

MICROECONOMICS 1A, 2021-22 - Exercises**QEM Master, Joint degree Erasmus Mundus, PhD**

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Exercises are for class corrections (practical sessions and TA sessions). During class corrections students can be selected randomly to propose their solution to starred () exercises.*

*Double starred (**) exercises are for PhD students. They will be solved in dedicated, additional, practical sessions.*

Exercises: 'Introduction' and 'Consumption'

I.1 What is microeconomics about? And why are you interested in microeconomics?

I.2 Discuss the following thoughts by Gerard Debreu on the mathematization and axiomatization of economic theory: "In its mathematical form, economic theory is open to an efficient scrutiny for logical errors... The greater logical solidity has enabled researchers to build on the work of their predecessors and to accelerate the cumulative process in which they are participating... But a Grand Unified Theory will remain out of the reach of economics, which will keep appealing to a large collection of individual theories. Each one of them deals with a certain range of phenomena that it attempts to understand and to explain. When it acquires an axiomatic form, its explicit assumptions delimit its domain of applicability and make illegitimate overstepping of its boundary flagrant" (Gerard Debreu, AER 1991)

C.1 (Ex 1.B.1 in MCWG): Suppose that $\succsim$ is rational. Prove that if $x \succ y \succsim z$, then $x \succ z$.

C.2* (Ex 1.B.2 in MCWG): Suppose that $\succsim$ is rational. Prove the following:

- a) $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive;
- b) $\sim$ is reflexive ($x \sim x$ for all x), transitive and symmetric (if $x \sim y$, then $y \sim x$).

C.3* Suppose that the choice structure $(\mathcal{B}, C(\cdot))$ with $\mathcal{B} = \{\{x, y\}, \{x, y, z\}\}$ and $C(\{x, y\}) = \{x\}$. Prove that if $(\mathcal{B}, C(\cdot))$ satisfies WARP, then it must hold one of the following choice rules: $C(\{x, y, z\}) = \{x\}$ or $C(\{x, y, z\}) = \{z\}$ or $C(\{x, y, z\}) = \{x, z\}$.

C.4 (Ex 1.B.4 in MCWG): Consider a rational preference relation $\succsim$. Show that if $u(x) = u(y)$ implies $x \sim y$ and if $u(x) > u(y)$ implies $x \succ y$, then $u(\cdot)$ is a utility function representing $\succsim$.

C.5* Suppose that $\succsim$ is rational. Show that if X is finite, then there exists an utility function $u(x)$ representing $\succsim$.

C.6* (Lexicographic preferences). For all $x = (x_1, x_2) \in \mathbb{R}_+^2$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}_+^2$, $x \succsim \bar{x} \iff "x_1 > \bar{x}_1" \text{ or } "x_1 = \bar{x}_1 \text{ and } x_2 \geq \bar{x}_2"$.

- For every $\bar{x} \in \mathbb{R}_+^2$, determine and draw the upper contour set $U(\bar{x})$.
- Show that every $\bar{x} \in \mathbb{R}_+^2$, the indifference set $I(\bar{x})$ is a singleton.
- Show that this preference relation is strongly monotone and strictly convex, but not continuous.

C.7* (Linear preferences). For all $x = (x_1, x_2) \in \mathbb{R}_+^2$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}_+^2$,

$$x \succsim \bar{x} \iff ax_1 + bx_2 \geq a\bar{x}_1 + b\bar{x}_2$$

with $a > 0$ and $b > 0$.

- For every $\bar{x} \in \mathbb{R}_+^2$, determine and draw the indifference set $I(\bar{x})$ and the upper contour set $U(\bar{x})$.
- Show that this preference relation is continuous, convex, strongly monotone, but not strictly convex.

C.8* (Leontief preferences). For all $x = (x_1, x_2) \in \mathbb{R}_+^2$ and $\bar{x} = (\bar{x}_1, \bar{x}_2) \in \mathbb{R}_+^2$,

$$x \succsim \bar{x} \iff \min\{x_1, x_2\} \geq \min\{\bar{x}_1, \bar{x}_2\}$$

- For every $\bar{x} \in \mathbb{R}_+^2$, determine and draw the indifference set $I(\bar{x})$ and the upper contour set $U(\bar{x})$.
- Show that this preference relation is continuous, convex, monotone, but it is not strictly convex and it is not strongly monotone.

C.9 (Ex 3.C.5a in MCWG). Prove the following result: a continuous $\succsim$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one; i.e. $u(\alpha x) = \alpha u(x)$ for all $\alpha > 0$.

C.10* Let $p = (p_1, p_2) \gg 0$ be a price system and let $w > 0$ be the wealth of the consumer. Using the definition of the demand of the consumer, determine graphically the demand of the consumer in the three following cases:

- a) Lexicographic preferences;
- b) Linear preferences.
- c) Leontief preferences.

C.11* For an utility function $u(x_1, x_2)$ representing rational preferences in $X = \mathbb{R}_+^2$ provide the graphical representations of the following solutions of an utility maximization problem (UMP) :

- a) internal solution $x(p, w)$ which is a global maximizer for $u(x_1, x_2)$ everywhere quasi-concave;
- b) corner solution $x(p, w)$ which is a global maximizer for $u(x_1, x_2)$ everywhere quasi-concave;
- c) internal solution $x(p, w)$ which is a global maximizer for $u(x_1, x_2)$ not everywhere quasiconcave;
- d) internal $x(p, w)$ which is a local but not global maximizer for $u(x_1, x_2)$.

C.12* Consider a twice continuously differentiable utility functions $u(x_1, x_2)$ representing a consumer's preference. Prove the following results:

- a) convexity of preference, that is quasi-concavity of $u(x_1, x_2)$, implies that at any bundle $\bar{x} = (\bar{x}_1, \bar{x}_2)$ the marginal rate of substitution $MRS_{12}(\bar{x}) = \frac{\partial u(\bar{x})/\partial x_1}{\partial u(\bar{x})/\partial x_2}$ is decreasing in x_1 .
- b) convexity of preference guarantees that the second order conditions for the utility maximization problem (UMP) are met.

C.13* (Cobb-Douglas utility function). For all $x = (x_1, x_2) \in \mathbb{R}_+^2$, the utility function representing Cobb-Douglas preferences takes the general form $u(x_1, x_2) = (x_1)^a (x_2)^b$ with $a > 0$ and $b > 0$

- a) For every $\bar{x} \in \mathbb{R}_+^2$, determine and draw the indifference sets $I(\bar{x})$ and the upper contour set $U(\bar{x})$ for the following three cases: i) $a = b$; ii) $a > b$; and iii) $b > a$.
 - b) Determine the following properties of $u(x_1, x_2)$: continuity, differentiability, strictly increasing, strictly quasiconcavity.
 - c) Consider the following real-valued function $f(z) = (z)^{\frac{1}{a+b}}$. Show that the utility function defined by $v(x_1, x_2) = f(u(x_1, x_2))$ represents the same Cobb-Douglas preferences for all $x = (x_1, x_2) \in \mathbb{R}_+^2$;
 - d) Consider now the real-valued function $g(z) = \ln(z)$. Show that the utility function defined by $\tilde{u}(x_1, x_2) = g(u(x_1, x_2))$ also represents the same Cobb-Douglas preferences for all $x = (x_1, x_2) \in \mathbb{R}_{++}^2$.
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- e) Let $p = (p_1, p_2) \gg 0$ be a price system and $w > 0$ be the wealth of the consumer with the above Cobb-Douglas preferences. Determine the ordinary demand of this consumer.
- f) Provide a graphical representation of the solution of the previous point in the (x_1, x_2) -space; assume in the diagram that the price of good x_1 changes. Show diagrammatically the offer curve. Assume that wealth changes and show the wealth-consumption path.

C.14* As usual, let $x(p_1, p_2, w) = (x_1(p_1, p_2, w), x_2(p_1, p_2, w))$ denote the demand of the consumer. For every commodity $l = 1, 2$, the demand of commodity l is given by

$$x_l(p_1, p_2, w) = \frac{w}{p_1 + p_2}$$

- a) Prove that this demand is homogeneous of degree zero.
- b) Prove that this demand satisfies Walras' Law.
- c) State the Weak Axiom of Revealed Preferences (WARP) in the framework of the demand.
- d) Prove that this demand satisfies WARP.

C.15* Let L be the number of commodities. Let $(-\infty, \infty) \times \mathbb{R}_+^{L-1}$ be the consumption set. The consumer has strictly convex preferences which are represented by a utility function $u(x) = x_1 + \phi(x_2, x_3, \dots, x_L)$. We assume $p \gg 0$, and we normalize $p_1 = 1$.

- a) Show that the demand for commodities $\{2, 3, \dots, L\}$ must be independent of wealth. How does demand for commodity 1 react to changes in wealth w ?
- b) Using the previous result, define the indirect utility function as usual, i.e. $v(p, w) := u(x^*)$, where x^* belongs to the demand, given p and w . Show that $v(p, w)$ is linear in wealth: $v(p, w) = w + \psi(p)$ for some function $\psi : \mathbb{R}_+^L \rightarrow \mathbb{R}$.
- c) Now let $L = 2$ and $\phi(x_2) = \alpha \ln(x_2)$. Solve the UMP as a function of (p, w) (Recall that we allow demand for commodity 1 to be negative).

C.16 (Ex 3.C.6 in MCWG). Suppose that in a two-commodity world, the consumer's utility function takes the form $u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho}$. This utility function is known as *constant elasticity of substitution* (CES) utility function.

- a) Show that when $\rho = 1$, indifference curves become linear.
 - b) Show that as $\rho \rightarrow 0$, this utility function comes to represent the same preferences as the (generalized) Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.
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- c) Show that as $\rho \rightarrow -\infty$, indifference curves become right angles; that is, this utility function has in the limit the indifference map of the perfect complement - Leontief utility function $u(x) = \min\{x_1, x_2\}$.

C.17 (Ex 2.E.1 in MCWG). Suppose $L = 3$ and consider the demand function $x(p, w)$ defined by:

$$\begin{aligned} x_1(p, w) &= \frac{p_2}{p_1 + p_2 + p_3} \frac{w}{p_1}, \\ x_2(p, w) &= \frac{p_3}{p_1 + p_2 + p_3} \frac{w}{p_2}, \\ x_3(p, w) &= \frac{\beta p_1}{p_1 + p_2 + p_3} \frac{w}{p_3}, \end{aligned}$$

Does this demand function satisfy homogeneity of degree zero and Walras' law when $\beta = 1$? What about when $\beta \in (0, 1)$?

C.18* (Ex 2.E.4 in MCWG): Show that if $x(p, w)$ is homogeneous of degree one with respect to w and satisfies Walras' law, then the elasticity of a generic commodity l to wealth is $\varepsilon_{lw}(p, w) = 1$. Interpret. Can you say something about the form of the Engel functions and curves in this case?

C.19 (Ex 2.E.7 in MCWG): A consumer in a two-good economy has a demand function $x(p, w)$ that satisfies Walras' law. His demand function for the first good is $x_1(p, w) = \alpha w/p_1$. Derive his demand function for the second good. Is his demand function homogeneous of degree 0?

C.20 (Ex 2.D.1 in MCWG). A consumer living for two periods consumes a single consumption good denoted c_1 and c_2 in period 1 and period 2, respectively. His wealth in period 1 is $w_1 > 0$ and in period 2 is $w_2 > 0$. There are perfect capital markets so that wealth can be transferred between the two periods at a constant interest rate r . Prices in the two periods are $p_1 = p_2 = 1$.

- What is the consumer's Walrasian (lifetime) budget set?
- Provide a graphical representation of the budget set.

C.21*. A consumer consumes one consumption good x and hours of leisure R . The time endowment is T while the consumer has no exogenous wealth endowment. The price of the consumption good is p and the consumer can work at a hourly wage rate of s .

- a) What is the consumer's Walrasian budget set?
- b) Assuming Cobb-Douglas preferences $u(x, R) = x^\alpha R^{1-\alpha}$, with $\alpha > 0$, compute the consumer's labour supply which is given by the time endowment minus the optimal hours of leisure R (solution of the UMP).
- c) Provide a graphical representation of the previous solution.
- d) Suppose now that the salary s increases. Compute analytically and show diagrammatically the change in the labour supply. What can you say about the substitution effect and the income effect generated by this salary change?
- e) Suppose now that, in addition to the time endowment T , the consumer has also an exogenous wealth endowment E . Repeat all steps a)-d) above.

C.22** (Ex 2.E.2 in MCWG): Show that the equations expressing the *Cournot aggregation* and the *Engel aggregation* lead to the following two elasticity formulas:

$$\sum_{l=1}^L b_l(p, w) \varepsilon_{lk}(p, w) + b_k(p, w) = 0, \quad \text{all } k = 1, \dots, L$$

and

$$\sum_{l=1}^L b_l(p, w) \varepsilon_{lw}(p, w) = 1,$$

where $b_l(p, w) = p_l x_l(p, w)/w$ is the budget share of the consumer's expenditure on good l given price p and wealth w .

C.23**. Consider a UMP in $L = 3$ with prices $p \in \mathbb{R}_{++}^3$ and wealth $w > 0$. The agent's preferences $\succeq$ over bundles (x_1, x_2, x_3) in $\mathbb{R}_+^3$ can be represented by the utility function:

$$u(x_1, x_2, x_3) = x_1 + x_2 \cdot x_3$$

- a) Given the definition of convex, strictly convex, and homothetic preferences,. Verify whether or not preferences in equation (1) are: (a.1) homothetic; (a.2) convex; (a.3) strictly convex.
- b) Let the price of commodity x_1 be $p_1 = 1$. Compute the optimal consumption bundle which maximizes the consumer's utility.
- c) Compute now the ordinary demands of the three goods. Do the ordinary demand functions satisfy: (c.i) the Walras' law? (c.ii) the so called property of *no-money illusion*?
- d) Compute the Slutsky substitution matrix and verify that is symmetric.

C.24**. In a two-commodity world and for utility functions: A) $u(x) = ax_1 + bx_2$, B) $u(x) = \text{Min}\{ax_1, bx_2\}$, C) $u(x) = x_1^a x_2^b$ (with $a > 0$ and $b > 0$ in all the three cases). Solve for the following:

- a) the compensated demand functions;
- b) the expenditure function and verify that the derivative of the expenditure function with respect to price of good l delivers the compensated demand of good l ;
- c) using duality, compute the ordinary demand function from the compensated demand functions and the indirect utility function from the expenditure function.

C.25** (Ex 3.G.6 in MWG). A consumer in a three-goods economy (goods denoted x_1 , x_2 , x_3 and prices p_1 , p_2 , p_3) with wealth $w > 0$ has demand functions for commodities 1 and 2 given by:

$$\begin{aligned}x_1(p, w) &= 100 - 5\frac{p_1}{p_3} + \beta\frac{p_2}{p_3} + \delta\frac{w}{p_3}, \\x_2(p, w) &= \alpha + \beta\frac{p_1}{p_3} + \gamma\frac{p_2}{p_3} + \delta\frac{w}{p_3},\end{aligned}$$

where Greek letters are nonzero constant.

- a) Indicate how to calculate the demand for good 3 (but do not actually do it).
 - b) Are the demand functions for x_1 and x_2 appropriately homogeneous?
 - c) Calculate the restrictions on the numerical values α , β , γ and δ implied by utility maximization.
 - d) Given your results in part (c), for a fixed level of x_3 draw the consumer's indifference curves in the (x_1, x_2) plane.
 - e) What does your answer to (d) imply about the form of the consumer's utility function $u(x_1, x_2, x_3)$?
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Exercises on ‘Production’

P.1* The basic properties of the production set Y are the following ones:

- Possibility of inaction
- Closedness
- Impossibility of free production (“no free lunch”)
- Free-disposal
- Irreversibility
- Convexity
- Increasing/decreasing/constant returns to scale.

Let $L = 2$ be the number of commodities. A firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha z$ with $\alpha > 0$ and $z \geq 0$.

- a) Determine, both formally and graphically, the production set Y that corresponds to the production function f .
- b) Determine if the production Y verifies the basic properties.

Now answer questions a) and b) for the two alternative production functions below:

- $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.
- $f(z) = \alpha z^2 + \beta z$ with $\alpha > 0$, $\beta > 0$ and $z \geq 0$.

P.2* For a general single output technology, show that the production set Y is convex if and only if the production function $q = f(z)$ is concave.

P.3* Let L be the finite number of commodities. A firm produces commodity L using the other $L - 1$ commodities as inputs. $z = (z_1, \dots, z_L, \dots, z_{L-1}) \in \mathbb{R}_+^{L-1}$ denotes a generic bundle of inputs. Show that if the production function $f : \mathbb{R}_+^{L-1} \rightarrow \mathbb{R}_+$ is concave, then the transformation function defined by

$$F(y) := y_L - f(z)$$

is quasi-convex on the convex set $A = \{y = (-z, y_L) \in \mathbb{R}_+^L : z \geq 0 \text{ and } y_L \geq 0\}$.

P.4* Let $L = 3$ be the number of commodities. The firm produces commodity 3 using commodities 1 and 2 as inputs. The production function is given by $f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta$ with $\alpha > 0$, $\beta > 0$, $z_1 \geq 0$ and $z_2 \geq 0$.

- a) Write the production set Y determined by the production function f .

- b) Determine if the production Y verifies the basic properties (same as in EX1).
- c) Discuss under which conditions of the scalars α and β the technology exhibits decreasing, increasing or constant return to scale.

P.5* (Ex 5.C.1 in MCWG). Show that, in general, if the production set Y exhibits non-decreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = +\infty$.

P.6* Let $L = 2$ be the number of commodities. The firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha z$ with $\alpha > 0$ and $z \geq 0$.

- a) Write the profit maximization problem of this firm.
- b) Consider the production set Y determined by the production function f . Using the shape of Y and the iso-profit lines, determine graphically the supply of this firm.
- c) Determine the profit function of this firm.

P.7* Let L be the finite number of commodities. Assume that the production set Y of the firm is represented by a transformation function F such that $Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$.

- a) State the profit maximization problem (PMP) of the firm.
- b) Let F be continuous and strictly quasi-convex. Show that if PMP has a solution for $p \gg 0$, then it must be unique.

P.8*. Let $L = 2$ be the number of commodities. The firm produces commodity 2 using commodity z as an input. The production function is given by $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.

- a) Write the transformation function and the profit maximization problem (PMP) of this firm.
- b) Show that if $\bar{y} = (\bar{y}_1, \bar{y}_2)$ belongs to the supply of the firm, then $\bar{y}_1 < 0$ and $\bar{y}_2 > 0$.
- c) Consider the open and convex set $A = \{y = (-z, y_2) \in \mathbb{R}^2 : z > 0 \text{ and } y_2 > 0\}$. Write the first order conditions associated with (PMP) on the set A .
- d) Compute the supply and the profit function of this firm.

P.9* (Ex 5.C.9 in MCWG). Derive the profit function $\pi(p)$ and supply function (or correspondence) $y(p)$ for the single-output technologies with production functions given by: **(a)** $f(z) = \sqrt{z_1 + z_2}$; **(b)** $f(z) = \sqrt{\min\{z_1, z_2\}}$; **(c)** $f(z) = (z_1^\rho + z_2^\rho)^{1/\rho}$.

P.10* Let L be the number of commodities. A firm uses a single-output technology to produce output q using $(L - 1)$ commodities as inputs. Let $z := (z_1, \dots, z_l, \dots, z_{L-1}) \in \mathbb{R}^{L-1}$ denote a generic bundle of inputs.

- b) Define the cost function $c(w, q)$ of the firm.
- b) Show that the cost function $c(w, q)$ is a concave function of the input price vector $w := (w_1, \dots, w_l, \dots, w_{L-1})$.
- c) Show that if the production function $f : \mathbb{R}_+^{L-1} \rightarrow \mathbb{R}_+$ is concave, then the cost function $c(w, q)$ is a convex function of the output level q .

P.11 A firm has the single-output technology, $f(z) = z_1^a + z_2$, with $0 < a < 1$, and where p is the price of output and w_1 and w_2 are the prices of inputs.

- a) Determine the conditional input demands, the profit and the supply function of the firm;
- b) How does your answer to point a) change when $a > 1$?

P.12* Let $L = 2$ be the number of commodities. The firm produces commodity 2 using commodity 1 as an input. The production function is $f(z) = \alpha(1 - \exp(-kz))$ with $k > 0$, $\alpha > 0$ and $z \geq 0$.

- a) Write and draw the production set Y determined by the production function f .
- b) For every level of output $y_2 \geq 0$, determine and draw the following set $Y(y_2) := \{z \in \mathbb{R}^2 : z \geq 0 \text{ and } f(z) \geq y_2\}$.
- c) Write the cost minimization problem of this firm.
- d) Determine the conditional demand of inputs and the cost function of the firm.

Now answer questions a) to d) for the two alternative production functions below:

- $f(z) = \alpha\sqrt{z}$ with $\alpha > 0$ and $z \geq 0$.
- $f(z) = \alpha z^2 + \beta z$ with $\alpha > 0$, $\beta > 0$ and $z \geq 0$.

P.13* A firm uses a single-output technology to produce output q using two commodities as inputs with prices $w = (w_1, w_2)$. The cost function is given by $c(w, q) = 2(q)^2(w_1)^{\frac{2}{3}}(w_2)^{\frac{1}{3}}$.

- a) Show that the cost function is homogeneous of degree one in the inputs prices w .
- b) Verify that this cost function is a convex function of the output level q .
- c) Compute the supply and the profit function of the firm.

P.14* Let $L = 3$ be the number of commodities. The firm produces commodity q using commodities 1 and 2 as inputs. The production function is given by

$$f(z_1, z_2) = (z_1)^\alpha (z_2)^\beta \text{ with } \alpha > 0, \beta > 0, z_1 \geq 0 \text{ and } z_2 \geq 0$$

- a) Determine the conditional input demands.

- b) Determine the cost function of the firm and verify that the derivatives of the cost function with respect to price of input $l = 1, 2$ deliver the conditional input demands.
- c) Discuss the conditions under which the technology admits a strictly positive, unique and finite level of output maximizing the firm's profit. For the case in which it is admitted, determine the supply and the profit function of the firm.

P.15* (Ex 5.D.1 in MCWG). Let $AC(\cdot)$ and $C'(\cdot)$ denote a firm's average cost function and marginal cost function, respectively. Show that $AC(\bar{q}) = C'(\bar{q})$ at any $\bar{q}$ satisfying $AC(\bar{q}) \leq AC(q)$ for all q . Does this result depend on differentiability of $C(\cdot)$ everywhere?

P.16 For the single-output technology, with two inputs, $q = (\text{Min}\{z_1; 2z_2\})^\lambda$, with $\lambda > 0$:

- a) Discuss under which conditions of λ the technology exhibits decreasing, increasing or constant return to scale.
 - b) Compute the cost function.
 - c) Derive the profit maximizing output for the firm for the different values of λ .
 - d) Suppose now that factor z_2 is fixed in the short run. Derive the cost function in the short run.
 - e) Provide a diagrammatic representation of the cost curves in the short run and use the diagram to discuss the profit maximizing output of the firm in the short run, taking care to consider the conditions under which the firm decides to operate in the market.
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Micro IB - Exercises with (partial) solutions¹

QEM Master Program - Ca' Foscari University of Venice

Lecturer: Pietro Dindo

Version: 200821

Note: I expect all students to be confident with the solution of unmarked questions and to be able to solve the questions marked with * with a bit more effort. Questions marked with ** are more involved. I encourage to answer these ** questions after you have covered all the others.

1 Edgeworth Box

Exercise 1

Consider an economy with two consumption goods, $L = \{1, 2\}$, and two consumers $I = \{a, b\}$. Consumer a has endowment $\omega_a = (3, 0)$ while consumer b has endowment $\omega_b = (1, 3)$. Both consumers have utility $U(x) = (x_1)^2(x_2)^3$.

1. Find the set of Pareto optimal allocations.
2. Determine the subset of Pareto optimal allocations where both consumers are at least as well-off as with their endowment (the contract curve).
3. Find the competitive equilibria. Are they Pareto optimal?

Exercise 2

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. Consumer 1 has utility $U_1(x_1) = x_{11} + \sqrt{x_{21}}$ and endowment $\omega_1 = (2, 0)$. Consumer 2 values only good 2, $U_2(x_2) = x_{22}$, and has endowment $\omega_2 = (0, 2)$.

- 1.* Find the set of Pareto optimal allocations (a graphical solution is enough).
- 2.* Find the competitive equilibria (a graphical solution is enough).
- 3.** Repeat the same analysis (both Pareto set and competitive equilibria) with the endowments $\omega_1 = (2, 1)$ and $\omega_1 = (0, 1)$.

¹Exercises have been gathered from various sources: exercise sessions of the QEM Micro1B course, Ca' Foscari University of Venice; the textbook "Microeconomic Theory" by Mas-Colell, Whinston, and Green [MWG]; past Microeconomics I exams of the QEM Joint Degree Program; Class Notes written by Jean-Marc Bonnisseau and Elena del Mercato for the QEM1 Microeconomics I course, Université Paris 1.

Note: This exercise is inspired by Figure 15.B.10(a) of the MWG textbook, you are encouraged to have a look at it.

Exercise 3

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. Consumer 1 has utility $U_1(x_1) = \sqrt{x_{11}} + x_{21}$. Consumer 2 values only good 1, $U_2(x_2) = x_{12}$. The total endowment is $\bar{\omega}$.

- 1.* Find the set of Pareto optimal allocations (a graphical solution is enough).
- 2.* Find the competitive equilibria when $\omega_{21} = 0$ (a graphical solution is enough).
- 3.* Find the competitive equilibria when $\omega_{12} = 0$ (a graphical solution is enough).

Note: This exercise is inspired by Figure 15.B.10(a) of the MWG textbook, you are encouraged to have a look at it.

Exercise 4

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. The endowments are ω_1 and ω_2 . Assume that both consumers have locally non satiated preferences. Prove that if the market for good l clears and $p \gg 0$, then also the market for good $l' \neq l$ clears.

Exercise 5

Consider an economy with two consumers, $I = \{1, 2\}$, and two commodities, $L = \{1, 2\}$. The total endowment is $\bar{\omega} = (4, 3)$. Find the set of Pareto optimal allocations in the following cases:

1. $U_1(x_1) = \sqrt{x_{11}}\sqrt{x_{21}}$ and $U_2(x_2) = (x_{12})^{\frac{1}{3}}(x_{22})^{\frac{2}{3}}$;
2. $U_1(x_1) = x_{11} + x_{21}$ and $U_2(x_2) = 2x_{12} + x_{22}$ (a graphical solution is enough);
- 3.* $U_1(x_1) = \sqrt{x_{11}}\sqrt{x_{21}}$ and $U_2(x_2) = x_{12} + 2x_{22}$.

Exercise 6

Consider an economy with two consumption goods, $L = \{1, 2\}$, and two consumers $I = \{1, 2\}$. Consumer 1 has endowment $\omega_1 = (1, 2)$ while consumer 2 has endowment $\omega_2 = (2, 1)$. Consumer 1 has utility $U_1(x_1) = x_{11}(x_{21})^2$ while 2 has utility $U_2(x_2) = x_{12}x_{22}$.

1. Find the set of Pareto optimal allocations.
2. Determine the subset of Pareto optimal allocations where both consumers are at least as well-off as with their endowment (the contract curve).

- Find whether the allocation on the Pareto set with $x_{11} = 2$ can be supported as a competitive equilibrium with transfers. If so, for which transfer using only good 1 ? Is there a transfer of only good 2 that achieves the same allocation?

Exercise 7 (*JD-QEM '20-'21*)

We consider an exchange economy with two consumers and two goods.

- Consumer 1 has consumption set $\mathbb{R}_+^2$, initial endowment $e_1 = (1, 1)$ and utility

$$u_1(x_{11}, x_{12}) = x_{11}^{1/3} (x_{12})^{2/3}$$

- Consumer 2 has consumption set $\mathbb{R}_+^2$, initial endowment $e_2 = (1, 3)$ and utility

$$u_2(x_{21}, x_{22}) = x_{21}^{1/2} x_{22}^{1/2}$$

- Represent in the Edgeworth box the initial endowment and the indifference curves going through the initial endowment for both agents.
- Represent in the Edgeworth box the set B of allocations that are better, in the sense of Pareto, than the initial endowments.
- Give the definition of a Pareto Optimum for this economy.
- Determine the set C of Pareto Optima of the economy.
- Represent (approximately) the set C in the Edgeworth box. Do B and C coincide ? Explain why/not ?
- Give the definition of a general equilibrium for this economy.
- Determine the general equilibrium of this economy (normalize to 1 the price of commodity 2).

Exercise 8

Solve Example 15.B.2 from the MWG textbook.

Exercise 9

Solve Exercise 15.B.2 from the MWG textbook.

Exercise 10

Solve Exercise 15.B.9 from the MWG textbook.

2 Robinson Crusoe Economy

Exercise 11

Solve Exercise 15.C.2 from the MWG textbook.

Exercise 12

Consider an economy with two periods, $t = 0, 1$, and a consumption good per period. The consumer has utility $U(x_0, x_1) = u(x_0) + \beta u(x_1)$ with $\beta \in (0, 1)$ and endowment k_0 of consumption good in $t = 0$. The economy has one firm, $J = 1$, which transforms the good in $t = 0$ (input, k_1), into an output (y) in $t = 1$. The firm production function is $f(k) = Ak^\alpha$ with $A > 0$ and $\alpha \in (0, 1]$. Assume that $u(x) = \log x$ and $\alpha \in (0, 1)$.

1. Find the competitive equilibrium.
2. Find the firm profit in the competitive equilibrium and discuss its dependence on α .
3. Find the Pareto optimal allocation of this economy and compare it with the allocation found in 1.

Exercise 13

Solve Exercise 12 when

- 1.* $u(x) = x$ and $\alpha \in (0, 1)$;
2. $u(x) = \log x$ and $\alpha = 1$;
- 3.** $u(x) = x$ and $\alpha = 1$.

Exercise 14

Consider an economy with two consumption goods, $L = 2$, one consumer, $I = 1$, and one firm, $J = 1$. The consumer has endowment $\omega = (4, 1)$ and utility $U_1(x) = x_1 x_2$. The firm has production set $Y_1 = \{y_2 \leq -ay_1, y_1 \leq 0\}$ with $a > 0$. The consumer is the owner of the firm.

1. Write down the definition of a competitive equilibrium for this economy.
2. Fix $a = 1$, find the set of competitive equilibria and the corresponding firm profits.
3. Are the the competitive equilibria found above Pareto optimal?
4. Find the Pareto optimal allocations for $a = 1$.
- 5.* Answer to 2. and 3. for each given $a > 0$.
- 6.* Find the Pareto optimal allocations for each given $a > 0$.

3 Competitive equilibrium and welfare theorems

Exercise 15

Consider an economy with L commodities, I consumers, and J firms. The endowments and ownership shares are $(\omega_i, (\theta_{ji})_{j=1}^J)_{i=1}^I$. Assume that all consumers have locally non satiated preferences. Prove that, for all choices of a good l' , if the markets for all goods $l \neq l'$ clears and $p \gg 0$, then also the market for good l' clears.

Exercise 16

Consider an exchange economy with two consumption goods, $L = 2$, and two consumers, $I = 2$. Consumer 1 has endowment $\omega_1 = (2, 0)$ and utility $U_1(x_1) = (x_{11})^\alpha + (x_{21})^\alpha$. Consumer 2, has endowment $\omega_2 = (1, 3)$ and utility $U_2(x_2) = x_{12}x_{22}$. Consider $\alpha = \frac{1}{2}$ first.

1. Find the competitive equilibria and the set of Pareto optimal allocations.
2. Discuss the validity of both Welfare Theorems.
3. The social planner has the possibility to implement some transfer between the initial endowments of commodity 2. Is there a transfer such that, in the competitive equilibrium achieved after trading, both agents consume the same amount of good 1. If so, find it. Can the same competitive equilibrium arise after a transfer of good 1? If so, find it.
4. Repeat the exercise taking $\alpha = 1$.

Exercise 17

Consider an economy with two consumption goods, $L = 2$, two consumers, $I = 2$, and one firm, $J = 1$. Consumer 1 has endowment $\omega_1 = (4, 1)$ and utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2 has endowment $\omega_2 = (1, 1)$ and utility $U(x_2) = x_{12} + x_{22}$. The firm has production set $Y = \{y_2 \leq \sqrt{-y_1}, y_1 \leq 0\}$. Consumer two is the owner of the firm.

1. Write down the definition of a competitive equilibrium in this economy.
2. Find the set of competitive equilibria and the corresponding firm profits.
3. Are the competitive equilibria allocation found above Pareto optimal?
- 4.** Find the set of Pareto optimal allocation of this economy.

Exercise 18

Consider three commodities, time for leisure/labor (commodity 1), durable good (commodity 2) and consumption good (commodity 3). There is a single consumer with utility

function $U(x_2, x_3) = x_2 + x_3$, so that the consumption of leisure does not have any impact on his utility. The consumer has an endowment of leisure equal to 1 and no endowment of durable and consumption good.

There are two firms. Firm 1 produces only durable goods using labor as input. Firm 1 production set is $Y_1 = \{-\sqrt{-y_{11}} + y_{21} \leq 0, y_{11} \leq 0\}$. Firm 2 produces only consumption good using labor as input. Firm 2 production set is $Y_2 = \{-2\sqrt{-y_{12}} + y_{32} \leq 0, y_{12} \leq 0\}$. The consumer is the owner of both firms.

1. Write down the definition of a competitive equilibrium in this economy.
2. Check that

$$\left\{ \frac{p_1}{p_2} = \frac{\sqrt{5}}{2}, \frac{p_3}{p_2} = 1, x = \left(0, \frac{1}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right), y_1 = \left(-\frac{1}{5}, \frac{1}{\sqrt{5}}\right), y_2 = \left(-\frac{4}{5}, \frac{4}{\sqrt{5}}\right) \right\}$$

is the only competitive equilibrium of this economy.

3. Is the competitive equilibria allocation found above Pareto optimal?
- 4.* Find the set of Pareto optimal allocation of this economy.

Exercise 19

Find the competitive equilibrium and the Pareto set in an economy as above with $U(x_2, x_3) = x_2 x_3$ and $Y_2 = \{y_{12} + y_{32} \leq 0, y_{12} \leq 0\}$ (firm 1 keeps the same technology).

Exercise 20

Consider an economy with two consumption goods, $L = 2$, two consumers, $I = 2$, and one firm, $J = 1$. Consumer 1, the worker, has endowment $\omega_1 = (1, 0)$ and utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2, the capitalist, is the owner of the firm, $\theta_{21} = 1$, and has utility $U_2(x_1) = x_{12}x_{22}$. The firm has production set $Y = \{y_1 \leq 2\sqrt{-y_2} \leq 0, y_1 < 0\}$.

1. Provide the definition of a competitive equilibrium for this economy.
2. Find the set of competitive equilibria and the set of Pareto optimal allocations.
3. Discuss the validity of both Welfare Theorems.
4. Find the competitive equilibrium with transfer such that both consumers consume the same amount of output. How can you implement such a transfer starting from the given endowment/ownership?

Exercise 21

Consider an exchange economy with two consumption goods, $L = 2$, and two consumers, $I = 2$. Consumer 1 has endowment $\omega_1 = (2, 2 - \delta)$ with $\delta \in (0, 2)$ and utility $U(x_1) = \epsilon x_{11} + x_{21}$ for $\epsilon > 0$. Consumer 2 has endowment $\omega_2 = (2, \delta)$ and utility $U(x_2) = x_{12} + x_{22}$.

1. Consider $\epsilon = \delta = 0$. Find the set of competitive equilibria and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.
2. Consider $\epsilon = \delta = 0$. State whether $p = (0, 1)$, $p = (1, 1)$, $p = (1, 2)$, $p(2, 1)$ support $x_1^* = (0, 2)$, $x_2^* = (4, 0)$ as a quasi-equilibrium transfer. If so, is each $\{p, x^*\}$ also a competitive equilibrium?
3. Consider $\epsilon > 0$ and $\delta = 0$. Find and plot the excess demand $z_1(p) = x_{11}(p) + x_{12}(p) - \bar{\omega}_1$ of good 1, the set of competitive equilibria, and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.
4. Consider $\epsilon = 0$ and $\delta \in (0, 2)$. Find and plot the excess demand function of good 1, the set of competitive equilibria, and the set of Pareto optimal allocations. Discuss the validity of both Welfare Theorems in this economy.

Exercise 22

Consider an economy with two consumption goods, $L = 2$, one consumer, $I = 1$, and one firm, $J = 1$. Consumer 1 has endowment $\omega_1 = (2, 1)$ and utility $U(x_1) = x_{11}x_{21}$. The firm has production set $Y = \{y_1 + y_2 \leq 0 \text{ if } y_1 \leq -a, y_2 \leq 0 \text{ if } y_1 \in (-a, 0]\}$ with $a \geq 0$. Consumer one is the owner of the firm. Fix $a = 1$.

1. Provide the definition of a competitive equilibrium for this economy.
2. Find the set of competitive equilibria.
3. Find the set of Pareto optimal allocations.
4. Discuss the validity of both Welfare Theorems.

Exercise 23

Solve Ex. 23 for each given $a \in [0, 1]$.

Exercise 24

Consider an exchange economy with $I = \{1, 2\}$, $L = \{1, 2\}$, $U_1(x_1) = x_{21}$, $U_2(x_2) = x_{12} + x_{22}$, and $\omega_1 = (1, 0)$, $\omega_2 = (0, 1)$.

- 1.* Find the utility possibility set
- 2.* Find the Pareto frontier
- 3.** Show that for some levels of $\bar{U}_2$ the solutions x^* of

$$\begin{aligned} &\text{Max } U_1(x_1) \\ &x \in \mathbf{R}_+^4 \\ &x_1 + x_2 \leq \bar{\omega} \\ &U_2(x_2) \geq \bar{U}_2 \end{aligned}$$

are not Pareto optimal (and thus $(U_1(x_1^*), U_2(x_2^*))$ does not belong to the Pareto frontier).

Exercise 25 (*Economics-QEM exam '20-'21 I*)

Consider an economy with two commodities, two consumers, and one firm. Consumer 1 has endowment $\omega_1 = (6, 0)$, owns half of the firm, and has utility $U_1(x_1) = x_{11}x_{21}$. Consumer 2 has endowment $\omega_2 = (4, 2)$, owns the other half of the firm, and has utility $U_2(x_2) = x_{12}x_{22}$. The firm uses commodity 1 to produce commodity 2 and has production function $f(z) = z$ with $z \geq 0$ (the amount of commodity 1).

1. Derive supply and profit of the firm.
2. Derive the demand of both consumers.
3. Provide the definition of a competitive equilibrium for this economy.
4. Compute the unique competitive equilibrium of this economy.
5. Is the competitive equilibrium found above Pareto optimal? Reply without computing the Pareto set.
6. Provide a definition of Pareto optimal allocation.
7. Explain how to find all the Pareto optimal allocation of this economy (write down the related maximization problem and the first order conditions for an interior solution).

4 Expected Utility

Exercise 26

Solve Exercise 6.B.2 in the MWG textbook.

Exercise 27

Show that a preference relation on lotteries that can be represented in an Expected Utility form satisfies transitivity.

Exercise 28

Consider the space of simple lotteries $L = \{C; (p_1, p_2, p_3)\}$. Show graphically that if a preference relation on lotteries satisfies the independence axiom, then indifference curves are parallel. (hint: show that if two indifference curves are not parallel, then the independence axiom is violated)

Exercise 29

Solve Exercise 6.B.4 in the MWG textbook.

Exercise 30

Solve Exercise 6.B.7 in the MWG textbook. (Note: a preference over lotteries is monotone if, given two real outcomes $C_1 > C_2$, then the lottery with sure outcome C_1 is strictly preferred to the lottery with sure outcome C_2)

Exercise 31

When faced with the choice between the lottery $A = \{(3500, 2800, 0); p = (0.3, 0.66, 0.04)\}$ and $B = \{3500; p = 1\}$, a decision maker chooses lottery B . When, instead, he is asked to choose between the lottery $A' = \{(3500, 2800, 0); p = (0.3, 0, 0.7)\}$ and lottery $B' = \{(3500, 2800, 0); p = (0, 0.34, 0.66)\}$, he chooses A' . Say whether the decision maker's preferences are consistent with the expected utility form, explaining how you have reached your conclusion. What if lottery B becomes $B'' = \{2800; 1\}$?

Exercise 32 (*JD-QEM exam '19-'20*)

We consider an expected-utility decision-maker facing the following possible professional occupations: working in the financial industry (A), working in the movie industry (B), working in the car industry (C). We further assume that the decision-maker can apply to three different schools.

- After School 1, he is certain to find a job in the car industry.
- After School 2, he is certain to find a job in the financial industry.

- After School 3, he would find a job in the movie industry with probability 0.1 and in the financial industry with probability 0.9.

We assume that School 2 is the least preferred option of the decision-maker and that he is indifferent between School 1 and School 3.

1. Give a representation of the utility function of this decision-maker.
2. We assume that a new school (School 4) opens. After school 4, a student would find a job in each industry with probability $\frac{1}{3}$. How would the new school rank compared to Schools 1, 2, 3?
3. There are too many application in School 4. The ministry of education decides that rather than applying to School 4 directly, the decision-maker must apply to a lottery that leads to admission in School 4 with probability $\alpha \in [0, 1]$ and to admission in School 2 with probability $1 - \alpha$. For which value of α does the decision-maker prefer to apply to School 1 rather than to the lottery that may lead to admission to School 4?

Exercise 33 (*JD-QEM exam '20-'21*)

We consider an expected-utility decision-maker with utility of the form $u(x) = x^a$ with $a < 1$.

1. Let X be a lottery whose outcome is uniformly distributed over $[0, 1]$. Determine $\mathbb{E}(u(X))$.
2. Let Y be a lottery whose outcome is 0 with probability $\frac{1}{3}$ and 1 with probability $\frac{2}{3}$. Determine $\mathbb{E}(u(Y))$.
3. Determine the value a^* (of a) for which the decision-maker is indifferent between X and Y .
4. If $a > a^*$, which lottery, X or Y , is preferred by the decision-maker?
5. Determine, as a function of a , the coefficient of absolute risk-aversion of the decision-maker.

Exercise 34

Solve Exercise 6.C.1 in MWG assuming $u(x) = \log(x)$ and find* for which value of q the agent does not insure.

Exercise 35

Consider an investor who has to choose between two assets. Asset A has payoff $x_A = (12, 6, 9)$, all with equal probability. Asset B is risk free with payoff $x_B = (\bar{x}, \bar{x}, \bar{x})$.

1. Assume $u(x) = \log(x)$. Find the certainty equivalent of x_B .
2. Find the level of $\bar{x}$ such that the investor is indifferent between asset A and asset B .
3. Assume now that $\bar{x} = E[x_A]$, find how many units h_A of asset A should be given to the agent to make him indifferent between h_A and one unit of the risk free asset. Provide an intuition.
4. Compute the coefficient and absolute risk aversion.

Exercise 36 (*Economics-QEM exam '20'21 I*)

A decision maker has preferences over lotteries represented by an expected utility with Bernoulli utility $u(x)$. Consider the lottery $L = \{(0, 4, 9); (1/6, 1/2, 1/3)\}$

1. Provide the definition of certainty equivalent of L given u .
2. Compute the certainty equivalent for L given $u(x) = \sqrt{x}$.
3. Compute the coefficient of absolute risk aversion given u and x .
4. Assume that the agent starts with $w = 12$ and owns the lottery. Find the minimum price he is willing to accept to sell the lottery.
5. Compare the minimum price found above and the certainty equivalent and provide an intuition of their difference based on the coefficient of absolute risk aversion.

Exercise 37

Solve Exercise 6.C.18 in MWG.

Exercise 38

Solve Exercise 6.C.16 in MWG assuming from the beginning the parametrization of point (d) with $p = 0.5$. Other than the proposed $u(x) = \sqrt{x}$, consider also $u(x) = x$ and $u(x) = \log x$. Give an interpretation in terms of the certainty equivalent and the change of the risk aversion coefficient with wealth.