollars and a calculator for 15 dollars. The salesman tells you that the calculator is on sale for 5 dollars less at the other branch of the store, located 20 minutes away. The stereo is the same price there. Would you make the trip to the other store?

It turns out that the fraction of respondents saying that they would travel to the other store for the 5 dollar discount is much higher than the fraction who say they would travel when the question is changed so that the 5 dollar saving is on the stereo. This is so even though the ultimate saving obtained by incurring the inconvenience of travel is the same in both

cases.[3] Indeed, we would expect indifference to be the response to the following question:

> Because of a stockout you must travel to the other store to get the two items, but you will receive 5 dollars off on either item as compensation. Do you care on which item this 5 dollar rebate is given?

If so, however, the individual violates transitivity. To see this, denote

x = Travel to the other store and get a 5 dollar discount on the calculator.
y = Travel to the other store and get a 5 dollar discount on the stereo.
z = Buy both items at the first store.

The first two choices say that $x \succ z$ and $z \succ y$, but the last choice reveals $x \sim y$. Many problems of framing arise when individuals are faced with choices between alternatives that have uncertain outcomes (the subject of Chapter 6). Kahneman and Tversky (1984) provide a number of other interesting examples.

Condorcet paradox and social choice

- Three alternatives A, B, C.
- Agent 1 : $A \succ B \succ C$
- Agent 2 : $B \succ C \succ A$
- Agent 3 : $C \succ A \succ B$
- One can always find a majority of agents against any given choice: social preferences can hardly be defined.
- Arrow impossibility theorem: generic impossibility to aggregate individual preference into a (transitive) social preference (unless aggregation rule is dictatorial).

Monotonicity properties of preferences

Definition

A preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is said to be[2]

- *Monotone if for all $x, y \in X$ one has: $y > x \Rightarrow y \succ x$.*
- *Strongly monotone if for all $x, y \in X$ one has:* $[y \geq x \wedge y \neq x] \Rightarrow y \succ x$.
- *Locally non-satiated if for all $x \in X$ and all $\epsilon > 0$ there exists $y \in X$ such that $\|y - x\| < \epsilon$ and $y \succ\!> x$*

Remark

Strong Monotonicity $\Rightarrow$ Monotonicity $\Rightarrow$ Local Nonsatiation.

[2] Given two vector $x, y \in \mathbb{R}^L$, one writes $x > y$ (resp $x \geq y$) if and only if for all $\ell = 1, \cdots, L$ $x_\ell > y_\ell$ (resp. $x_\ell \geq y_\ell$)

Convexity properties of preferences

Definition

A preference relation $\succsim$ on $X = \mathbb{R}^L_+$ is said to be

- *convex if for all $x, y, z \in X$ one has:*

$$[y \succsim x \wedge z \succsim x] \Rightarrow \forall\, \lambda \in [0,1]\ \lambda y + (1-\lambda) z \succsim x.$$

- *strictly convex if for all $x, y, z \in X$ such that $y \neq z$ one has:*

$$[y \succsim x \wedge z \succsim x] \Rightarrow \forall\, \lambda \in]0,1[\ \lambda y + (1-\lambda) z \succ x.$$

Discussion of convexity

- Convexity $\Rightarrow$ Taste for diversification.
- Convexity $\Rightarrow$ Diminishing marginal rates of substitution: it takes increasingly larger amount of one commodity (bundle) to compensate for losses of another:
  - Assume $\succsim$ strictly convex and $x, y \in X$ such that $x + \delta \sim x$ (think $\delta = a - b$).
  - One has $x + \delta = \frac{1}{2}(x + 2\delta) + \frac{1}{2}x$.
  - One must have $x \succ x + 2\delta$ as otherwise one would have $x + \delta \succ x$.

Continuity or preferences

Definition

A preference relation $\succsim$ *on* $X = \mathbb{R}_+^L$ *is said to be continuous if for every pair of converging sequences* $(x_n)_{n\in\mathcal{N}}, (y_n)_{n\in\mathcal{N}} \in (\mathbb{R}^L)^{\mathcal{N}}$ *such that for all* $n \in \mathcal{N}$, $x_n \succsim y_n$, *one has* $\lim_{n\to+\infty} x_n \succsim \lim_{n\to+\infty} y_n$

Problem set

- Exercises C1, C2, C6, C7, C8, proofs of proposition on slide 12 and Remark on slide 16.
- Reading list for next class: MWG 1A-B, 2A-C, 3A-C.

Lecture 2 Utility functions

- Objectives: Introduce the notion of utility function and the utility maximization problem of the consumer
- References: MWG 1B, 3C,

Utility function

- A natural way to represent preferences is to “measure" consumption bundles, i.e. to assign a numerical value to bundles and compare them on this basis.

Definition

A utility function on X is a mapping from X to $\mathbb{R}$.

- *One can associate to an utility function $u : X \to \mathbb{R}$ a preference relation $\succsim_u$ on X by letting by $x \succsim_u y$ if and only if $u(x) \geq u(y)$.*
- *Conversely, given a preference relation $\succsim$, one says if is represented by the utility function u if $x \succsim y \Leftrightarrow u(x) \geq u(y)$*

Examples of utility function

- Linear utility functions of the form $u(x_1, \cdots, x_L) = \sum_{\ell=1}^{L} a_\ell x_\ell$ where $a_\ell \in \mathbb{R}_+$
- Leontieff utility functions of the form $u(x_1, \cdots, x_L) = \min(a_1 x_1, \cdots, a_L x_L)$ where $a_\ell \in \mathbb{R}_{++}$
- Cobb-Douglas utility functions of the form $u(x_1, \cdots, x_L) = \prod_{\ell=1}^{L} x_\ell^{a_\ell}$ where $a_\ell \in \mathbb{R}_+$ and $\sum_{\ell=1}^{L} a_\ell = 1$

N.B. Graphical representations.

C.E.S Utility functions

- A Constant Elasticity of Substitution (C.E.S) utility function is of the form
$$u(x_1, \cdots, x_L) = [\sum_{\ell=1}^{L} a_\ell x_\ell^\theta]^{1/\theta}$$
where $a_\ell \in \mathbb{R}_+$, $\theta \in \mathbb{R}$ and $\sigma := 1/(1-\theta)$ is the elasticity of substitution.
- If $\theta = 1$, the utility function is linear.
- As θ tends towards 0, C.E.S tends towards Cobb-Douglas, namely $\lim_{\theta \to 0}[\sum_{\ell=1}^{L} a_i x_i^\theta]^{1/\theta} = \prod_{\ell=1}^{L} x_\ell^{a_\ell}$
- As θ tends towards $-\infty$, C.E.S tends towards Leontieff, namely $\lim_{\theta \to -\infty}[\sum_{\ell=1}^{L} a_i^\theta x_i^\theta]^{1/\theta} = \min(a_1 x_1, \cdots, a_L x_L)$

From CES to Cobb-Douglas

- Proof in the case of 2 goods, we search for

$$\lim_{\theta \to 0} (a_1 x_1^\theta + a_2 x_2^\theta)^{1/\theta}$$

- Taking the log, we consider

$$L = \lim_{\theta \to 0} \log[(a_1 x_1^\theta + a_2 x_2^\theta)^{1/\theta}]$$

$$L = \lim_{\theta \to 0} \frac{1}{\theta} \log[(a_1 x_1^\theta + a_2 x_2^\theta)]$$

- Using L'Hôpital's rule (with respect to θ) and the fact that $a_i x_i^\theta = a_i e^{\theta \log(x_i)}$ this yields:

$$L = \lim_{\theta \to 0} \frac{a_1 x_1^\theta \log(x_1) + a_2 x_2^\theta \log(x_2)}{(a_1 x_1^\theta + a_2 x_2^\theta)}$$

From CES to Cobb-Douglas

- From

$$L = \lim_{\theta \to 0} \frac{a_1 x_1^\theta \log(x_1) + a_2 x_2^\theta \log(x_2)}{(a_1 x_1^\theta + a_2 x_2^\theta)}$$

- One gets

$$L = \frac{\log(x_1^{a_1} x_2^{a_2})}{(a_1 + a_2)}$$

which ends the proof consideirng $a_1 + a_2 = 1$.

From CES to Leontieff

- Proof in the case of 2 goods, we search for

$$\lim_{\theta \to +\infty} (a_1^\theta x_1^\theta + a_2^\theta x_2^\theta)^{1/\theta}$$

- Taking the log, we consider

$$L = \lim_{\theta \to +\infty} \log[(a_1^\theta x_1^\theta + a_2^\theta x_2^\theta)^{1/\theta}]$$

$$L = \lim_{\theta \to +\infty} \frac{1}{\theta} \log[(a_1^\theta x_1^\theta + a_2^\theta x_2^\theta)]$$

- Using L'Hôpital's rule (with respect to θ) and the fact that $a_i^\theta x_i^\theta = e^{\theta \log(a_i x_i)}$ this yields:

$$L = \lim_{\theta \to +\infty} \frac{a_1^\theta x_1^\theta \log(a_1 x_1) + a_2^\theta x_2^\theta \log(a_2 x_2)}{(a_1^\theta x_1^\theta + a_2^\theta x_2^\theta)}$$

From CES to Leontieff II

- From
$$L = \lim_{\theta \to +\infty} \frac{a_1^\theta x_1^\theta \log(a_1 x_1) + a_2^\theta x_2^\theta \log(a_2 x_2)}{(a_1^\theta x_1^\theta + a_2^\theta x_2^\theta)}$$
- Assuming wlog that $a_1 x_1 < a_2 x2$, we get, by dividing the numerator and denominator by $a_1^\theta x_1^\theta$:
$$L = \lim_{\theta \to +\infty} \frac{\log(a_1 x_1) + \frac{a_2^\theta x_2^\theta}{a_1^\theta x_1^\theta} \log(a_2 x_2)}{(1 + \frac{a_2^\theta x_2^\theta}{a_1^\theta x_1^\theta})}$$
- As $a_1 x_1 < a_2 x2$, we have $\lim_{\theta \to +\infty} \frac{a_2^\theta x_2^\theta}{a_1^\theta x_1^\theta} = 0$, so that:
$$L = \log(a_1 x_1)$$
which ends the proof.

Ordinal and cardinal properties

Remark

Utility function representing preferences are not unique. In fact for any increasing function $f : \mathbb{R} \to \mathbb{R}$, u and $f \circ u$ represent the same preferences.

- Example of useful transformation: from multiplicative (Cobb-Douglas) $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$ to log-linear utility functions $\log \circ u(x_1, x_2) = \alpha \log(x_1) + (1 - \alpha) \log(x_2)$.
- A property that is invariant to a monotone transformation of the utility function is said to be ordinal. Example: “being preferred to".
- A property that depends on the specific utility function is said to be cardinal. Example: “yielding double the utility of".

Existence of a utility representation

Proposition

A preference relation can be represented by a utility function only if it is rational.

- Remark: not all rational preferences can be represented by a utility function: the lexicographic preference gives a counter-example.
- Idea of the proof: otherwise one can construct a monotonic mapping from $\mathbb{R}$ to $\mathbb{Q}$.
- Namely,
  - for every $x \in \mathbb{R}$, there exists $q(x) \in \mathbb{Q}$ such that $u(x, 1) < q(x) < u(x, 2)$.
  - one necessarily has also $x > y \Rightarrow q(x) > q(y)$.

Utility representation theorem

Definition

A preference relation $\succsim$ on $X = \mathbb{R}_+^L$ is said to be continuous if for every pair of converging sequences $(x_n)_{n\in\mathcal{N}}, (y_n)_{n\in\mathcal{N}} \in (\mathbb{R}^L)^{\mathcal{N}}$ such that for all $n \in \mathcal{N}$, $x_n \succsim y_n$, one has $\lim_{n\to+\infty} x_n \succsim \lim_{n\to+\infty} y_n$

Theorem (Utility representation Theorem)

A preference relation that is rational and continuous can be represented by a continuous utility function.

Sketch of the proof of the utility representation Theorem

Case of a monotonous, continuous, transitive and complete relation.

- Let $e = (1, \cdots, 1) \in \mathbb{R}^L_+$.
- For every $x \in \mathbb{R}^L_+$, there exists $\alpha(x) \in \mathbb{R}$ such that $x \sim \alpha(x)e$ (using continuity and monotonicity, not true in general).
- The mapping α represents the preferences and it is continuous.

Basic properties of utility functions

- A utility function is locally non satiated (resp. monotonic, strongly monotonic) if the associated preference relation is non satiated (resp. monotonic, strongly monotonic). Namely
  - Monotone if for all $x, y \in X$ one has: $y > x \Rightarrow u(y) > u(x)$.
  - Strongly monotone if for all $x, y \in X$ one has: $[y \geq x \wedge y \neq x] \Rightarrow u(y) > u(x)$.
  - Locally non-satiated if for all $x \in X$ and all $\epsilon > 0$ there exists $y \in X$ such that $\|y - x\| < \epsilon$ and $u(y) > u(x)$

Basic properties of utility functions

- A utility function is continuous if and only the preference relation $\succsim_u$ is continuous.
- The preference relation $\succsim_u$ is convex if and only if u is quasi-convave, i.e:

$$\forall x, y \in X \; \forall \lambda \in [0,1] \; u(\lambda x + (1-\lambda)y) \geq \min(u(x), u(y))$$

- The preference relation $\succsim_u$ is strictly convex if and only if u is strictly quasi-convave, i.e:

$$\forall x, y \in X \; \forall \lambda \in]0,1[\; u(\lambda x + (1-\lambda)y) > \min(u(x), u(y))$$

Problem set

- Utility: C4, C5, C13 (up to d), C16
- Proof of limit-case for CES functions.

Lecture 3 Utility maximization and consumer choice

- Objectives: Introduce the notion of utility function and the utility maximization problem of the consumer
- References: MWG 2D, 3D

Prices

- We consider the behavior of consumers within a complete system of markets .
- Namely, there is a publicly posted price for each commodity and the consumer does not have an effect on the price (price-taking behavior).
- Formally, there exists a price vector $p = (p_1, \cdots, p_L) \in \mathbb{R}^L_+$.

Budget set

We consider the behavior of a consumer with given utility function u and budget $w \in \mathbb{R}_+$, facing a price vector $p \in \mathbb{R}^L_+/\{0\}$.

Definition

The budget set of a consumer with budget w facing a price p is the set of consumption bundles that the consumer can afford given this budget and the price, namely

$$B(p, w) := \{x \in X \mid p \cdot x \leq w\}$$

N.B. Graphical representation of the budget set with two commodities.

Properties of the budget set

Proposition

Let $p \in \mathbb{R}^L_+/\{0\}$, *and* $w \geq 0$, *one has:*

1. $B(p, w)$ *is a non-empty, closed and convex subset of* R^L
2. $B(p, w)$ *is bounded if* $p > 0$.
3. *For all* $t \in \mathbb{R}^*_+$, $B(tp, tw) = B(p, w)$.

The consumer's problem

- We consider the behavior of a consumer with given utility function u and budget $w \in \mathbb{R}_+$, facing a price vector $p \in \mathbb{R}^L_+/\{0\}$.
- The consumer is "rational" in the sense that he selects the optimal choice among the alternative he faces.
- In our context, he chooses a consumption bundle in his budget set maximising his utility, that is a solution of the optimization problem:

$$\mathcal{P}(p, w) := \begin{cases} \max & u(x) \\ s.t & p \cdot x \leq w \\ & x \geq 0 \end{cases}$$

The consumer's problem II

- The consumer/ utility maximization problem (UMP):

$$\mathcal{P}(p,w) := \begin{cases} \max & u(x) \\ s.t & p \cdot x \leq w \\ & x \geq 0 \end{cases}$$

- The set of solutions of $\mathcal{P}(p,w)$, that we denote by $d(p,w)$ is called the Walrasian demand correspondence. The value $v(p,w)$ of the problem $\mathcal{P}(p,w)$, i.e the value $u(x)$ for $x \in d(p,w)$, is called the indirect utility function.
- Remark: the Walrasian demand can be defined even if the consumer preferences are not represented by a utility function. It is the set of consumption bundles $x \in B(p,w)$ such that for all $y \in B(p,w)$ one has $x \succsim y$.

Properties of the Walrasian correspondence

Proposition

Suppose that u is continuous and locally non-satiated, then the Walrasian demand correspondence satisfies the following properties

1. *If* $p > 0$, $d(p, w)$ *is non-empty.*
2. *Homogeneity of degree zero: for all* $p \in \mathbb{R}^L_+/\{0\}$, $w \in \mathbb{R}_+$ *and* $t \in \mathbb{R}_+$ *one has* $d(tp, tw) = d(p, w)$.
3. *Walras' law: for all* $x \in d(p, w)$ $p \cdot x = w$
4. *Convexity: if u moreover is quasi-concave* ($\succsim_u$ *convex), then* $d(p, w)$ *is convex.*
5. *Strict convexity/continuity: if u morever is strictly quasi-concave* ($\succsim_u$ *strictly convex), then* $d(p, w)$ *is single-valued (i.e. a function) and continuous.*

Remark on price normalization

Remark

As highlighted by homogeneity of degree 0 *of the demand, the absolute value of prices is irrelevant in our setting, i.e. only relative prices matter. In fact, shifting from a price* $p = (p_1, \cdots, p_L)$ *to* $tp = (tp_1, \cdots, tp_L)$ *amounts to change the unit of account (e.g. from euros to cents). Thus one often chooses by convention a normalization of prices, either by setting the price of a particular good (the numeraire) to* 1 *or by assuming all prices sum to* 1.

Differential characterization of demand

- If u is differentiable then one can characterize $x^* \in d(p, w)$ using KKT conditions.
- Namely, if $x^* \in d(p, w)$, there exists $\lambda \geq 0$ such that for all $\ell = 1, \cdots L$: $\frac{\partial u}{\partial x_\ell}(x^*) \leq \lambda p_\ell$, with equality if $x_\ell^* > 0$.
- Hence, if $x \in d(p, w)$, is such that $x^* > 0$, one must have for some $\lambda > 0$

$$\nabla u(x^*) = \lambda p \tag{1}$$

Characterization of demand via marginal rates of substitutions

- If $\nabla u(x^*) > 0$, Equation (1) is equivalent to the requirement that for all $k, \ell \in \{1, \cdots, L\} : \frac{\partial u / \partial x_k (x^*)}{\partial u / \partial x_\ell (x^*)} = \frac{p_k}{p_\ell}$
- In other words, marginal rates of substitutions are equal to price ratios.

Second order conditions

The above conditions are sufficient to ensure $x^* \in d(p, w)$ if

- u is concave (the Hessian matrix is semi-definite positive)
- or u is quasi-concave, monotonic and satisfies for all $x \in \mathbb{R}_+$, $\nabla u(x) > 0$.

Problem set

- Utility maximization problem for Leontieff, Cobb-Douglas and CES.
- Demand: C10, C12, C13, C17, C19, C20, C21
- Reading list: 2D, 3D, 1C-D,2F

Lecture 3bis: Revealed preferences

- Objectives: Introduce revealed preferences.
- References: MWG 1C, 1D, 2F(up to page 30).

Criticisms of rational choice

- Agents do not maximize preferences (because of altruism, lack of computing ability, imperfect information,..)
- Preferences are unobservable $\Rightarrow$ Revealed preference theory as a partial answer.

Choice structures and choice rules

Definition

A choice structure consists in :

- *A family $\mathcal{B}$ of subsets of X (sets of observed possible choices, e.g. different consumption sets)*
- *A choice rule $c : \mathcal{B} \to 2^X$ that associates to every set $B \in \mathcal{B}$ a subset $c(B) \subset B$ of admissible choices.*

Exemples

- Choice sets are all possible budget sets: $\mathcal{B} := \{B(p, w) \mid p \in \mathbb{R}^L_+/\{0\},\ w \in \mathbb{R}_+\}$
- Choice rule is Walrasian demand $c(B(p, w)) = d(p, w)$

Weak axiom of revealed preferences

Definition

The choice structure $(\mathcal{B}, c)$ satisfies the weak axiom of revealed preferences if the following property holds:

if for some $B \in \mathcal{B}$ with $x, y \in B$ one has $x \in c(B)$ then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in c(B')$ one must have $x \in c(B')$

Definition

A Walrasian demand function satisfies the weak axiom of revealed preferences if for all $(p, w), (, p', w') \in \mathbb{R}_+^\ell \setminus \{0\} \times \mathbb{R}_+$, one has:

$$[p' \cdot d(p, w) \leq w' \wedge d(p', w') \neq d(p, w)] \Rightarrow p \cdot d(p', w') > w.$$

Rationalizable choice rule

- For every family $\mathcal{B}$ of subsets of X, one can associate to a preference relation $\succsim$ the choice rule $c_{\succsim}$ such that

$$c_{\succsim}(B) = \{x \in B \mid \forall z \in B \; x \succsim z\}$$

Proposition

If $\succsim$ is a rational preference relation then for every family $\mathcal{B}$ of subsets of X, $c_{\succsim}$ satisfies the weak axiom of revealed preferences.

- Conversely, if c is a choice rule on $\mathcal{B}$ and $\succsim$ is such that for all $B \in \mathcal{B}$, one has $c(B) = c_{\succsim}(B)$, $\succsim$ is said to rationalize c.
- Given a choice rule (in particular a demand function), is there a unique preference relation that rationalizes it ?

Rationalizable choice rule 2

Proposition

If $(\mathcal{B}, c)$ is a choice structure such that:

1. *the weak axiom is satisfied*
2. *$\mathcal{B}$ includes all subsets of X up to three elements*

Then there exists a unique rational preference relation that rationalizes c.

- Proof considers the relation $x \succeq_c y$ if and only if there exists $B \in \mathcal{B}$ such that $x, y \in B$ and $x \in c(B)$.

Rationalizable choice rule 3

- counter-example to rationalizable choice-rule satisfying warp:
  - $X = \{x, y, z\}$, $\mathcal{B} = \{\{x, y\}, \{y, z\}, \{x, z\}\}$
  - $c(\{x, y\}) = \{x\}$, $c(\{y, z\}) = \{y\}$, $c(\{x, z\}) = \{z\}$,
- there exists demand functions satisfying warp that are not rationalizable (because the family of budget sets does not entail sufficient restrictions).
- Revealed preference theory "compensates" for unobservable preferences only to the extent that one accepts the maximization principle.

Problem Set

- Exercise C3, C14
- Reading list for next class: MWG 5A-B

Lecture 4-5 Expected utility and decision under risk (refer to prof. Bloch for updates)

- Objectives: introduce the basic model of decision-making under uncertainty.
- References: MWG 6A, 6B, 6C.

Setting

- A set of potential outcomes (consequences) C: consumption bundles, monetary returns.
- C assumed to be finite, $C = \{c_1, \cdots, c_N\}$
- There is (objective) uncertainty about the actual outcome.
- The decision-maker faces lotteries (probability distributions) over C. Namely:
  - A simple lottery L is a vector $L = (p_1, \cdots, p_N) \in \mathbb{R}_+^N$ such that $\sum_{n=1}^N p_n = 1$ where p_n is the probability of outcome n happening (i.e. a probability distribution over C).
  - Given K simple loteries and a probability vector $(\alpha_1, \cdots, \alpha_K)$ over K, the compound lottery $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$, also denoted by $\sum \alpha_k L_K$, is the risky alternative that yields the lottery k with probability α_k (i.e. the convex combination of the distributions L_k.).

Consequentialist premise

- One can associate to a compound lottery $(L_1, \cdots, L_K; \alpha_1, \cdots, \alpha_K)$, the reduced lottery l such that for all n
$$p_n = \sum_{k=1}^{K} \alpha_k p_n^k$$
where $L_k = (p_1^k, \cdots, p_n^k)$
- The consequentialist premise postulates that the decision maker only cares about this reduced form lottery, i.e. abut the “consequences" of his choice.
- Accordingly, we focus in the following on a decision-maker that has preferences $\succsim$ defined over the set $\mathcal{L}$ of simple lotteries.

Expected utility

- How should/could a decision-maker order risky alternatives?
- St-Petersburg Paradox. Consider the following variants of heads and tail:
  - One tosses a coin until heads come out.
  - If heads come out at first round, you win 2 euros.
  - If heads come out at second round, you win 4 euros.
  - If heads come out at nth round, you win 2^n euros.
  - How much are you willing to pay to enter such a game ?

Extended expected utility framework

- The expected utility framework can be used to represent preferences on arbitrary set of risky alternatives.
- Formally, for any set of random variables $\mathcal{C}$ on a set C, given a utility function $u : \mathbb{R} \to \mathbb{R}$ (called the Bernoulli utility function in this context), one can define the (von Neumann-Morgenstern) expected utility of any random variable X as

$$U(X) = \mathbb{E}(u(X)) = \int_C u(x) dP_X(x)$$

  where dP_X is the law of the random variable X.
- In the following, we focus on the set of real random variables, i.e. on random monetary payments and assume u is increasing and continuous.

Risk Aversion

When they face risk or uncertainty (e.g. risk of accident, uncertain financial returns), economic agents are usually risk-averse. In the expected utility framework, risk-aversion can be defined formally as follows.

Definition

A decision-maker with expected utility u is risk-averse if for every random variable X, one has

$$u(\mathbb{E}(X)) \geq \mathbb{E}(u(X))$$

- Accordingly, the decision-maker is risk-neutral if for all X, one has $u(\mathbb{E}(X)) = \mathbb{E}(u(X))$ and risk-loving if $u(\mathbb{E}(X)) \leq \mathbb{E}(u(X))$.

Characterization of risk-aversion

- According to Jensen inequality (a standard result in convex analysis), the inequality $u(\mathbb{E}(X)) \geq \mathbb{E}(u(X))$ is satisfied for all random variable X if and only if u is concave.
- According to the intermediate value theorem, every random variable X has a certainty equivalent c_X such that $\mathbb{E}(u(X)) = u(c_X)$. The decision-maker is risk-averse if and only if for all random variable X, $c_X \leq \mathbb{E}(X)$.
- The risk premium associated to X is then defined by $\rho_X = \mathbb{E}(X) - c_X$.

Examples of micro-economic implications of risk-aversion

- demand for insurance
- demand for a risky asset

Absolute risk-aversion I

Proposition (Arrow-Pratt)

For a random variable $X = x + \epsilon$ *with* $E[\epsilon] = 0$ *and* $VAR(\epsilon) = \sigma^2$, *one has:* $\rho_X \simeq \frac{\sigma^2}{2}\frac{-u''(x)}{u'(x)}$

Accordingly $A_u(x) = \frac{-u''(x)}{u'(x)}$ *is called the coefficient of absolute risk-aversion (for the wealth* x*) of the decision-maker.*

Proof:

Taking Taylor developments, one has for $||\epsilon||$ *sufficiently small:*

$$u(X) = u(x+\epsilon) \simeq u(x) + u'(x)\epsilon + u''(x)\frac{\epsilon^2}{2} \Rightarrow E[u(X)] \simeq u(x) + u''(x)\frac{\sigma^2}{2}$$

$$u(c_X) = u(x - \rho_X) \simeq u(x) - u'(x)\rho_X$$

One concludes by identifying $u(c_X)$ *and* $E[u(X)]$.

Absolute risk-aversion II

The coefficient of absolute risk-aversion, $R_u(x)$, characterizes risk aversion in the sense of the following proposition:

Proposition

Given two E.U decision makers (characterized by u_1 and u_2 respectively), the following propositions are equivalent;

1. *For each random variable X, $\rho_X^1 \geq \rho_X^2$*
2. *There exists $\phi : \mathbb{R} \to \mathbb{R}$ increasing and concave such that $u_1 = \phi \circ u_2$*
3. *For all $x \in \mathbb{R}$, $A_{u_1}(x) \geq A_{u_2}(x)$*

Absolute risk-aversion III

- Up to an affine transformation, the fonction $u(x) = 1 - e^{-\alpha x}$ is the only one exhibiting constant absolute risk-aversion
- The fonction $v(x) = \log(x)$ exhibits decreasing absolute risk-aversion
- The quadratic utility fonction $w(x) = ax + bx - cx^2$ exhibits increasing absolute risk aversion.

Relative risk aversion

- A priori the aversion to a certain loss often decreases with the wealth of the agent.
- In order to obtain a measure of risk aversion relative to the wealth, let us consider a random variable of the form $X = x + x\epsilon$ with $E[\epsilon] = 0$ and $VAR(\epsilon) = \sigma^2$.
- Let us on the other hand consider τ_X such that

$$u(x(1-\tau_X)) = E[u(X)]$$

- Taking Taylor developments, one has for $\|\epsilon\|$ sufficiently small:

$$u(X) = u(x + x\epsilon) \simeq u(x) + u'(x)x\epsilon + u''(x)\frac{x^2\epsilon^2}{2} \Rightarrow E[u(X)] \simeq u(x) + u''(x)\frac{x^2\sigma^2}{2}$$

$$u(x(1-\tau_X)) \simeq u(x) - u'(x)x\tau_X$$

- One has $\tau_X \simeq \dfrac{\sigma^2}{2}\dfrac{-xu''(x)}{u'(x)}$
- Hence $\dfrac{-xu''(x)}{u'(x)}$ provides a measure of risk aversion relative to the current wealth. It is called the coefficient of relative risk aversion.

First-order stochastic dominance

If a random variable X is preferred to Y by every EU decision-maker, it can be consider better in the following sense

Definition

A real random variable X first-order stochastically dominates Y if and only if for every increasing utility u, one has

$$\mathbb{E}(u(X)) \geq \mathbb{E}(u(Y))$$

Proposition

A real random variable X first-order stochastically dominates Y if and only if for all $x \in \mathbb{R}$ $F_X(x) \leq F_Y(x)$

Proof first-order stochastic dominance (case of positive valued variable)

- Lemma: if Z has positive values $E(Z) = \int_0^{+\infty}(1 - F_Z(z))dz = \int_0^{+\infty} P(Z \geq z)dz$
- Assume X first-order stochastically dominates Y, one has for all $x \in \mathbb{R}$, $P(X \geq x) \geq P(Y \geq x)$. Let us then consider u increasing. One has

$$E(u(X)) = \int_0^{+\infty} P(u(X) \geq t)dt = \int_0^{+\infty} P(X \geq u^{-1}(t))dt \geq$$

$$\int_0^{+\infty} P(Y \geq u^{-1}(t))dt = \int_0^{+\infty} P(u(Y) \geq t)dt = E(u(Y))$$

- Let us show the converse by contraposition. Hence, assume there exists $\bar{x}$ such that $F_X(\bar{x}) > F_Y(\bar{x})$. Consider the utility function $u(x) = 1_{[\bar{x},+\infty]}$, one clearly has $E(u(X)) = \int_{\bar{x}}^{+\infty} dF_X(x) = 1 - F_X(\bar{x}) < 1 - F_Y(\bar{x}) = E(u(Y))$.

Second order stochastic dominance

A natural extension of risk-dominance in economic environments is to specialize the definiton to risk-averse decision-makers

Definition

Given two real random variable X and Y with same mean,X second-order stochastically dominates Y if for every increasing and concave utility u, one has: $\mathbb{E}(u(X)) \geq \mathbb{E}(u(Y))$.

Proposition

A real random variable X second-order stochastically dominates Y, with the same mean, if and only if for all $x \in \mathbb{R}$

$$\int_0^x F_X(t)dt \leq \int_0^x F_Y(t)dt$$

Problem set

- Exercises 26 to 31

Expected utility representation

- A preference $\succsim$ over $\mathcal{L}$ admits an expected utility representation if there exists a function $v : C \to \mathbb{R}$ such that $\succsim$ is represented by $U(p_1, \cdots, p_N) = \sum_{n=1}^{N} p_i v(c_i)$
- In other words, $U(L)$ is the expected utility (for the utility v) of a random variable with distribution L.
- Under which conditions does a preference over $\mathcal{L}$ admits an expected utility representation ?

von Neumann-Morgenstern representation theorem

Theorem

if the preference relation $\succsim$ over $\mathcal{L}$ is rational (transitive and complete) and satisfies the continuity and independence axioms then it admits an expected utility representation.

Proposition

If $\succsim$ admits an expected utility representation, the utility function on C (i.e. the function v) is unique up to affine transformation.

Continuity axiom

Definition

The preference relation $\succsim$ on $\mathcal{L}$ is continuous if for any $L, L', L'' \in \mathcal{L}$, the sets:

- $\{\alpha \in [0,1] \mid \alpha L + (1-\alpha)L' \succsim L''\}$
- $\{\alpha \in [0,1] \mid L'' \succsim \alpha L + (1-\alpha)L'\}$

are closed.

- Alternative statement: the preference relation $\succsim$ on $\mathcal{L}$ is continuous if for any $L \succsim L' \succsim L''$ there exists $p \in]0,1[$ such that $pL + (1-p)L'' \sim L'$

Intepretation of the continuity axiom

- The continuity axiom rules out that small changes in probabilities change the ordering between two lotteries.
- e.g. if you prefer driving than walking to work then you prefer “driving or dying in a car accident with small probability" than walking (and arriving safely).
- In other words, continuity axioms rules out lexicographic preferences of the "safety first" type.

Independence (of irrelevant alternatives) axiom

Definition

The preference relation $\succsim$ on $\mathcal{L}$ satisfies the independence axiom if for any $L, L', L'' \in \mathcal{L}$, and $\alpha \in [0,1]$ one has

$$L \succsim L' \;\Leftrightarrow \alpha L + (1-\alpha)L'' \succsim \alpha L' + (1-\alpha)L''$$

The independence axiom implies the ranking between two lotteries is independent of other (independent alternatives) available

Allais' paradox

- Consider three potential outcomes in euros (2500000, 500000,0) and the lotteries:
  - $L_1 = (0, 1, 0)$ vs $L_1' = (0.1, 0.89, 0.01)$;
  - $L_2 : (0, 0.11, 0.89)$ vs $L_2' : (0.1, 0, 0.9)$

Allais' paradox

- Consider three potential outcomes in euros (2500000, 500000,0) and the lotteries:
  - $L_1 = (0, 1, 0)$ vs $L'_1 = (0.1, 0.89, 0.01)$;
  - $L_2 : (0, 0.11, 0.89)$ vs $L'_2 : (0.1, 0, 0.9)$
- Most of the people prefer L_1 over L'_1 and L'_2 over L_2.

Allais' paradox

- Consider three potential outcomes in euros (2500000, 500000,0) and the lotteries:
  - $L_1 = (0, 1, 0)$ vs $L_1' = (0.1, 0.89, 0.01)$;
  - $L_2 : (0, 0.11, 0.89)$ vs $L_2' : (0.1, 0, 0.9)$
- Most of the people prefer L_1 over L_1' and L_2' over L_2.
- However, if one lets $L = (10/11, 0, 1/11)$ and $L_0 = (0, 0, 1)$ one has:
  - $L_1 = 0.11L_1 + 0.89L_1$, $L_1' = 0.11L + 0.89L_1$, $L_2 = 0.11L_1 + 0.89L_0$, $L_2' = 0.11L + 0.89L_0$
- Thus under the independence axiom, one shall have:
  - $L_1 \succ L_1' \Rightarrow L_1 \succ L$
  - $L_2' \succ L_2 \Rightarrow L \succ L_1$
- Hence Allais' paradox.

Machina's paradox

- Three possible outcomes: going to Venice $\succ$ watching a movie about Venice $\succ$ staying home.
- Yet going to Venice with probability 0.99 and staying home with probability 0.01,is arguably better then going to Venice with probability 0.99 and watching the Venice movie with probability 0.01 because you are likely to hate the Venice movie because of the disappointment to have missed the trip.
- In other words, the independence axiom discards regret.

Problem set

- Exercises 32 to 38
- Complete missing proofs (in particular proof of the utility representation theorem).

Lecture 6: Producer theory (refer to prof. Bloch for updates)

- Objective: introduce the building blocks for the representation of producers/firms in micro-economics
- References: MWG 5A, 5B

Production plans

- In general equilibrium, firms are mainly considered as technological/technical actors.
- The action of the firms consists in transforming input (one or many) into output (one or many).
- Formally, these actions are represented by a production plan $y \in \mathbb{R}^L$.
- The negative coordinates of y correspond to inputs, the positive ones to outputs.

Production set

- In general, a given firm can implement different productive actions.
- Different scales, different techniques, different outputs.
- The set of possible production plans for the firm is called its production set.
- Generally, it is denoted by $Y \subset \mathbb{R}^L$ and a production set y is called feasible if $y \in Y$.
- Exemples and graphical representation.

Production and Transformation functions

- If there is a single output, e.g. good L, one can generally define the production set through a production function $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ that gives the maximum quantity of output $f(z)$ that can be produced using a vector $z \in \mathbb{R}_+^{L-1}$ of inputs.
- $Y = \{(-z_1, \cdots, -z_{L-1}, q) \mid q \leq f(z_1, \cdots, z_{L-1})\}$
- More generally, production set be described by a transformation function $F : \mathbb{R}^L \to \mathbb{R}$

$$Y = \{y \in \mathbb{R}^L \mid F(y) \leq 0\}$$

Exemples

- Leontieff production function
  $f(y_1, \cdots, y_K) = \min_{\ell=1,\cdots,K}(\frac{y_\ell}{a_\ell})$
- Linear production function $f(y_1, \cdots, y_K) = \sum_{\ell=1}^{K} a_\ell y_\ell$
- Cobb-Douglas production function
  $f(y_1, \cdots, y_K) = [\prod_{\ell=1}^{K} y_\ell^{a_\ell}]$ where $\Sigma a_\ell = 1$
- More generally C.E.S production function

$$f(y_1, \cdots, y_K) = [\sum_{\ell=1}^{K} a_\ell y_\ell^{\theta}]^{1/\theta}$$

Properties of the corresponding production sets ?

Properties of production sets

Usual properties of a production set $Y \subset \mathbb{R}^L$:

- Possibility of inaction: $0 \in Y$
- Impossibility of free production : $Y \cap \mathbb{R}^\ell_+ \subset \{0\}$.
- Irreversibility : $Y \cap -Y \subset \{0\}$.
- Free disposability : $Y - \mathbb{R}^L_+ \subset Y$
- Convexity : Y is convex.
- Closedness : Y is a closed subset of $\mathbb{R}^L$.

Returns to scale

Definition

Let Y be a production set. The production exhibits:

- increasing returns to scale if for all $y \in Y$ and for all $t \geq 1$, $ty \in Y$.
- decreasing returns to scale if for all $y \in Y$ and for all $t \in [0, 1]$, $ty \in Y$.
- constant returns to scale if for all $y \in Y$ and for all $t \in \mathbb{R}_+$, $ty \in Y$.

Proposition

If the production set is convex and satisfies the possibility of inaction, then the production has decreasing returns to scale. If the production set is convex with constant returns to scale, then the production set is a convex cone.

Efficient production sets

Definition

Let Y be a production set.

- A production $y \in Y$ is efficient if it does not exist a production $y' \in Y$ such that $y' \geq y$ and $y' \neq y$. In other words, $(\{y\} + \mathbb{R}^\ell_+) \cap Y = \{y\}$.
- A production y in Y is weakly efficient if it does not exist a production $y' \in Y$ such that $y\prime \gg y$. In other words, $(\{y\} + \mathbb{R}^\ell_{++}) \cap Y = \emptyset$. We denote by $E_f(Y)$ the set of weakly efficient productions of Y.

Problem set

- P1,P2, P3, P4, P5 and following (as time permits)

Lecture 7-8: Producer choice (refer to prof. Bloch for updates)

- Objective: understand rational choice model of firms' behavior.
- References: MWG 5C, 5D

The producer/firm problem

- We consider the behavior of a producer/firm with given production set Y facing a price vector $p \in \mathbb{R}^L_+/\{0\}$
- The producer is “rational" in the sense that he selects the optimal choice among the alternative he faces. Namely he maximizes profit.
- Formally, the producer's problem writes

$$\mathcal{Q}(p) := \begin{cases} \max & p \cdot y \\ s.t & y \in Y \end{cases}$$

- The set of solutions of this problem is called the supply correspondence of the firm and denoted by $s(p)$ (or $y(p)$ and the value of the problem is the profit of the firm, denoted by $\pi(p)$.
- N.B this perspective discards issue related to the shareholding/financial structure of the firm as well as those that pertain to its internal organization.

Graphical representation

- isoprofit line
- graphical representation of a production set and supply function.

Properties of the supply function

Proposition

Let $p >> 0$ and Y a non-empty production set.

- *Every element $y \in s(p)$ is efficient.*
- *If Y is closed and convex, $s(p)$ is closed and convex.*
- *For every $t \in \mathbb{R}_+$, $s(tp) = s(p)$.*
- *If Y is closed, satisfies free-disposal and is "strictly convex" in the sense that for all $y, y' \in \partial Y$ and for all $t \in]0, 1[$ $ty + (1-t)y' \in int(Y)$, then $s(p)$ is single-valued for all $p \in \mathbb{R}_+$ such that $s(p) \neq \emptyset$.*
- *π is homogeneous of degree one and convex (result from convex analysis)*

Case of constant returns to scale

Proposition

If Y has constant returns to scale, then for all $p \in \mathbb{R}_+$, one has:

- *$s(p) \neq \emptyset$ if and only if $\pi(p) = 0$*
- *$y \in s(p)$ is and only if for all $\lambda \geq 0$, $\lambda y \in s(p)$.*

Differential characterization of supply

Proposition

Let Y be a production set of $\mathbb{R}^\ell$. Let $\bar{y} \in Y$ and $p \in \mathbb{R}^\ell_{++}$. We assume that Y is locally representable by a transformation function t in a neighborhood of $\bar{y}$, t is differentiable and there exists at least one commodity k such that $D_{y_k} t(\bar{y}) > 0$.

1) *If $\bar{y} \in s(p)$, then there exists $\mu > 0$ such that $p = \mu \nabla t(\bar{y})$ and $t(\bar{y}) = 0$.*

2) *Conversely, we furthermore assume that t is quasi-convex. If there exists $\mu > 0$ such that $p = \mu \nabla t(\bar{y})$ and $t(\bar{y}) = 0$, then $\bar{y} \in s(p)$.*

This result means that $\bar{y}$ is the supply of the firm if the marginal productivities and the marginal rate of substitution between inputs are equal to the relative prices.

First-order conditions continued

- Assume Y is representable by a production function, i.e. $Y := \{(x, -z) \in \mathbb{R}_+ \times \mathbb{R}_-^{L-1} \mid x \leq f(z)\}$, then the problem of the consumer is given by

$$\max_{z \in \times \mathbb{R}_+^{L-1}} qf(z) - p \cdot z$$

- If z^* is optimal, then one has for all $\ell = 2, \cdots, L$, $q\frac{\partial f}{\partial z_\ell}(z^*) \leq p_\ell$ with equality if $z_\ell^* > 0$.
- The condition is necessary if Y is convex (I.e. f concave)
- In particular, if $z_k^* > 0$ and $z_\ell^* > 0$, one has

$$\frac{\frac{\partial f}{\partial z_k}(z^*)}{\frac{\partial f}{\partial z_\ell}(z^*)} = \frac{p_k}{p_\ell}$$

Properties of supply and profit functions

Proposition

Let $p \gg 0$ and Y a nonempty closed production set satisfying the free-disposal assumption. One has:

- *(Hotelling Lemma) If $s(\overline{p})$ is single-valued, then π is differentiable at $\overline{p}$ and $\nabla\pi(\overline{p}) = s(\overline{p})$.*
- *If s is differentiable at $\overline{p}$, then $Ds(\overline{p}) = D^2\pi(\overline{p})$ is symmetric and positive semi-definite with $Ds(\overline{p})\overline{p} = 0$ (law of supply).*

Cost minimization

- If Y is representable by a production function, i.e. $Y := \{(y, -z) \in \mathbb{R}_+ \times \mathbb{R}_-^{L-1} \mid y \leq f(z)\}$, then an auxiliary problem to profit maximization is cost minimization:

$$\mathcal{C}(p, x) := \begin{cases} \min & p \cdot z \\ s.t. & f(z) \geq x \end{cases}$$

- The value $c(p, x)$ of $\mathcal{C}(p, x)$ is called the cost function and its solution $z(q, x)$ is called the conditional factor demand.

Properties of the cost function

Proposition

Let $p \gg 0$ and Y a nonempty closed production set satisfying the free-disposal assumption. One has:

- *c is homogeneous of degree one in p and non-decreasing in x.*
- *c is a concave function of p. If f is concave, c is a convex function of x.*
- *z is homogeneous of degree 0.*
- *(Sheppard's lemma) If $z(\overline{p}, x)$ is single-valued, then c is differentiable with respect to p at $\overline{p}$ and $\nabla_p c(\overline{p}, x) = z(\overline{p}, x)$.*

Envelope Theorem

- Consider the problem

$$\begin{cases} \max_{x\in\mathbb{R}^M} f(x,q) \\ g_1(x,q)=0 \\ \vdots \\ g_n(x,q)=0 \end{cases}$$

- Let $\lambda_1,\cdots,\lambda_n$ denote the associated multipliers, ($x(q)$ a solution at q and $v(q)$ the value function. Then

$$\frac{\partial v}{\partial q_s}(q)=\frac{\partial f}{\partial q_s}(x(q),q)-\sum_{i=1}^{n}\lambda_i\frac{\partial g_i}{\partial q_s}(x(q),q)$$

Problem set

- P6,P8, P12, P13,P9, P10,P11, P5, P7, P14 and following (as time permits)

Lecture 9: competitive equilibrium in an exchange economy

- Economic agents do not act in isolation...
- Investigate the coordination of agents' choices and actions.
- Assuming rational and competitive behavior and that coordination takes place through a system of prices:
- i.e. general competitive equilibrium.
- References: MWG 15A, 15B, 10B

The framework

- Exchange economy: a set of consumers with preferences and initial endowments.
- One investigates the competitive allocation of goods among consumers
- abstracting away from the production process

N.B a stylized model but often used to analyze financial markets, intertemporal allocation of ressources and risks.

The framework

- An exchange economy with a finite number ℓ of commodities labeled by the subscript $h = 1, \ldots, \ell$ and a finite number m of consumers labeled by the subscript $i = 1, \ldots, m$
- The preferences of each consumer i are represented by a utility function u_i from $\mathbb{R}^\ell_+$ to $\mathbb{R}$.
- Each consumer has an initial endowments $e_i \in \mathbb{R}^\ell_+$.
- The total initial endowments of the economy is then $e = \sum_{i=1}^m e_i \in \mathbb{R}^\ell_+$.

Allocations

An allocation is a vector $(x_i) \in (\mathbb{R}^{\ell}_{+})^m$ representing the allocation of goods to each agent in the economy.

Definition

An allocation $(x_i) \in (\mathbb{R}^{\ell}_{+})^m$ is said to be feasible with respect to the total initial endowments e if $\sum_{i=1}^{m} x_i = e$. Accordingly, the set of feasible allocations is:

$$A(e) = \{(x_i) \in (\mathbb{R}^{\ell}_{+})^m \mid \sum_{i=1}^{m} x_i = e.\}$$

Basic properties of the set of feasible allocations

Proposition

If $e \in \mathbb{R}^{\ell}_{+}$, the set of feasible allocations $A(e)$ is non-empty, bounded, closed and convex.

Proof:

- *Non-empty as it contains* $(e, 0, \cdots, 0)$
- *Bounded as for all* i, *one must have* $0 \leq x_i$ *and thus* $0 \leq x_i \leq e$.
- *Closed because of the continuity of the sum.*
- *Convex as if* $(x_i), (y_i) \in A(e)$, *one has for all* $\lambda \in [0, 1]$: $\sum_{i=1}^{m}(\lambda x_i + (1-\lambda)y_i) = \lambda \sum_{i=1}^{m} x_i + (1-\lambda)\sum_{i=1}^{m} y_i = \lambda e + (1-\lambda)e = e$ *and thus* $(\lambda x_i + (1 - \lambda y_i)) \in A(e)$

N.B: Closed+ Bounded $\Rightarrow$ Compact

Solution concept

Which feasible allocation(s) should emerge from agents' interactions ?

- Normative approach: Pareto-optimal allocation (no agent can be made better of without making another worst off, see below).
- Cooperative game theory: core allocation (no coalition of agent can improve upon the allocation).
- Non-cooperative game theory: Nash equilibrium of a bargaining game.
- General equilibrium theory: competitive equilibrium, individually rational choices made compatible by a system of prices.

Competitive equilibrium: underlying economic assumptions

- Existence of institutions (system of markets, stock-exchange, Walrasian auctioneer,...) that provides public prices for commodities.
- Each agent acts in a competitive manner: has no influence on price, takes the price as given.
- Given a price $p \in \mathbb{R}^{\ell}_{+}$, each agent can compute his income $p \cdot e_i$.

Competitive equilibrium: underlying economic assumptions

- Existence of institutions (system of markets, stock-exchange, Walrasian auctioneer,...) that provides public prices for commodities.
- Each agent acts in a competitive manner: has no influence on price, takes the price as given.
- Given a price $p \in \mathbb{R}_+^\ell$, each agent can compute his income $p \cdot e_i$.
- Then, we assume each agent is rational: he maximizes his utility given his budget $p \cdot e_i$.Hence agent i demands $d_i(p, p \cdot e_i)$.
- A competitive equilibrium is a situation where the price system makes the demands of all agents compatible.

Competitive equilibrium: formal definition

Definition

A Walrasian (competitive) equilibrium of the economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$ is a price $p^* \in \mathbb{R}_+^\ell$ and allocations $(x_i^*) \in (\mathbb{R}_+^\ell)^m$ satisfying :

a) For all $i = 1, \ldots, m$, x_i^* is a solution of the optimization problem :

$$\begin{cases} \text{Maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i \\ x_i \geq 0 \end{cases}$$

and

b) (Market Clearing Conditions) $\sum_{i=1}^m x_i^* = \sum_{i=1}^m e_i$.

Competitive equilibrium: equivalent definition

Definition

A Walrasian (competitive) equilibrium of the economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$ is a system of price $p^* \in \mathbb{R}_+^\ell$ and allocations $(x_i^*) \in (\mathbb{R}_+^\ell)^m$ satisfying :

a) For all $i = 1, \ldots, m$, $x_i^* \in d_i(p^*, p^* \cdot e_i)$.

b) (Market Clearing Conditions) $\sum_{i=1}^m x_i^* = \sum_{i=1}^m e_i$.

N.B. If the demand functions are single-valued, p^* is an equilibrium price if and only if

$$\sum_{i=1}^m d_i(p^*, p^* \cdot e_i) = \sum_{i=1}^m e_i.$$

Basic properties of competitive equilibrium

Proposition

Let $(p^*, (x_i^*))$ be a Walras equilibrium of the economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$.

i) For all $t > 0$, $(tp^*, (x_i^*))$ is a Walras equilibrium of the economy $\mathcal{E}$.

ii) If the preferences are monotonic, for all $i = 1, \ldots, m$, $p^* \cdot x_i^* = p^* \cdot e_i$.

iii) If the preferences of one consumer are strictly monotonic, then the equilibrium price p^* belongs to $\mathbb{R}^\ell_{++}$.

iv) For all $i = 1, \ldots, m$, $u_i(x_i^*) \geq u_i(e_i)$.

Proof: Properties of the solutions of the consumers' problems.
N.B: (i) implies that prices can be normalized.

The 2×2 case: the Edgeworth box.

- An economy with 2 consumers and 2 *goods*.
- Can be represented graphically through the Edgeworth box.

The 2×2 case: the Edgeworth box.

- An economy with 2 consumers and 2 *goods*.
- Can be represented graphically through the Edgeworth box.
- The allocation of agent 1 is represented in a standard coordinate system with origin $(0, 0)$.

The 2×2 case: the Edgeworth box.

- An economy with 2 consumers and 2 *goods*.
- Can be represented graphically through the Edgeworth box.
- The allocation of agent 1 is represented in a standard coordinate system with origin $(0, 0)$.
- The allocation of agent 2 is represented in the coordinate system with origin e and with inverted direction of axes.

The 2×2 case: the Edgeworth box.

- An economy with 2 consumers and 2 *goods*.
- Can be represented graphically through the Edgeworth box.
- The allocation of agent 1 is represented in a standard coordinate system with origin $(0, 0)$.
- The allocation of agent 2 is represented in the coordinate system with origin e and with inverted direction of axes.
- The set of feasible allocations correspond to the inside of the "box" formed by $(0, 0)$, e and the four coordinate axes.

The 2×2 case: the Edgeworth box.

- An economy with 2 consumers and 2 *goods*.
- Can be represented graphically through the Edgeworth box.
- The allocation of agent 1 is represented in a standard coordinate system with origin $(0,0)$.
- The allocation of agent 2 is represented in the coordinate system with origin e and with inverted direction of axes.
- The set of feasible allocations correspond to the inside of the “box" formed by $(0,0)$, e and the four coordinate axes.
- A point $x_1 \in \mathbb{R}^2_+$ in the Edgeworth box corresponds to the allocation to agent 1 with respect to the origin $(0,0)$ and to the allocation $x_2 = e - x_1 \in \mathbb{R}^2_+$ to agent 2 with respect to the origin e.

The Edgeworth box (continued).

- The budget line of both agents at price $p \in \mathbb{R}^2_+$ is represented by a line going through $e_1 \in \mathbb{R}^2_+$ and with slope $-{p_1}/{p_2}$

The Edgeworth box (continued).

- The budget line of both agents at price $p \in \mathbb{R}^2_+$ is represented by a line going through $e_1 \in \mathbb{R}^2_+$ and with slope $-p_1/p_2$
- The preferences can be represented by drawing indifference curves for both consumers in the box.

The Edgeworth box (continued).

- The budget line of both agents at price $p \in \mathbb{R}^2_+$ is represented by a line going through $e_1 \in \mathbb{R}^2_+$ and with slope $-{}^{p_1}\!/_{p_2}$
- The preferences can be represented by drawing indifference curves for both consumers in the box.
- The set of allocations preferred to x_1 by agent 1 is the upper contour set of the indifference curve through x_1

The Edgeworth box (continued).

- The budget line of both agents at price $p \in \mathbb{R}^2_+$ is represented by a line going through $e_1 \in \mathbb{R}^2_+$ and with slope $-p_1/p_2$
- The preferences can be represented by drawing indifference curves for both consumers in the box.
- The set of allocations preferred to x_1 by agent 1 is the upper contour set of the indifference curve through x_1
- The set of allocations preferred to x_2 by agent 1 is the lower contour set of the indifference curve through x_2

The Edgeworth box (continued).

The Edgeworth box (continued).

- The demand of consumer 1 (resp. 2) corresponds to the point(s) where the budget line is tangent to the indifference curve.

The Edgeworth box (continued).

- The demand of consumer 1 (resp. 2) corresponds to the point(s) where the budget line is tangent to the indifference curve.
- The offer curve of consumer 1 is the curve that represents the demand in the Edgeworth box as the price varies.

The Edgeworth box (continued).

- The demand of consumer 1 (resp. 2) corresponds to the point(s) where the budget line is tangent to the indifference curve.
- The offer curve of consumer 1 is the curve that represents the demand in the Edgeworth box as the price varies.
- An equilibrium materializes in the Edgeworth box as a point where the demand of both agents coincide, i.e. offer curves coincide. Indeed, one then has:

$$d_1(p, p \cdot e_1) = e - d_2(p, p \cdot e_2).$$

N.B. The equilibrium is not necessarily unique.

The Edgeworth box (example).

- Two consumers with initial endowments $e_1 = (1, 2)$ and $e_2 = (2, 1)$ Cobb-Douglas utility $u_i(x_{i,1}, x_{i,2}) = x_{i,1}^{\alpha} x_{i,2}^{1-\alpha}$.
- Price of good 2 set equal to 1

The Edgeworth box (example).

- Two consumers with initial endowments $e_1 = (1, 2)$ and $e_2 = (2, 1)$ Cobb-Douglas utility $u_i(x_{i,1}, x_{i,2}) = x_{i,1}^{\alpha} x_{i,2}^{1-\alpha}$.
- Price of good 2 set equal to 1
- Offer curve of consumer 1 is given by

$$d_1(p, p \cdot e_1) = (\alpha \frac{(p_1 + 2)}{p_1}, (1 - \alpha)(p_1 + 2)), \ p_1 \in \mathbb{R}_+$$

The Edgeworth box (example).

- Two consumers with initial endowments $e_1 = (1, 2)$ and $e_2 = (2, 1)$ Cobb-Douglas utility $u_i(x_{i,1}, x_{i,2}) = x_{i,1}^{\alpha} x_{i,2}^{1-\alpha}$.
- Price of good 2 set equal to 1
- Offer curve of consumer 1 is given by

$$d_1(p, p \cdot e_1) = (\alpha \frac{(p_1 + 2)}{p_1}, (1 - \alpha)(p_1 + 2)),\ p_1 \in \mathbb{R}_+$$

- Offer curve of consumer 2 is given by

$$d_2(p, p \cdot e_2) = (\alpha \frac{(2p_1 + 1)}{p_1}, (1 - \alpha)(2p_1 + 1)),\ p_1 \in \mathbb{R}_+$$

The Edgeworth box (example).

- Two consumers with initial endowments $e_1 = (1, 2)$ and $e_2 = (2, 1)$ Cobb-Douglas utility $u_i(x_{i,1}, x_{i,2}) = x_{i,1}^{\alpha} x_{i,2}^{1-\alpha}$.
- Price of good 2 set equal to 1
- Offer curve of consumer 1 is given by

$$d_1(p, p \cdot e_1) = (\alpha \frac{(p_1 + 2)}{p_1}, (1-\alpha)(p_1 + 2)),\ p_1 \in \mathbb{R}_+$$

- Offer curve of consumer 2 is given by

$$d_2(p, p \cdot e_2) = (\alpha \frac{(2p_1 + 1)}{p_1}, (1-\alpha)(2p_1 + 1)),\ p_1 \in \mathbb{R}_+$$

- Equilibrium determined by market clearing conditions:

$$\alpha^{(p_1+2)}/_{p_1} + \alpha^{(2p_1+1)}/_{p_1} = 3 \Rightarrow p_1^* = {}^{\alpha}/_{(1-\alpha)}$$
$$(1-\alpha)(p_1^* + 2) + (1-\alpha)(2p_1^* + 1) = 3 \Rightarrow p_1^* = {}^{\alpha}/_{(1-\alpha)}$$

A primer on Pareto optimality

Definition

An allocation x in the Edgeworth box is Pareto optimal if there is no other allocation x' such that for all i $x_i \succsim x_i'$ and for one i, $x_i \succ x_i'$ where $i = 1, 2$.

- Graphically, an allocation x is Pareto optimal if both indifference curves are tangent at x.
- The set of all Pareto optimal allocations is known as the Pareto set.
- The contract curve corresponds to the part of the Pareto set where both agents do at least as well as at their initial endowments.

Problem set

- Micro 1B exercise list: 1, 2, 3,4.

Supplementary material : Existence of a competitive equilibrium in an exchange economy

- Objectives: understand sufficient conditions for the existence of a competitive equilibrium
- References: MWG 15C, 16E, 16F

Existence of a competitive equilibrium in an exchange economy

Theorem

The exchange economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$ has a Walras equilibrium with a strictly positive equilibrium price if the following sufficient condition are satisfied :

(i) For all $i = 1, \dots, m$, u_i is continuous, quasi-concave and strictly monotonic;

(ii) For all $i = 1, \dots, m$, $e_i \geq 0$ and $\sum_{i=1}^m e_i \gg 0$.

Excess demand

Definition

In the exchange economy $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$:

1. *The excess demand of consumer i is the function* $z_i : \mathbb{R}_+^\ell \to \mathbb{R}_+^\ell$ *defined by* $z_i(p) = d_i(p, p \cdot e_i) - e_i$
2. *The excess demand of the economy is the function* $z : \mathbb{R}_+^\ell \to \mathbb{R}_+^\ell$ *defined by* $z(p) = \sum_{i=1}^m z_i(p)$

Proposition

The price $p \in \mathbb{R}_+^\ell$ *is an equilibrium price of the economy* $\mathcal{E} = ((u_i, e_i)_{i=1}^m)$: *if and only if* $z(p) = 0$.

Properties of excess demand

Proposition

Assume that For all $i = 1, \ldots, m$, u_i is continuous, quasi-concave and strictly monotonic while $e_i \geq 0$ and $\sum_{i=1}^{m} e_i \gg 0$, then:

1. *z is continuous,*
2. *z is homogeneous of degree zero ,i.e for all $t \geq 0$ and all $p \in \mathbb{R}_{+}^{\ell}$, $z(tp) = z(p)$*
3. *$p \cdot z(p) = 0$ (Walras law),*
4. *z is bounded below*
5. *If $\lim_{n \to +\infty} p^n = p$ with $p \neq 0$ and $p_h = 0$ for some h, then $\lim_{n \to +\infty} \|z(p^n)\| = +\infty$*

Existence of competitive equilibrium I

With two commodities:

- According to Walras law, it suffices to prove there exists an equilbrium on the market for good 1 and one can normalize $p_2 = 1$
- For p_1 sufficiently small, one must have $z_1(p1, 1) \geq 0$
- For p_1 sufficiently large, one must have $z_1(p1, 1) \leq 0$
- By continuity there must thus exist $p_1 \in \mathbb{R}_+$ such that $z_1(p1, 1) = 0$

In the general case, the proof relies on more complex mathematical techniques such as Kakutani theorem or degree theory.

Existence of competitive equilibrium II

Theorem

Let C be a compact an convex subset of $\mathbb{R}^d$ and $f : C \to C$. There exists $x^* \in C$ such that $f(x^*) = x^*$.

Theorem (Varian, Micro-economic analysis, p. 321)

If $z : S \to \mathbb{R}^\ell$ is continuous and satisfies Walras law, there exists some p^* such that $z(p*) \leq 0$

N.B The original demand might have to be modified to ensure continuity of z on the frontier of S.

Remark

If $z(p^*) \leq 0$ and $p^* >> 0$ then one must have $z(p^*) = 0$

Proof of existence following Varian

- Consider $g : S \to S$ defined by

$$g_h(p) = \frac{p_h + \max(0, z_h(p))}{1 + \sum_{j=1}^{\ell} \max(0, z_j(p))}$$

- Brouwer theorem implies there is a fixed point p^*. One gets successively:
  - $p_h^* \sum_{j=1}^{\ell} \max(0, z_j(p^*)) = \max(0, z_h(p^*))$
  - $z_h(p^*) p_h^* \sum_{j=1}^{\ell} \max(0, z_j(p^*)) = z_h(p^*) \max(0, z_h(p^*))$
  - $[\sum_{j=1}^{\ell} \max(0, z_j(p^*))] \sum_h z_h(p^*) p_h^* = \sum_h z_h(p^*) \max(0, z_h(p^*))$
  - This yields using Walras law $0 = \sum_h z_h(p^*) \max(0, z_h(p^*))$
- The last equation implies for all h, $z_h(p*) \leq 0$ and thus $z(p^*) \leq 0$

Counter-example to the existence of a competitive equilibrium

- We consider an economy with 2 consumers and 2 goods.
- Consumer 1 has utility $u_1(x_1, x_2) = x_2$ and initial endowment $e_1 = (2, 2)$ and thus his demand is given by

$$d_1(p) = (0, \frac{2p_1 + 2p_2}{p_2})$$

- Consumer 2 has utility $u_2(x_1, x_2) = x_1 + x_2$ and initial endowment $e_2 = (2, 0)$ and thus his demand is given by

$$d_2(p) = \begin{cases} (2, 0) & \text{if} p_1 < p_2 \\ (t, 2 - t)_{t \in [0,2]} & \text{if} p_1 = p_2 \\ (0, 2\frac{p_1}{p_2}) & \text{if } p_2 < p_1 \end{cases}$$

- $\Rightarrow$ No equilibrium without interiority of endowment or strict monotinicity

Counter-example to the existence of a competitive equilibrium II

We consider the economy with the following characteristics:

- $\ell = 2$, $m = n = 1$.
- $u_1(x_1, x_2) = x_1 x_2$ and $e_1 = (2, 1)$.
- $Y = \{(y_1, y_2) \in \mathbb{R}^2 \mid y_1 \leq 0, y_2 \leq 0 \text{ if } y_1 > -1, y_2 \leq 1 \text{ if } y_1 < -1\}$,

This economy doesn't have an equilibrium $\Rightarrow$ the convexity assumption is a fundamental requirement for the existence of a Walras equilibrium.

Lecture 10: Private ownership production economies

- Objectives: analyze the interactions between firms and consumers.
- References: MWG 15C, 16E, 16F

Definition

Definition

A private ownership production economy consists in:

- *A finite space of commodities* ($\mathbb{R}^\ell$),
- *A finite number n of producers characterized by their production set* $Y_j \subset \mathbb{R}^\ell$,
- *A finite number of consumers characterized by:*
  - *their utility function* $u_i : \mathbb{R}^\ell_+ \to \mathbb{R}$,
  - *their initial endowment* $e_i \in \mathbb{R}^\ell$
  - *their portfolio of shares in firms* $(\theta_{ij})_{j=1}^n$ *where* $\theta_{ij} \in [0,1]$ *and for all* j, $\sum_{i=1}^m \theta_{ij} = 1$.

To summarize, a production economy is a collection

$$\mathcal{E} = \left(\mathbb{R}^\ell, (u_i, e_i)_{i=1}^m, (Y_j)_{j=1}^n, (\theta_{ij})_{i=1,j=1}^{i=m,j=n}\right)$$

Equilibrium

Definition

A Walras equilibrium of the private ownership economy $\mathcal{E}$ is an element $((x_i^*), (y_j^*), p^*)$ of $(\mathbb{R}_+^\ell)^m \times (\mathbb{R}^\ell)^n \times \mathbb{R}_+^\ell$ such that

(a) [Profit maximization] for every j, y_j^* is a solution of

$$\begin{cases} \text{maximize } p^* \cdot y_j \\ y_j \in Y_j \end{cases}$$

(b) [Preference maximization] for every i, x_i^* is a solution of

$$\begin{cases} \text{maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^* \\ x_i \geq 0 \end{cases}$$

(c) [Market Clearing Conditions]

$$\sum^m x_i^* = \sum^m e_i + \sum^n y_j^*$$

Basic properties of a Walrasian equilibrium

Proposition

If preferences are monotonic and $((x_i^*), (y_j^*), p^*)$ is a Walras equilibrium of the economy $\mathcal{E}$, then

1. *for every $t > 0$, $((x_i^*), (y_j^*), tp^*)$ is also a Walras equilibrium;*
2. *for every i, $p^* \cdot x_i = p^* \cdot e_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^*$;*
3. *if for every $j \in J$, $0 \in Y_j$, then $u_i(x_i^*) \geq u_i(e_i)$ for all i;*
4. *$((x_i^*), p^*)$ is a Walras equilibrium of the pure exchange economy $\tilde{\mathcal{E}} = \left(u_i, e_i + \sum_{j=1}^n \theta_{ij} y_j^*\right)_{i=1}^m$*

Walras law

Proposition

Let us assume that all the preferences are monotonic and strictly monotonic for at least one consumer. Let $(p^*, (x_i^*), (y_j^*)) \in \mathbb{R}^{\ell}_{++} \times (\mathbb{R}^{\ell}_{+})^m \times \prod_{j=1}^n Y_j$ *be such that:*

(a) [Profit maximization] for every j, y_j^* *is a solution of*

$$\left\{ \begin{array}{l} \text{maximize } p^* \cdot y_j \\ y_j \in Y_j \end{array} \right.$$

(b) [Preference maximization] for every i, x_i^* *is a solution of*

$$\left\{ \begin{array}{l} \text{maximize } u_i(x_i) \\ p^* \cdot x_i \leq p^* \cdot e_i + \sum_{j=1}^n \theta_{ij} p^* \cdot y_j^* \\ x_i \geq 0 \end{array} \right.$$

(c) for all commodities $h = 1, \ldots, \ell - 1$, $\sum_{i=1}^m x*_{i,h} = \sum_{i=1}^m e_{i,h} + \sum_{j=1}^n y_{j,h}^*$.

Then, $(p^*, (x_i^*))$ *is a Walrasian equilibrium of the economy.*

Example

Production economy with two commodities, one producer and one consumer :

- $u_1(x_{1,1}, x_{1,2}) = (x_{1,1})^{1/2}(x_{1,2})^{1/2},\ e_1 = (2,1),\ \theta_{11} = 1$
- $Y_1 = \{(y_{1,1}, y_{1,2}) \in \mathbb{R}^2 \mid y_{1,1} \leq 0; y_{1,2} \leq \sqrt{|y_{1,1}|}\}$;

Given price $(p, 1)$, the demand of the consumer, the supply and the profit of the producer are given respectively by:

- $d_1((p,1), w) = (\frac{w}{2p}, \frac{w}{2})$
- $s_1(p,1) = (-\frac{1}{4p^2}, \frac{1}{2p})$
- $\pi_1(p,1) = \frac{1}{4p}$

market clearing for good 1 is then given by

$$\frac{1}{2p}(2p + 1 + {}^{1}/_{4p}) - 2 + \frac{1}{4p^2} =$$

$$(p^*, x_1^*, y_1^*) = \left((1, 2(\sqrt{3}-1)), (\frac{2+3\sqrt{3}}{2}, \frac{2+3\sqrt{3}}{4(\sqrt{3}-1)}), (-\frac{2-\sqrt{3}}{2}, \sqrt{3}-1)\right)$$

is the unique equilibrium of this economy up to price normalization.

Existence of a competitive equilibrium

Theorem

The economy $\mathcal{E} = ((u_i, e_i), (Y_j), (\theta_{ij}))$ has a Walras equilibrium if:

1. *For all $i = 1, \ldots, m$, u_i is continuous, quasi-concave and monotonic;*
2. *For all $i = 1, \ldots, m$, $e_i \gg 0$.*
3. *for all $j = 1, \ldots, n$, Y_j is closed, convex and satisfies the possibility of inactivity.*
4. *The total production set $Y = \sum_{j=1}^{n} Y_j$ satisfies the irreversibility condition $Y \cap -Y = \{0\}$ and the impossibility of free production $Y \cap \mathbb{R}_+^\ell \subset \{0\}$.*

Sketch of the proof at the end of the class (if time permits). See Debreu, Theory of Value (1984) for a comprehensive account.

Characterization of equilibrium by first-order conditions

Assume:

- for all i, u_i is quasi-concave , continuous on $\mathbb{R}^{\ell}_{+}$, differentiable on $\mathbb{R}^{\ell}_{++}$ and for all $x \in \mathbb{R}^{\ell}_{++}$, $\nabla u_i(x) \in \mathbb{R}^{\ell}_{++}$.
- for all j, Y_j is representable by a quasi-convex and differentiable transformation function t_j, strictly increasing in one of its coordinate.

Proposition

$(p^*, (x_i^*), (y_j^*)) \in \mathbb{R}^{\ell}_{++} \times (\mathbb{R}^{\ell}_{++})^m \times \prod_{j=1}^{N} Y_j$ is a Walras equilibrium of the economy, is and only if there exists $(\lambda_i) \in \mathbb{R}^m_{++}$ and $(\lambda_i) \in \mathbb{R}^n_{++}$ such that

- *for all i, $\nabla u_i(x_i^*) = \lambda_i p^*$ and $p^* \cdot x_i^* = p^* \cdot e_i + \sum_{j=1}^{n} y_{j,h}^*$.*
- *for all j, $\nabla t_j(y_j^*) = \lambda_j p^*$ and $t_j(y_j^*) = 0$;*
- *for all commodities $h = 1, \ldots, \ell - 1$,*

$$\sum_{i=1}^{m} x_{i,h}^* = \sum_{i=1}^{m} e_{i,h} + \sum_{j=1}^{n} y_{j,h}^*$$

Characterization via marginal rates of substitution and transformation

The previous characterization is equivalent to the set of conditions:

- for all $i = 1, \cdots m$ and all $h, k = 1, \cdots m$, one has

$$\frac{\partial u_i / \partial x_{i,h}(x_i^*)}{\partial u_i / \partial x_{i,k}(x_i^*)} = \frac{p_h^*}{p_k^*} \text{ and } p^* \cdot x_i^* = p^* \cdot e_i$$

- for all $j = 1, \cdots n$ and all $h, k = 1, \cdots m$, one has

$$\frac{\partial t_j / \partial y_{j,h}(y_j^*)}{\partial t_j / \partial x_{j,k}(y_j^*)} = \frac{p_h^*}{p_k^*} \text{ and } t_j(y_j^*) = 0$$

- for all commodities $h = 1, \ldots, \ell - 1$, $\sum_{i=1}^m x_i^{*h} = \sum_{i=1}^m e_i^h$.

Problem set

- Exercises 11-14, 15,16,17,18,25
- One-consumer, one-producer economy

The one-consumer, one-producer economy (Mas-Colell, 15C)

- Two goods: labor (good 1) and consumption good (good 2).
- 2 agents: 1 consumer, 1 firm.
- Consumer has continuous, convex and strongly monotone preferences over $\mathbb{R}^2_+$ represented by the utility function u and initial endowment $(L, 0)$.
- Firm produces consumption good from labor using strictly concave and increasing production function $f(z)$.
- Price of consumption good is denoted by p, wage is denoted by w.

Behavior of agents

- Problem of the firm:

$$\max_{y \geq 0} pf(z) - wz$$

labor input denoted by $z(p, w)$, output $q(p, w)$ and profit $\pi(p, w)$.

- Problem of the household:

$$\begin{cases} \max u(x_1, x_2) & \\ \quad s.t. & px_2 \leq w(L - x_1) + \pi(p, w) \\ & L - x_1 \geq 0 \\ & x_2 \geq 0 \end{cases}$$

Demand denoted by $(x_1(p, w), x_2(p, w))$.

Equilibrium

- Equilibrium
$$L = x_1(p^*, w^*) + z(p^*, w^*).$$
and
$$x_2(p^*, w^*) = q(p^*, w^*)$$

Graphical representation

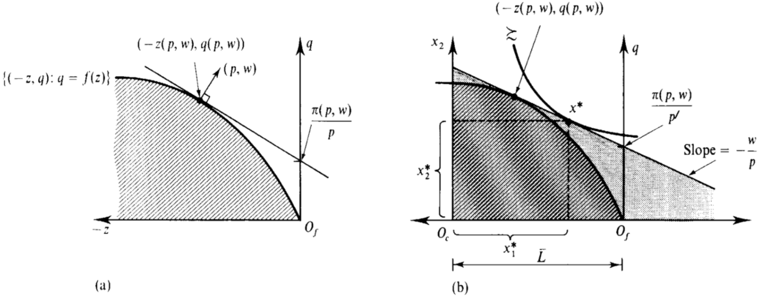
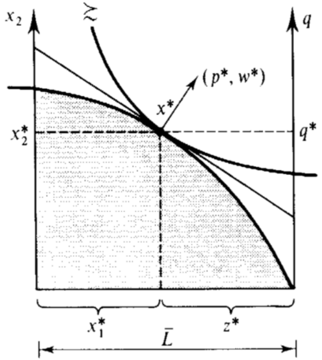


Figure 15.C.1 (a) The firm's problem. (b) The consumer's problem.

Graphical representation of equilibrium



- Equilibrium if and only if utility is maximized given technological constraints.
- First and second welfare theorems.

Non-convexities and welfare theorems

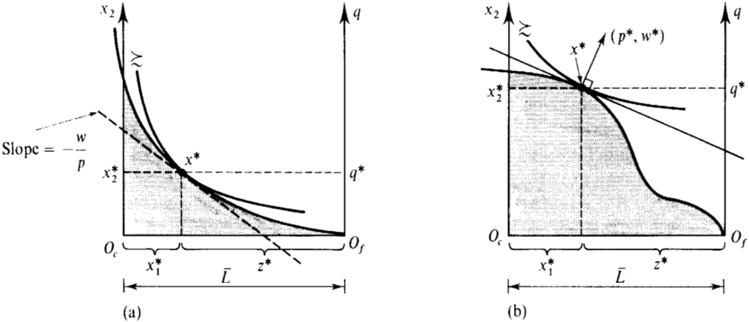


Figure 15.C.3 (a) Failure of the second welfare theorem with a nonconvex technology. (b) The first welfare theorem applies even with a nonconvex technology.

Lecture 11 : Pareto Optimality in production economies

- Objectives: investigate the welfare properties of market equilibrium.
- References: MWG 16C, 16D.

Outline

Attainable allocations in an exchange economy

Definition

In the exchange economy $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i)_{i=1}^m, e\right)$, *the set of attainable/feasible allocations is:*

$$\mathcal{A}(e) = \left\{ (x_i) \in (\mathbb{R}_+^\ell)^m \mid \sum_{i=1}^m x_i = e \right\}$$

Attainable allocations in a production economy

Definition

In the production economy $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i)_{i=1}^m, (Y_j)_{j=1}^n, e\right)$, the set of attainable/feasible allocations of is:

$$\mathcal{A}(e) = \left\{ ((x_i), (y_j)) \in (\mathbb{R}_+^\ell)^m \times \prod_{j=1}^n Y_j \mid \sum_{i=1}^m x_i = e + \sum_{j=1}^n y_j \right\}$$

With a slight abuse of notation, one says that $(x_i) \in (\mathbb{R}_+^\ell)^m$ is feasible if there exists $(y_j)) \in \prod_{j=1}^n Y_j$ such that $((x_i), (y_j)) \in \mathcal{A}(e)$

Definition

Definition

An allocation $(x_i) \in (\mathbb{R}^\ell_+)^m$ *is preferred in the sense of Pareto to an allocation* $(x'_i) \in (\mathbb{R}^\ell_+)^m$ *if for all* i, $u_i(x_i) \geq u_i(x'_i)$ *and if for at least one* i_0, $u_{i_0}(x_{i_0}) > u_{i_0}(x'_{i_0})$.

Definition

An allocation $(\bar{x}_i) \in (\mathbb{R}^\ell_+)^m$ is a Pareto optimum if it is feasible and if there does not exist a feasible allocation (x_i) which is preferred to $(\bar{x}_i)$ in the sense of Pareto.

Comment: the notion of Pareto optimum is a“minimum" notion of efficiency, it doesn’t embed any notion of social choice.

Reminder: Pareto optimum in the Edgeworth box

Definition

An allocation x in the Edgeworth box is Pareto optimal if there is no other allocation x' such that for all i $x_i \succsim x'_i$ and for one i, $x_i \succ x'_i$ where $i = 1, 2$.

- The set of allocations preferred to x_1 by agent 1 is the upper contour set of the indifference curve through x_1
- The set of allocations preferred to x_2 by agent 1 is the lower contour set of the indifference curve through x_2
- Graphically, an allocation x is Pareto optimal if both indifference curves are tangent at x.
- The set of all Pareto optimal allocations is known as the Pareto set.
- The contract curve corresponds to the part of the Pareto set where both agents do at least as well as at their initial endowments (it is also the core of the economy).

Characterization of Pareto optima

Proposition

Assume that for all $i = 1, \dots, m$, u_i is continuous and strictly monotonic on $\mathbb{R}^\ell_+$. An allocation $(\bar{x}_i)$ is Pareto optimal if and only if $(\bar{x}_i)$ is a solution of the following problem.

$$\max_{(x_i)\in(\mathbb{R}^\ell_+)^m} \quad u_1(x_1)$$

$$\text{subject to} \quad \begin{cases} u_i(x_i) \geq u_i(\bar{x}_i) \text{ for all } i = 2, \dots, m, \\ \sum_{i=1}^m x_i = e \end{cases}$$

Proof:

One uses the fact that if $(\bar{x}_i)$ is a Pareto optimum the allocation to consumer 1 can not be improved upon and that the converse is true if the utilities are strictly increasing.

Differential characterization of Pareto optima

Proposition

Assume that for all $i = 1, \dots, m$, u_i is differentiable and quasi-concave on $\mathbb{R}^\ell_+$, and $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ for all $x \in \mathbb{R}^\ell_{++}$.
An interior allocation $(\bar{x}_i) \in (\mathbb{R}^\ell_{++})^m \cap A(e)$ is Pareto optimal if and only if there exists $(\lambda_2, ..., \lambda_i, ..., \lambda_m) \in \mathbb{R}^{m-1}_{++}$ such that

$$\nabla u_1(\bar{x}_1) = \lambda_i \nabla u_i(\bar{x}_i), \;\; \forall\, i = 2, \dots, m$$

In other words, at an interior Pareto optimum $\bar{x}$, all the marginal rates of substitution are equal, i.e. for all $i = 1, \cdots m$ and all $h, k = 1, \cdots m$, one has:

$$\frac{\partial u_i / \partial x_{i,h}(x_i^*)}{\partial u_i / \partial x_{i,k}(x_i^*)} = \frac{\partial u_1 / \partial x_{1,h}(x_1^*)}{\partial u_1 / \partial x_{1,k}(x_1^*)}$$

Characterization of Pareto optima in a production economy

Assume that for all $i = 1, \ldots, m$, u_i is continuous and strictly increasing on $\mathbb{R}^\ell_+$ and for all j, Y_j is represented by a continuous transformation function t_j.

Proposition

An allocation $((\bar{x}_i), (\bar{y}_j)($ is Pareto optimal if and only if it is a solution of the following problem.

$$\max_{(x_i)\in(\mathbb{R}^\ell_+)^m} \quad u_1(x_1)$$

$$\textit{subject to} \quad \begin{cases} u_i(x_i) \geq u_i(\bar{x}_i) \textit{ for all } i = 2, \ldots, m, \\ t_j(y_j) \leq 0 \textit{ for all } j = 1, \cdots, n \\ \sum_{i=1}^m x_i = e + \sum_{j=1}^n y_j \end{cases}$$

Differential characterization of Pareto optimality in Production economies

Assume that

- for all i, u_i is continuous and quasi-concave on $\mathbb{R}^\ell_+$, differentiable on $\mathbb{R}^\ell_{++}$ and $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ for all $x \in \mathbb{R}^\ell_{++}$.
- for all j t_j is differentiable and quasi-convex.

Proposition

A feasible allocation $((\bar{x}_i), (\bar{y}_j)) \in (\mathbb{R}^\ell_{++})^m \times \prod_{j=1}^n Y_j$ *is Pareto optimal if and only if:*

1. *For all* $i = 2, \ldots, m$, *there exists* $\lambda_i > 0$ *such that* $\nabla u_1(\bar{x}_1) = \lambda_i \nabla u_i(\bar{x}_i)$ *and*
2. *for all* $j = 1, \ldots, n$, $t_j(y_j) = 0$ *and there exists* $\lambda_j > 0$ *such that* $\nabla u_1(\bar{x}_1) = \lambda_j \nabla t_j(\bar{y}_j)$

Characterization via marginal rates of substitution and transformation

The previous characterization is equivalent to

- for all $i, i' = 1, \cdots m$ and all $h, k = 1, \cdots m$, one has

$$\frac{\partial u_i / \partial x_{i,h}(\bar{x}_i)}{\partial u_i / \partial x_{i,k}(\bar{x}_i)} = \frac{\partial u_j / \partial x_{i',h}(\bar{x}_{i'})}{\partial u_{i'} / \partial x_{i',k}(\bar{x}_{i'})}$$

- for all $j, j' = 1, \cdots n$ and all $h, k = 1, \cdots m$, one has

$$\frac{\partial t_j / \partial y_{j,h}(\bar{y}_j)}{\partial t_j / \partial y_{j,k}(\bar{y}_j)} = \frac{\partial t_{j'} / \partial y_{j',h}(\bar{y}_j)}{\partial t_{j'} / \partial y_{j',k}(\bar{y}_j)}$$

- for all $i, = 1, \cdots m$, all $j = 1, \cdots n$ and all $h, k = 1, \cdots m$, one has:

$$\frac{\partial u_i / \partial x_{i,h}(\bar{x}_i)}{\partial u_i / \partial x_{i,k}(\bar{x}_i)} = \frac{\partial t_j / \partial y_{j,h}(\bar{y}_j)}{\partial t_j / \partial y_{j,k}(\bar{y}_j)}$$

Feasible utilities

Definition

The set of feasible utilities is
$U(A) = \{(u_1(x_1), \ldots, u_m(x_m)) \in \mathbb{R}^m_+ \mid (x_i) \in A(e)\}$

Lemma

If for all $i = 1, \ldots, m$, u_i is continuous, the set of feasible utilities is compact.

Characterization of Pareto optimal through Negishi weights

For all $\lambda \in R^m_+$, we consider the problem

$$\mathcal{P}_\lambda : \begin{cases} \max & \sum_{i=1}^m \lambda_i u_i(x_i) \\ s.t & (x_i) \in A(e) \end{cases}$$

Proposition

- *If $(\bar{x}_i)$ is a solution to $\mathcal{P}_\lambda$ for some $\lambda \in R^m_{++}$, then it is a Pareto optimum.*
- *Conversely, if $U(A)$ is convex, $(\bar{x}_i)$ is a pareto optimum if there exists $\lambda \in R^m_+$ such that $(\bar{x}_i)$ is a solution to $\mathcal{P}_\lambda$.*

Proof:

The first statement is straightforward. The converse relies on the supporting hyperplan theorem: the vector of utility at a Pareto point is at the boundary of $u(A) - \mathbb{R}^\ell_+$.

The (Negishi) weights quantify the weight assigned to the utility of each agent

Supporting Hyperplan theorem

Theorem

Let $C \subset \mathbb{R}^N$ be a convex set and $x \notin int(C)$. There exists $p \in \mathbb{R}^N$ such that for all $y \in C$, one has:

$$p \cdot x \geq p \cdot y$$

Existence of a Pareto optimal allocation

Proposition

The production economy $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i)_{i=1}^m, (Y_j)_{j=1}^n, e\right)$ *has a Pareto optimal allocation if the utility functions are continuous and* $(\sum_{j=1}^n Y_j + e) \cap \mathbb{R}_+^\ell$ *is bounded and closed.*

Existence of a Pareto optimal allocation: proof

It suffices to remark that the following maximization problem has a solution $((\bar{x}_i), \bar{y})$.

$$\max_{(x_i)\in(\mathbb{R}^\ell_+)^m} \quad \sum_{i=1}^m u_i(x_i)$$

$$\textit{subject to} \quad \left\{ \begin{array}{l} x_i \in \mathbb{R}^\ell_+ \text{ for all } i = 2, \ldots, m, \\ t_j(y_j) \leq 0 \text{ for all } j = 1, \cdots, n \\ \sum_{i=1}^m x_i = e + \sum_{j=1}^n y_j \end{array} \right.$$

Outline

Statement of the problem

- Is coordination by the market efficient ?
- First welfare theorem: competitive equilibria are Pareto optimal
- Given a certain social objective, can it be implemented by the market ?
- Second welfare theorem: Pareto optima can be decentralized as competitive equilibria.

First welfare theorem

Theorem

If $((x_i^*), p^*)$ is a Walras equilibrium of the economy $\mathcal{E} = (u_i, e_i)_{i=1}^m$ and for all i, u_i is monotonic, then the equilibrium allocation (x_i^*) is Pareto optimal.

Proof:

Assume that (x_i^*) is not Pareto optimal and consider an allocation $(\bar{x}_i)$ which is Pareto better than (x_i^*). One can then show that

$$p^* \cdot \sum_{i=1}^m \bar{x}_i > p^* \cdot \sum_{i=1}^m x_i^*$$

which leads to a contradiction with

$$\sum_{i=1}^m x_i^* = \sum_{i=1}^m \bar{x}_i = \sum_{i=1}^m e_i.$$

Second welfare theorem

Assume that for all i, u_i is continuous, strictly quasi-concave on $\mathbb{R}^\ell_+$, differentiable on $\mathbb{R}^\ell_{++}$ and $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ for all $x \in \mathbb{R}^\ell_{++}$.

Proposition

If $(x_i^*) \in (\mathbb{R}^\ell_{++})^m$ is a Pareto optimal allocation of $\mathcal{E} = (u_i, e_i)_{i=1}^m$, then there exists $p^* \in \mathbb{R}^\ell_{++}$ such that $(p^*, (x_i^*)) \in \mathbb{R}^\ell_{++} \times (\mathbb{R}^\ell_{++})^m$ is an equilibrium of the economy $\mathcal{E}^* = (u_i, x_i^*)_{i=1}^m$ and one has

$$p^* = \nabla u_1(x_1^*)$$

Reminder: differential characterization of Pareto optima

Under the assumptions of the second welfare theorem:

Proposition

An interior allocation $(\bar{x}_i) \in (\mathbb{R}^{\ell}_{++})^m \cap A(e)$ *is Pareto optimal if and only if there exists* $(\lambda_2, ..., \lambda_i, ..., \lambda_m) \in \mathbb{R}^{m-1}_{++}$ *such that*

$$\nabla u_1(\bar{x}_1) = \lambda_i \nabla u_i(\bar{x}_i), \;\; \forall\, i = 2, \dots, m$$

Proposition

$(p^*, (x_i^*)) \in A(e)$ *is a Walras equilibrium of the economy, if and only there exists* $(\lambda_i) \in \mathbb{R}^m_{++}$ *such that*

- • *for all* i, $\nabla u_i(x_i^*) = \lambda_i p^*$ *and* $p^* \cdot x_i^* = p^* \cdot e_i$;

Decentralization of Pareto optima

Corollary

Under the assumptions of the second welfare theorem, any Pareto optimal allocation $(x_i^*) \in (\mathbb{R}^\ell_{++})^m$ *can be decentralized as a Walras equilibrium using transfers* $(t_i) \in \left(\mathbb{R}^\ell\right)^m$ *such that* $\sum_{i=1}^m t_i = 0$ *and* $p^* \cdot (e_i + t_i) = p^* \cdot x_i^*$ *for all* $i = 1, \ldots, m$*. The equilibrium price* p^* *is called the supporting price for the Pareto optimum.*

Decentralization of Pareto optima

Remark

If the Pareto optima of an economy are known as well as the associated supporting prices, one can compute the equilibrium for all initial endowments. Indeed, the equilibrium associated to the initial endowments (e_i) *are the elements* $(p^*, (x_i^*))$ *such that* (x_i^*) *is a Pareto optimum,* p^* *is the supporting price and* $p^* \cdot x_i^* = p^* \cdot e_i$ *for all* $i = 1, \ldots, m-1$*.*

Implications of the welfare theorems

- The welfare theorems convey the idea that the state shall only be concerned about the redistribution of revenues (endowments) and then let the market operate freely to reach an equilibrium allocation.
- This however rests on a number of explicit (continuous and concave utility functions) and implicit assumptions (no strategic behavior of agents, no external effects, no public goods,...)

First welfare theorem for production economies

Proposition

If $((x_i^*),(y_j^*),p^*)$ is a Walras equilibrium of the private ownership economy $\mathcal{E} = \left(\mathbb{R}^\ell, (u_i, e_i)_{i=1}^m, (Y_j)_{j=1}^n, (\theta_{ij})_{i=1,j=1}^{i=m,j=n}\right)$ and for all i, u_i is monotonic, then the equilibrium allocation $((x_i^*),(y_j^*))$ is Pareto optimal.

First welfare theorem for production economies: proof

The proof is very similar to this in the exchange economy:

- We consider an equilibrium $((x_i^*), (y_j^*))$ and assume that if is Pareto dominated by $((x_i), (y_j))$.
- One must then have

$$p^* \cdot (\sum_j y_j + e) = p^* \cdot \sum_{i=1}^{m} x_i > p^* \cdot \sum_{i=1}^{m} x_i^* = p^* \cdot (\sum_j y_j^* + e)$$

- Thus for at least one j_0, $p^* \cdot y_{j_0} > p^* \cdot y_{j_0}^*$.
- This contradicts the fact that the producer j_0 maximizes its profit for the price p^* at $y_{j_0}^*$

Second welfare theorem for production economies

Assume that

- for all i, u_i is continuous and quasi-concave on $\mathbb{R}^\ell_+$, differentiable on $\mathbb{R}^\ell_{++}$ and $\nabla u_i(x) \in \mathbb{R}^\ell_{++}$ for all $x \in \mathbb{R}^\ell_{++}$.
- for all j t_j is differentiable and quasi-convex.

Proposition

If $((\bar{x}_i), (\bar{y}_j)) \in (\mathbb{R}^\ell_{++})^m \times \prod_{j=1}^n Y_j$ is Pareto optimal, there exists a price $\bar{p} \in \mathbb{R}^\ell_{++}$ such that

1. *for every j, $\bar{y}_j$ is a solution of* $\begin{cases} \text{Maximize } \bar{p} \cdot y_j \\ y_j \in Y_j \end{cases}$
2. *for all i, $\bar{x}_i$ is a solution of* $\begin{cases} \text{Maximize } u_i(x_i) \\ \bar{p} \cdot x_i \leq \bar{p} \cdot \bar{x}_i \\ x_i \geq 0 \end{cases}$
3. $\sum_{i=1}^m \bar{x}_i = e + \sum_{j=1}^n \bar{y}_j$.

Problem set

- Exercise 4-7, 19-24 in Micro1B.
- Parteo optimality and welfare theorems in the Edgeworth box.