Optimization B – QEM1 and IMMAEF Slides 1 - One week and half: October 20, October 23 and November 3, 2025

Elena del Mercato – Université Paris 1 Panthéon-Sorbonne, CES & PSE

Outline

- Optimization problems : Basic notions
- Existence of a solution
- Uniqueness of the solution
- Unconstrained optimization
- Equality constrained optimization
- About the Lagrange multipliers

Textbooks

Sydsaeter K., Hammmond P., Seierstadt A., Strom A. (2005) : Further Mathematics for Economic Analysis, Prentice Hall.

Sundaram R.K. (1999) : A First Course in Optimization Theory, Cambridge University Press, Cambridge.

Basic notions

D is a subset of $\mathbb{R}^n$ with $n \in \mathbb{N}$. Let f be a function from D to $\mathbb{R}$:

$$f : x \in D \subseteq \mathbb{R}^n \to f(x) \in \mathbb{R}.$$

Let S be a subset of the domain D. The set S is the set of **feasible points** (or **admissible points**), and sometimes it can be described by a finite number of constraints.

An optimization problem consists in finding the maximum (respectively, the minimum) of f on the set S. It is denoted by :

$$(\mathcal{P}) \max_{x \in S} f(x) \qquad \textit{resp. } (\mathcal{Q}) \min_{x \in S} f(x)$$

In both cases, f is called the **objective function.**

Solutions and value

Definition

- The point $\overline{x}$ is a **solution** of problem $(\mathcal{P})$ if $\overline{x} \in S$ and for all x in S :

$$f(x) \leq f(\overline{x}).$$

- The point $\overline{x}$ is a **solution** of problem $(\mathcal{Q})$ if $\overline{x} \in S$ and for all x in S :

$$f(\overline{x}) \leq f(x).$$

$\mathcal{S}ol(\mathcal{P})$ denotes the set of solutions of problem $(\mathcal{P})$, and $\mathcal{S}ol(\mathcal{Q})$ denotes the set of solutions of problem $(\mathcal{Q})$.

Definition

- The value of problem $(\mathcal{P})$ is the supremum of the set $\{f(x) \mid x \in S\}$.
- The value of problem $(\mathcal{Q})$ is the infimum of the set $\{f(x) \mid x \in S\}$.

If $\overline{x}$ is a solution of problem $(\mathcal{P})$, then $f(\overline{x}) = \max\{f(x) \mid x \in S\}$ and $f(\overline{x})$ is called **the maximum value** of f on S.

If $\overline{x}$ is a solution of problem $(\mathcal{Q})$, then $f(\overline{x}) = \min\{f(x) \mid x \in S\}$ and $f(\overline{x})$ is called **the minimum value** of f on S.

Remark

If problem $(\mathcal{P})$ (respectively, problem $(\mathcal{Q})$) has ***several solutions****, for instance, $\overline{x} \in S$ and $\widetilde{x} \in S$ with $\overline{x} \neq \widetilde{x}$, then it must be that :*

$$f(\overline{x}) = f(\widetilde{x}).$$

That is, ***the maximum value*** *(respectively,* ***the maximum value****) of f on S is* ***unique****. This is a simple consequence of the definition of solution.*

Local solutions

Let $d : D \times D \to \mathbb{R}_+$ be a distance on $D \subseteq \mathbb{R}^n$, for instance, the **Euclidean distance.** The **open ball** in D of center $\bar{x}$ and radius $r > 0$ is :

$$B(\bar{x}, r) = \{x \in D \mid d(x, \bar{x}) < r\}.$$

Definition

• The point $\overline{x}$ is a **local solution** of problem $(\mathcal{P})$ if $\overline{x} \in S$ and there exists $r > 0$ such that for all x in S such that $d(x, \overline{x}) < r$, we have that $f(x) \leq f(\overline{x})$.

• The point $\overline{x}$ is a **local solution** of problem $(\mathcal{Q})$ if $\overline{x} \in S$ and there exists $r > 0$ such that for all x in S such that $d(x, \overline{x}) < r$, we have that $f(\overline{x}) \leq f(x)$.

Consumer behavior.

The utility function u represents the preferences of the consumer on consumption bundles $x = (x_1, ..., x_L) \in \mathbb{R}^L_+$.

Let $p = (p_1, ..., p_L)$ be a price system and let $w \in \mathbb{R}_+$ be the wealth of the consumer. The demand of the consumer is the set of solutions of the following maximization problem :

$$\begin{array}{l} \max u(x_1, ..., x_L) \\ (x_1, ..., x_L) \in S \end{array}$$

where the set of **feasible points** is :

$$S = \left\{ (x_1, ..., x_L) \in \mathbb{R}^L : \left\{ \begin{array}{l} x_\ell \geq 0, \forall \ell = 1, ..., L \\ p_1 x_1 + ... + p_L x_L \leq w \end{array} \right. \right\}$$

Cost minimization.

We consider a firm that produces the good L, by using the goods $1, ..., L-1$ as inputs. Its production function is $f : (z_1, ..., z_{L-1}) \in \mathbb{R}_+^{L-1} \to f(z_1, ..., z_{L-1}) \in \mathbb{R}_+$.

Let $p = (p_1, ..., p_{L-1})$ be the price system of the inputs and let $q \geq 0$ be a level of **output**. The cost minimization problem is :

$$\begin{array}{l} \min p_1 z_1 + \ldots + p_{L-1} z_{L-1} \\ (z_1, ..., z_{L-1}) \in S \end{array}$$

where the set of **feasible points** is :

$$S = \left\{ (z_1, ..., z_{L-1}) \in \mathbb{R}^{L-1} : \left\{ \begin{array}{l} z_\ell \geq 0, \forall \ell = 1, ..., L-1 \\ f(z_1, ..., z_{L-1}) \geq q \end{array} \right. \right\}$$

The demand of inputs of the firm is the set of solutions of the cost minimization problem. The cost function $c(p, q)$ of the firm is the value of the cost minimization problem.

Game theory.

The best response of a player is the set of solutions of the maximization of his payoff function in his own strategies, by taking as given the strategies of the other players.

Consider a game with two players. For each player $i = 1, 2$, the set S_i is the set of strategies of player i and

$$u_i : (s_1, s_2) \in S_1 \times S_2 \to u_i(s_1, s_2) \in \mathbb{R}$$

is the payoff function of player i.

The best response of player i for a given strategy $\bar{s}_j \in S_j$ of player j, with $j \neq i$, is the set of solutions of the following maximization problem :

$$\begin{array}{l} \max u_i(s_i, \bar{s}_j) \\ s_i \in S_i \end{array}$$

Extreme Value Theorem (or Weierstrass Theorem)

Let f be a function from $D \subseteq \mathbb{R}^n$ to $\mathbb{R}$. The following theorem provides sufficient conditions for the **existence** of a solution.

Theorem

Problem $(\mathcal{P})$ (respectively, problem $(\mathcal{Q})$) has ***at least*** *a solution if the set of feasible points S is a* ***non-empty closed and bounded*** *subset of $\mathbb{R}^n$, and f is* ***continuous*** *on S.*

• The set S is **closed** in $\mathbb{R}^n$ if and only if it coincides with its **closure** $\mathrm{cl} S$, where :

$$\mathrm{cl} S = \{x \in \mathbb{R}^n \mid \exists\ (x_k)_{k \in \mathbb{N}} \subseteq S \text{ such that } \lim_{k \to \infty} x_k = x\}.$$

• The set S is **bounded** in $\mathbb{R}^n$ if and only if it is included in some closed ball of $\mathbb{R}^n$.

Uniqueness of the solution

As we have already remarked, an optimization problem might have several solutions.

The following proposition provides sufficient condition for the **uniqueness** of the solution.

Proposition

*Assume that the set of feasible points S is **convex** and problem $(\mathcal{P})$ (respectively, problem $(\mathcal{Q})$) has **at least** a solution $\bar{x}$.*

*• if f is **strictly quasi-concave** on S, then $\bar{x}$ is the **unique** solution of problem $(\mathcal{P})$.*

*• if f is **strictly quasi-convex** on S, then $\bar{x}$ is the **unique** solution of problem $(\mathcal{Q})$.*

Open sets and interior

Let U be a subset of $\mathbb{R}^n$. The set U is **open** in $\mathbb{R}^n$ if and only if for all $\bar{x} \in U$, there exists an **open ball** $B(\bar{x}, r)$ in $\mathbb{R}^n$ such that :

$$B(\bar{x}, r) \subseteq U.$$

Proposition

a) A finite intersection of open sets is open.

b) A union of finitely many or infinitely many open sets is open.

Let A be a subset of $\mathbb{R}^n$. The **interior** of A is the **largest open** set in $\mathbb{R}^n$ that is included in A. The interior of A is denoted by $\mathrm{int}A$. In other words, $\mathrm{int}A$ is the **union** of all the open sets in $\mathbb{R}^n$ included in A.

Hence, A is **open** in $\mathbb{R}^n$ if and only if $A = \mathrm{int}A$.

First order necessary conditions

Let U be an **open** subset of $\mathbb{R}^n$. Let f be function that has all the partial derivatives in every point of U. A **stationary point** of f is a point where all the first derivatives are equal to 0.

Consider now the two following problems, where we maximize (respectively, minimize) the function f on the **open** set U.

$$(\mathcal{P}) \max_{x \in U} f(x) \qquad \textit{resp.}\ (\mathcal{Q}) \min_{x \in U} f(x)$$

Theorem

*If $\bar{x} \in U$ is a **solution** of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$), then $\nabla f(\bar{x}) = 0$, i.e.,*

$$\frac{\partial f}{\partial x_i}(\bar{x}) = 0, \ \textit{for all } i = 1, \ldots, n.$$

*That is, $\bar{x}$ is a **stationary point** of f.*

Sketch of the proof

We prove the theorem above for a solution $\bar{x} \in U$ of problem $(\mathcal{P})$. We assume that f is differentiable. Then,

1. the **directional derivative** of f at $\bar{x}$ exists for every direction $h \in \mathbb{R}^n$, $||h|| = 1$, i.e.,

$$D_h f(\bar{x}) = \lim_{t \to 0^+} \frac{f(\bar{x} + th) - f(\bar{x})}{t} \text{ exists and it is finite,}$$

2. $D_h f(\bar{x}) = \nabla f(\bar{x}) \cdot h$.

Claim 1. $\nabla f(\bar{x}) \cdot h \leq 0$ for all $h \in \mathbb{R}^n$, $h \neq 0$.

Otherwise, assume that there is a direction $\bar{h} \neq 0$ such that :

$$\nabla f(\bar{x}) \cdot \bar{h} > 0.$$

Then, using the definition of directional derivative, for $t > 0$ sufficiently small, the points $(\bar{x} + t\bar{h})$ belong to some open ball of center $\bar{x}$ and

$$f(\bar{x} + t\bar{h}) > f(\bar{x}).$$

But, the inequality above contradicts the fact that $\bar{x}$ solves problem $(\mathcal{P})$. Hence, Claim 1 is completely proved.

Claim 2. $\nabla f(\bar{x}) \cdot h = 0$ for all $h \in \mathbb{R}^n$.

Pick any $h \in \mathbb{R}^n$, $h \neq 0$, and consider the opposite direction $-h \in \mathbb{R}^n$.

By Claim 1, we get $\nabla f(\bar{x}) \cdot h \leq 0$ and $\nabla f(\bar{x}) \cdot (-h) \leq 0$.

Hence, we have that $\nabla f(\bar{x}) \cdot h = 0$.

We then conclude that $\nabla f(\bar{x}) \cdot h = 0$ for all $h \in \mathbb{R}^n$, and consequently it must be that $\nabla f(\bar{x}) = 0$. ■

If $\bar{x} \in U$ is a **stationary point** of f, $\bar{x}$ **is not** necessarily a **solution** of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$).

However, under additional conditions one gets the following result.

Let U be an **open and convex** subset of $\mathbb{R}^n$ and let f be a **continuously differentiable** function from U to $\mathbb{R}$.

Theorem

- *If f is* ***concave*** *in U and $\nabla f(\bar{x}) = 0$, then $\bar{x} \in U$ is a solution of problem $(\mathcal{P})$.*
- *If f is* ***convex*** *in U and $\nabla f(\bar{x}) = 0$, then $\bar{x} \in U$ is a solution of problem $(\mathcal{Q})$.*

Second order necessary conditions for local solutions

We now consider a function f that is $\mathcal{C}^2$ on U. We then get information also on the second derivatives of f, that is, on the Hessian matrix $D^2 f(\bar{x})$ of f at any local solution $\bar{x}$.

Theorem

*If $\bar{x}$ is a **local solution** of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$), then $\nabla f(\bar{x}) = 0$ and the Hessian matrix $D^2 f(\bar{x})$ of f at $\bar{x}$ is **negative semi-definite** (respectively, **positive semi-definite**).*

Second order sufficient conditions for local solutions

Let f be a $\mathcal{C}^2$ function on U.

Theorem

If $\bar{x} \in U$ satisfies $\nabla f(\bar{x}) = 0$ and the Hessian matrix $D^2 f(\bar{x})$ is ***negative definite*** *(respectively,* ***positive definite****), then $\bar{x}$ is a* ***local solution*** *of problem $(\mathcal{P})$ (respectively, of problem $(\mathcal{Q})$).*

Optimization problems with equality constraints

Let U be an **open** subset of $\mathbb{R}^n$. The functions f and $g_1, \ldots, g_i, \ldots, g_p$ are defined on U. We consider the following optimization problems with **equality constraints**.

$$(\mathcal{P}) \begin{cases} \max\limits_{x \in U} f(x) \\ g_i(x) = 0,\ i = 1, \ldots, p \end{cases} \qquad (\mathcal{Q}) \begin{cases} \min\limits_{x \in U} f(x) \\ g_i(x) = 0,\ i = 1, \ldots, p \end{cases}$$

Definition

Assume that $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^1$ on U. Let $\bar{x} \in U$ be a point such that $g_i(\bar{x}) = 0$ for all $i = 1, \ldots, p$. The **constraint qualification condition** is satisfied at $\bar{x}$ if all the gradient vectors $\nabla g_1(\bar{x}), \ldots, \nabla g_i(\bar{x}), \ldots, \nabla g_p(\bar{x})$ exist and are **linearly independent**.

Theorem

Assume that the functions f and $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^1$ on U.

Let $\bar{x} \in U$ be a ***solution*** *of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$) that satisfies the* ***constraint qualification condition****.*

Then, there exists a vector of Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_i, \ldots, \bar{\lambda}_p) \in \mathbb{R}^p$ such that :

$$\nabla f(\bar{x}) - \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0$$

A counterexample

Remark

The previous result does not hold true if the constraint qualification condition is not satisfied. Indeed, consider the following minimization problem :

$$\begin{cases} \min\limits_{(x,y)\in\mathbb{R}^2} f(x,y) = x + y \\ g_1(x,y) = (x-1)^2 + y^2 - 1 = 0 \\ g_2(x,y) = (x+1)^2 + y^2 - 1 = 0 \end{cases}$$

$\{(x,y) \in \mathbb{R}^2 \mid g_1(x,y) = g_2(x,y) = 0\} = \{(0,0)\} =$ *Set of solutions.*

$\nabla f(0,0) = (1,1)$, $\nabla g_1(0,0) = (-2,0)$, $\nabla g_2(0,0) = (2,0)$.

$\nabla g_1(0,0)$ *and* $\nabla g_2(0,0)$ *are* ***not linearly independent****, because* $\nabla g_2(0,0) = -\nabla g_1(0,0)$.

$\nexists(\lambda_1,\lambda_2) \in \mathbb{R}^2$ *such that* $\nabla f(0,0) = \lambda_1 \nabla g_1(0,0) + \lambda_2 \nabla g_2(0,0)$.

Lagrangian function

Definition

The **Lagrangian function** $\mathcal{L}$ associated with problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$) is the function from $U \times \mathbb{R}^p$ to $\mathbb{R}$ defined by :

$$\mathcal{L}(x, \lambda) = f(x) - \sum_{i=1}^{p} \lambda_i g_i(x),$$

where $\lambda = (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_p) \in \mathbb{R}^p$.

Notice that for every $(\bar{x}, \bar{\lambda}) \in U \times \mathbb{R}^p$:

$$\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = \nabla f(\bar{x}) - \textstyle\sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}), \text{ and}$$

$$\forall i = 1, \ldots, p, \ \frac{\partial \mathcal{L}}{\partial \lambda_i}(\bar{x}, \bar{\lambda}) = g_i(\bar{x}).$$

First order sufficient conditions

Let U be an **open and convex** subset of $\mathbb{R}^n$.

Theorem

Assume that the functions f and $g_1, \dots, g_i, \dots, g_p$ are $\mathcal{C}^1$ on U. Let $\bar{x} \in U$ be a point such that $g_i(\bar{x}) = 0$ for all $i = 1, \dots, p$.

If there exists a vector of Lagrange multipliers $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_i, \dots, \bar{\lambda}_p) \in \mathbb{R}^p$ such that

$$\nabla f(\bar{x}) - \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0,$$

*and the Lagrangian function $\mathcal{L}$ is **concave** (resp., **convex**) in x, then $\bar{x}$ is a **solution** of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$).*

Second order necessary conditions for local solutions

Assume that f and $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^2$ on U. Consider $\bar{x} \in U$ and the following set :

$$Z(\bar{x}) = \{z \in \mathbb{R}^n \mid \nabla g_i(\bar{x}) \cdot z = 0, \forall i = 1, \ldots, p\}.$$

Denote by $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ the **partial Hessian matrix of the Lagrangian function** $\mathcal{L}(\cdot, \bar{\lambda})$ with respect to x at $\bar{x}$.

Theorem

*Let $\bar{x}$ be a **local solution** of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$) that satisfies the **constraint qualification condition**, and $\bar{\lambda} \in \mathbb{R}^p$ such that $\nabla_x \mathcal{L}(\bar{x}, \bar{\lambda}) = 0$.*

*Then, $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ is **negative semi-definite** (resp., **positive semi-definite**) on $Z(\bar{x})$, i.e.,*

$$z^T D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda}) z \leq 0 \ (\textit{resp.} \geq 0), \ \textit{for all } z \in Z(\bar{x}).$$

Second order sufficient conditions for local solutions

Assume that f and $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^2$ on U.

Theorem

Let $\bar{x} \in U$ such that $g_i(\bar{x}) = 0$ for all $i = 1, ..., p$, and $\nabla f(\bar{x}) - \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x}) = 0$, for some $\bar{\lambda} \in \mathbb{R}^p$. If

$$z^T D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda}) z < 0 \text{ (resp. } > 0), \ \text{for all } z \in Z(\bar{x}) \setminus \{0\},$$

that is, the partial Hessian matrix of the Lagrangian function $\mathcal{L}(\cdot, \bar{\lambda})$ at $\bar{x}$, i.e., $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ is ***negative definite*** *(resp.,* ***positive definite****) definite on the set $Z(\bar{x}) \setminus \{0\}$, then $\bar{x}$ is a* ***local solution*** *of problem $(\mathcal{P})$ (resp., problem $(\mathcal{Q})$).*

Conditions on the border Hessian determinants

We assume that the **constraint qualification condition** is satisfied at $\bar{x} \in U$.

Consider the mapping $g = (g_1, \ldots, g_i, \ldots, g_p)$ defined by :

$$g : x \in U \subseteq \mathbb{R}^n \longrightarrow g(x) = (g_1(x), \ldots, g_i(x), \ldots, g_p(x)) \in \mathbb{R}^p$$

We rank the components of x in such a way that the first p columns of the Jacobian matrix $Dg(\bar{x})$ are linearly independent.

This is possible because the Jacobian matrix $Dg(\bar{x})$ has rank p, since its rows are the gradients of the constraint functions.

For every $r = p+1, \ldots, n$, define the following **border Hessian determinant** :

$$B_r(\bar{x}) = \begin{vmatrix} 0 & \ldots & 0 & \frac{\partial g_1(\bar{x})}{\partial x_1} & \ldots & \frac{\partial g_1(\bar{x})}{\partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \frac{\partial g_p(\bar{x})}{\partial x_1} & \ldots & \frac{\partial g_p(\bar{x})}{\partial x_r} \\ \frac{\partial g_1(\bar{x})}{\partial x_1} & \ldots & \frac{\partial g_p(\bar{x})}{\partial x_1} & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_1^2} & \ldots & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_1 \partial x_r} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_1(\bar{x})}{\partial x_r} & \ldots & \frac{\partial g_p(\bar{x})}{\partial x_r} & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_r \partial x_1} & \ldots & \frac{\partial^2 \mathcal{L}(\bar{x})}{\partial x_r^2} \end{vmatrix}$$

Proposition

- *If for all* $r = p+1, \ldots, n$, $(-1)^r B_r(\bar{x}) > 0$*, then the partial Hessian matrix of the Lagrangian function* $\mathcal{L}(\cdot, \bar{\lambda})$ *at* $\bar{x}$*, i.e.,* $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ *is* ***negative definite*** *on* $Z(\bar{x}) \setminus \{0\}$.

- *If for all* $r = p+1, \ldots, n$, $(-1)^p B_r(\bar{x}) > 0$*, then the partial Hessian matrix of the Lagrangian function* $\mathcal{L}(\cdot, \bar{\lambda})$ *at* $\bar{x}$*, i.e.,* $D_x^2 \mathcal{L}(\bar{x}, \bar{\lambda})$ *is* ***positive definite*** *on* $Z(\bar{x}) \setminus \{0\}$.

Interpretation of the Lagrange multipliers

Consider $b = (b_1, ..., b_i, ..., b_p) \in \mathbb{R}^p$ and the “perturbed" problem $(\mathcal{P}_b)$ of problem $(\mathcal{P})$:

$$(\mathcal{P}_b) \begin{cases} \max\limits_{x \in U} f(x) \\ g_i(x) = b_i,\ i = 1, \ldots, p \end{cases}$$

Assume that the value $v(b)$ of problem $(\mathcal{P}_b)$ is a **well-defined** function around $0 \in \mathbb{R}^p$. That is, there exists an open ball $B \subseteq \mathbb{R}^p$ of center 0 such that for all $b \in B$, problem $(\mathcal{P}_b)$ has at least a solution in U.

For all $b \in B$, the **value function** $v(b)$ is then defined by :

$$v(b) = \max\{f(x) \colon x \in U \text{ and } g_i(x) = b_i, \forall i = 1, \ldots, p\}$$

Also assume that the value function v is **differentiable** on B.

Let $\bar{x}$ be a solution of problem $(\mathcal{P}_0)$. Consider the mapping $g = (g_1, \ldots, g_i, \ldots, g_p)$ from U to $\mathbb{R}^p$ and notice that $g(\bar{x}) = 0$.

For all x in a open neighborhood of $\bar{x}$, define V as follows :

$$V(x) := f(x) - v(g(x)) \leq 0 \text{ and } V(\bar{x}) = 0.$$

Then $\nabla V(\bar{x}) = 0$, because $\bar{x}$ maximizes the function V in an open set.

From the **chain rule** for differentiable mappings, one gets :

$$0 = \nabla V(\bar{x}) = \nabla f(\bar{x}) - [Dg(\bar{x})]^T \nabla_b v(0).$$

Equivalently,

$$\nabla f(\bar{x}) = \sum_{i=1}^{p} \frac{\partial v}{\partial b_i}(0) \nabla g_i(\bar{x}).$$

Let $\bar{\lambda} = (\bar{\lambda}_1, \ldots, \bar{\lambda}_i, \ldots, \bar{\lambda}_p) \in \mathbb{R}^p$ the Lagrange multipliers associated with the solution $\bar{x}$ of problem $(\mathcal{P}_0)$. That is,

$$\nabla f(\bar{x}) = \sum_{i=1}^{p} \bar{\lambda}_i \nabla g_i(\bar{x})$$

Hence, for each $i = 1, \ldots, p$, the Lagrange multiplier $\bar{\lambda}_i$ is equal to the partial derivative $\dfrac{\partial v}{\partial b_i}(0)$ of the value function v at $b = 0$.