Optimization B – QEM1 and IMMAEF Slides 2 - One week and half: November 6, November 10 and November 13, 2025

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Outline

- Inequality constraints
- Karush-Kuhn-Tucker (KKT) conditions
- KKT as necessary conditions
- KKT as sufficient conditions
- KKT as necessary and sufficient conditions with mixed constraints
- Comparative statics and the Envelope Theorems

Inequality constraints

Let U be an **open** subset of $\mathbb{R}^n$. The functions f and $h_1, \ldots, h_j, \ldots, h_m$ are defined on U.

We study the **maximization** problem $(\mathcal{I})$ with the following **inequality constraints** (i.e., ≤ 0).

$$(\mathcal{I}) \left\{ \begin{array}{l} \max_{x \in U} f(x) \\ h_j(x) \leq 0, \, j = 1, \ldots, m \end{array} \right.$$

Remark. The simple adaptation of the following study to minimization problems of a function a, or optimization problems with inequality constraints described by the inequality $c_j(x) \geq 0$ is left to the reader, by remarking that :

1. $\min a(x) = \max f(x)$, with $f(x) := -a(x)$.
2. $c_j(x) \geq 0$ if and only if $h_j(x) \leq 0$, with $h_j(x) := -c_j(x)$.

Binding constraints

Definition

Let $x^* \in U$, we say that the constraint j is **binding** at x^* if $h_j(x^*) = 0$. We denote :

1. $J(x^*)$ the set of all binding constraints at x^*, that is :

$$J(x^*) := \{j = 1, ..., m : h_j(x^*) = 0\},$$

2. $m^* \leq m$ the number of elements of $J(x^*)$, and
3. $h^* := (h_j)_{j \in J(x^*)}$ is the following mapping :

$$h^* : x \in U \subseteq \mathbb{R}^n \longrightarrow h^*(x) = (h_j(x))_{j \in J(x^*)} \in \mathbb{R}^{m^*}$$

Karush-Kuhn-Tucker (KKT) conditions

From now on, f and $h_1, \ldots, h_j, \ldots, h_m$ are $\mathcal{C}^1$ on U.

KKT conditions associated with the maximization problem $(\mathcal{I})$:

$$(KKT)\begin{cases} \nabla f(x) = \sum_{j=1}^{m} \mu_j \nabla h_j(x), \\ \forall j = 1, ..., m,\ \mu_j \in \mathbb{R}_+ \text{ and } h_j(x) \leq 0, \\ \forall j = 1, ..., m,\ \mu_j h_j(x) = 0 \text{ (complementary slackness).} \end{cases}$$

That is, at x :
1) The gradient of the objective function is a linear combination of the gradients of the constraint functions, with **positive coefficients** $\mu_j \geq 0$.
2) All the constraints are satisfied.
3) If $\mu_j > 0$, then the constraint j is **binding** at x. If x belongs to the **interior** of the constraint j (i.e., $h_j(x) < 0$), then $\mu_j = 0$.

Linearized problem

Let $x^* \in U$ be a solution of problem $(\mathcal{I})$.

The **main idea** to prove that KKT conditions are **necessary conditions** to solve problem $(\mathcal{I})$ is to replace problem $(\mathcal{I})$ with the ***linearized problem*** $(\mathcal{L}^*)$:

$$(\mathcal{L}^*) \begin{cases} \max_{x \in \mathbb{R}^n} \nabla f(x^*) \cdot (x - x^*) \\ \nabla h_j(x^*) \cdot (x - x^*) \leq 0, \; j \in J(x^*) \end{cases}$$

Notice that what really matters in problem $(\mathcal{L}^*)$ is the use of the set $J(x^*)$ of **binding constraints** at x^*.

Generalized constraint qualification condition

Definition

Let $x^* \in U$ be a **solution** of problem $(\mathcal{I})$ such that $h_j(x^*) = 0$ for all $j \in J(x^*)$. The generalized constraint qualification (GCQ) condition is satisfied at x^* if x^* is **also** a solution of problem $(\mathcal{L}^*)$.

Remark that condition GCQ is not always satisfied.

One can easily find examples where $x^* \in \mathcal{S}ol(\mathcal{I})$, but $x^* \notin \mathcal{S}ol(\mathcal{L}^*)$.

Existence of Lagrange multipliers

Theorem

Assume that f and $h_1, \ldots, h_j, \ldots, h_m$ are $\mathcal{C}^1$ on U.

*If $x^* \in U$ is a **solution** of problem $(\mathcal{I})$ and x^* satisfies condition GCQ, then there exists $\mu^* = (\mu_1^*, ..., \mu_j^*, ..., \mu_m^*) \in \mathbb{R}_+^m$ such that the vector $(x^*, \mu^*) \in U \times \mathbb{R}_+^m$ satisfies the KKT conditions associated with problem $(\mathcal{I})$.*

Sketch of the proof

If $\nabla f(x^*) = 0$, take $\mu_j^* = 0$ for all $j = 1, ..., m$.

Assume now that $\nabla f(x^*) \neq 0$. Since x^* solves problem $(\mathcal{L}^*)$, there is no $x \neq x^*$ such that :

$$\nabla f(x^*) \cdot (x - x^*) > 0 = \nabla f(x^*) \cdot (x^* - x^*),$$

and

$$\nabla h_j(x^*) \cdot (x - x^*) \leq 0, \ \forall j \in J(x^*).$$

Take $b = \nabla f(x^*)$, and $a^j = \nabla h_j(x^*)$ for all $j \in J(x^*)$.

By **Farkas' Lemma**, there exists $\mu^* = (\mu_j^*)_{j \in J(x^*)} \in \mathbb{R}_+^{m^*}$ such that :

$$b = \sum_{j \in J(x^*)} \mu_j^* a^j.$$

For all $j \notin J(x^*)$, take $\mu_j^* = 0$.

By construction, we get $\mu_j^* h_j(x^*) = 0$ for all $j = 1, ..., m$, and
$$\nabla f(x^*) = \sum_{j=1}^{m} \mu_j^* \nabla h_j(x^*).$$

Further, $h_j(x^*) \leq 0$ for all $j = 1, ..., m$, because x^* is a solution of problem $(\mathcal{I})$.

Hence, (x^*, μ^*) satisfies the KKT conditions associated with problem $(\mathcal{I})$. ■

Sufficient conditions for generalized constraint qualification

Theorem

Assume that f and $h_1, \dots, h_j, \dots, h_m$ are $\mathcal{C}^1$ on U.

1. *If h_j **is linear or affine** for all $j = 1, ..., m$, then condition GCQ is satisfied.*
2. **(Slater's condition)** *Assume that U is also **convex** and :*
   - *the constraint functions h_j is **convex** for all $j = 1, ..., m$,*
   - *there exists $\widetilde{x} \in U$ such that $h_j(\widetilde{x}) < 0$ for all $j = 1, ..., m$.*

   Then, condition GCQ is satisfied.
3. **(Rank condition)** *If all the gradients $(\nabla h_j(x^*))_{j \in J(x^*)}$ are **linearly independent**, i.e., the rank of the Jacobian matrix $Dh^*(x^*)$ is equal to m^* (full row rank), then condition GCQ is satisfied.*

Two remarks

Remark 1. In the Rank condition above, one easily recognizes the classical constraint qualification condition given for optimization problems with **equality constraints**.

Remark 2. In Slater's condition, the convexity of h_j can be weakened by another assumption, that is, h_j is *pseudo-convex*.

It is well know that :

1. A $\mathcal{C}^1$ convex function is *pseudo-convex*.
2. A $\mathcal{C}^1$ quasi-convex function, with gradient different from zero everywhere, is *pseudo-convex*. Hence, in Slater's condition, the convexity of h_j can be replaced with the following assumption : h_j is **quasi-convex** with $\nabla h_j(x) \neq 0$ for all $x \in U$.

KKT as necessary conditions

As a consequence of the previous two theorems one gets the following theorem.

Theorem

Assume that f and $h_1, \dots, h_j, \dots, h_m$ are $\mathcal{C}^1$ on U.

*Let $x^* \in U$ be a **solution** of problem $(\mathcal{I})$.*

*Assume that **one** of the following three conditions is satisfied.*

1. *If h_j **is linear or affine** for all $j = 1, ..., m$.*
2. **Slater's condition.**
3. **Rank condition.**

*Then, there exists $\mu^* = (\mu_1^*, ..., \mu_j^*, ..., \mu_m^*) \in \mathbb{R}_+^m$ such that $(x^*, \mu^*) \in U \times \mathbb{R}_+^m$ **satisfies the KKT conditions** associated with problem $(\mathcal{I})$.*

KKT as sufficient conditions

Let U be an **open and convex** subset of $\mathbb{R}^n$.

Theorem

Assume that f and $h_1, \ldots, h_j, \ldots, h_m$ are $\mathcal{C}^1$ on U.

*If there exists $\mu^* = (\mu_1^*, ..., \mu_j^*, ..., \mu_m^*) \in \mathbb{R}_+^m$ such that $(x^*, \mu^*) \in U \times \mathbb{R}_+^m$ **satisfies the KKT conditions** associated with problem $(\mathcal{I})$, and the following condition (C) holds true, then x^* is a **solution** of problem $(\mathcal{I})$.*

***Condition (C) :** The function $\mathcal{L}(x) = f(x) - \sum_{j=1}^m \mu_j^* h_j(x)$ is **concave** in x.*

Let U be an **open and convex** subset of $\mathbb{R}^n$. Assume that f and $h_1, \ldots, h_j, \ldots, h_m$ are $\mathcal{C}^1$ on U.

Proposition

*The previous theorem still holds true if **Condition (C)** is replaced by **one** of the following two conditions.*

1. *The objective function f is **concave** and the constraint functions h_j are **quasi-convex** for all $j = 1, ..., m$.*
2. *The objective function f is **quasi-concave** with $\nabla f(x) \neq 0$ for all $x \in U$, and the constraint functions h_j are **quasi-convex** for all $j = 1, ..., m$.*

Hence, in order to check if KKT conditions are sufficient conditions to solve problem $(\mathcal{I})$, we have to verify also some properties of the **objective function** f.

Sketch of the proof

Without loss of generality, f is *pseudo-concave* on U.

Assume that there exists $\mu^* = (\mu_1^*, ..., \mu_j^*, ..., \mu_m^*) \in \mathbb{R}_+^m$ such that $(x^*, \mu^*) \in U \times \mathbb{R}_+^m$ satisfies the KKT conditions associated with problem $(\mathcal{I})$.

If $\nabla f(x^*) = 0$, then $f(x) \leq f(x^*)$ for all $x \in U$ (because U is **open** and f is pseudo-concave on U). Hence, $f(x) \leq f(x^*)$ for all $x \in U$ such that $h_j(x) \leq 0$ for all $j = 1, ..., m$. Further, $h_j(x^*) \leq 0$ for all $j = 1, ..., m$. Then, x^* solves problem $(\mathcal{I})$.

Assume now that $\nabla f(x^*) \neq 0$.

By contradiction, if x^* is not a solution of problem $(\mathcal{I})$, then there is $x \in U$, $x \neq x^*$, such that $h_j(x) \leq 0$ for all $j = 1, ..., m$, and $f(x) > f(x^*)$. By pseudo-concavity of f, one gets :

$$\nabla f(x^*) \cdot (x - x^*) > 0.$$

Since h_j is quasi-convex and $h_j(x) \leq 0 = h_j(x^*)$ for all $j \in J(x^*)$, we have that $\nabla h_j(x^*) \cdot (x - x^*) \leq 0$ for all $j \in J(x^*)$. Then, we get $\mu_j^* \nabla h_j(x^*) \cdot (x - x^*) \leq 0$ for all $j \in J(x^*)$, because $\mu_j^* \geq 0$.

If $j \notin J(x^*)$, then $\mu_j^* = 0$, because of complementary slackness.

Hence, we get :

$$\mu_j^* \nabla h_j(x^*) \cdot (x - x^*) \leq 0, \ \forall j = 1, ..., m.$$

Summing over $j = 1, ..., m$, we have :

$$\sum_{j=1}^{m} \mu_j^* \nabla h_j(x^*) \cdot (x - x^*) < \nabla f(x^*) \cdot (x - x^*).$$

That is impossible, because $\sum_{j=1}^{m} \mu_j^* \nabla h_j(x^*) = \nabla f(x^*)$. We then conclude that x^* must be a solution of problem $(\mathcal{I})$. ■

Sufficient conditions for uniqueness

Let U be an **convex** subset of $\mathbb{R}^n$.

We provide below sufficient conditions for the uniqueness of the solution of problem $(\mathcal{I})$.

Proposition

*Assume that the objective function f is **striclty quasi-concave** and the constraint functions h_j are **quasi-convex** for all $j = 1, ..., m$. If problem $(\mathcal{I})$ admits a solution $x^* \in U$, then x^* is the unique solution.*

The proof of this proposition is left as an exercise.

(Useful) Mathematical digression 1

Farkas' Lemma is a consequence of one of the Separation Theorems, and it is often used in **mathematical programming.**

Let $A = \{a^1, ..., a^j, ..., a^m\}$ be a set of m points of $\mathbb{R}^n$.

$K(A)$ denotes the set of all linear combinations of elements of A with positive coefficients :

$$K(A) = \left\{ z = \sum_{j=1}^{m} \mu_j a^j \in \mathbb{R}^n : \mu_j \geq 0 \text{ and } a^j \in A, \ \forall j = 1, ..., m \right\}.$$

That is, $K(A)$ is the smallest (in the sense of inclusion) **convex cone** of vertex 0 generated by $a^1, ..., a^j, ..., a^m$.

Theorem (Farkas' Lemma)

Let $A = \{a^1, ..., a^j, ..., a^m\}$ be a set of m points of $\mathbb{R}^n$. Consider any point $b \in \mathbb{R}^n$.

*Then, **only one** of the following two alternatives holds true.*

1. *There exists $\mu = (\mu_1, ..., \mu_j, ..., \mu_m) \in \mathbb{R}^m_+$ such that :*

$$b = \sum_{j=1}^{m} \mu_j a^j.$$

2. *There exists $p \in \mathbb{R}^n$ with $p \neq 0$ such that :*

$$p \cdot b > 0 \ \text{ and } \ p \cdot a^j \leq 0, \ \forall\, j = 1, ..., m.$$

(Useful) Mathematical digression 2

Let U be an **open and convex** subset of $\mathbb{R}^n$, f is a $\mathcal{C}^1$ on U.

Definition (Pseudo-concavity)

f is pseudo-concave on U if for all x and $\overline{x}$ in U with $x \neq \overline{x}$,

$$f(x) > f(\overline{x}) \Longrightarrow \nabla f(\overline{x}) \cdot (x - \overline{x}) > 0$$

A function g is pseudo-convex on U if and only if the function $f = -g$ is pseudo-concave on U.

Proposition

1. *If f is concave on U, then f is pseudo-concave on U.*
2. *If f is quasi-concave on U and $\nabla f(x) \neq 0$ for all $x \in U$, then f is pseudo-concave on U.*

Mixed constraints

Let U be an **open** subset of $\mathbb{R}^n$. From now on, the functions f, $g_1, \dots, g_i, \dots, g_p$, and $h_1, \dots, h_j, \dots, h_m$ are $\mathcal{C}^1$ on U.

We consider the following maximization problem $(\mathcal{M})$ that includes both **equality and inequality constraints.**

$$(\mathcal{M}) \begin{cases} \max\limits_{x \in U} f(x) \\ g_i(x) = 0, \, i = 1, \dots p \\ h_j(x) \leq 0, \, j = 1, \dots, m \end{cases}$$

Consider $x^* \in U$, as we have previously seen :

$J(x^*) = \{j = 1, ..., m : h_j(x^*) = 0\}$, m^* is the number of elements of $J(x^*)$, and $h^* = (h_j)_{j \in J(x^*)}$.

Also define the mapping $g = (g_i)_{i=1,\dots,p}$ from U to $\mathbb{R}^p$.

KKT conditions with mixed constraints

The Karush-Kuhn-Tucker (KKT) conditions associated with the maximization problem $(\mathcal{M})$ are :

$$(KKT)\begin{cases} \nabla f(x) = \sum_{i=1}^{p} \lambda_i \nabla g_i(x) + \sum_{j=1}^{m} \mu_j \nabla h_j(x), \\ \forall i = 1, ..., p,\ g_i(x) = 0, \\ \forall j = 1, ..., m,\ \mu_j \in \mathbb{R}_+ \text{ and } h_j(x) \leq 0, \\ \forall j = 1, ..., m,\ \mu_j h_j(x) = 0 \text{ (complementary slackness).} \end{cases}$$

• $\lambda = (\lambda_i)_{i=1,...,p} \in \mathbb{R}^p$ is the vector of Lagrange multipliers associated with the **equality constraints**, and
• $\mu = (\mu_j)_{j=1,...,m} \in \mathbb{R}_+^m$ is the vector of Lagrange multipliers associated with the **inequality constraints.**

Remark. Notice that λ_i is **not required to be positive.** This is not surprising, because an equality constraint can be written as two inequality constraints.

KKT as necessary conditions with mixed constraints

As a consequence of the previous results, one gets the following theorems.

Theorem

*Let $x^* \in U$ be a **solution** of problem $(\mathcal{M})$.*
*Assume that **one** of the following two conditions is satisfied.*

1. *The functions g_i and h_j **are linear or affine** for all $i = 1, ..., p$ and all $j = 1, ..., m$.*
2. **(Rank condition)** *All the gradients $(\nabla g_i(x^*))_{i=1,...,p}$ and $(\nabla h_j(x^*))_{j \in J(x^*)}$ are **linearly independent**. That is,*
$$\mathrm{rank} \begin{bmatrix} Dg(x^*) \\ Dh^*(x^*) \end{bmatrix} = p + m^*.$$

*Then, there exist $\lambda^* = (\lambda_i^*)_{i=1,...,p} \in \mathbb{R}^p$ and $\mu^* = (\mu_j^*)_{j=1,...,m} \in \mathbb{R}_+^m$ such that (x^*, λ^*, μ^*) **satisfies the KKT conditions** associated with problem $(\mathcal{M})$.*

KKT as sufficient conditions with mixed constraints

Now U is an open and **convex** subset of $\mathbb{R}^n$. We remind that the functions f, $g_1, \ldots, g_i, \ldots, g_p$, $h_1, \ldots, h_j, \ldots, h_m$ are $\mathcal{C}^1$ on U.

Theorem

*If there exist $\lambda^* = (\lambda_i^*)_{i=1,\ldots,p} \in \mathbb{R}^p$ and $\mu^* = (\mu_j^*)_{j=1,\ldots,m} \in \mathbb{R}_+^m$ such that $(x^*, \lambda^*, \mu^*) \in U \times \mathbb{R}^p \times \mathbb{R}_+^m$ **satisfies the KKT conditions** associated with problem $(\mathcal{M})$, and and the following condition (G) holds true, then x^* is a **solution** of problem $(\mathcal{M})$.*

Condition (G) : *The function $\mathcal{L}(x) = f(x) - \sum_{i=1}^{p} \lambda^* g_i(x) - \sum_{j=1}^{m} \mu_j^* h_j(x)$ is **concave** in x.*

Proposition

*The previous theorem still holds true if **Condition (G)** is replaced by **one** of the following two conditions.*

1. *The objective function f is **concave**, the functions g_i are **linear or affine** for all $i = 1, ..., p$, and the functions h_j are **quasi-convex** for all $j = 1, ..., m$.*
2. *The objective function f is **quasi-concave** with $\nabla f(x) \neq 0$ for all $x \in U$, the functions g_i are **linear or affine** for all $i = 1, ..., p$, and the functions h_j are **quasi-convex** for all $j = 1, ..., m$.*

Parameterized optimization problems

We now consider the following parameterized maximization problem with **equality constraints** only.

Problem $(\mathcal{P}_r)$ depends on some parameters $r = (r_1, ..., r_\ell, ..., r_k) \in \mathbb{R}^k$, because now the value of the objective function and the values of the constraint functions might change according to some parameters.

$$(\mathcal{P}_r) \begin{cases} \max\limits_{x \in U} f(x, r) \\ g_i(x, r) = 0,\ i = 1, \ldots, p \end{cases}$$

The value of problem $(\mathcal{P}_r)$ is denoted by $v(r)$.

Let $\bar{r} = (\bar{r}_1, ..., \bar{r}_\ell, ..., \bar{r}_k) \in \mathbb{R}^k$ some reference parameters.

We assume that $v(\cdot)$ is a **well-defined** function around $\bar{r}$. That is, there exists an open ball $B \subseteq \mathbb{R}^k$ of center $\bar{r}$ such that for all $r \in B$, problem $(\mathcal{P}_r)$ has **at least** a solution.

For all $r \in B$, the **value function** is then defined as :

$$v(r) = \max\{f(x, r) \colon x \in S(r)\},$$

where $S(r) = \{x \in U \colon g_i(x, r) = 0, \forall i = 1, \ldots, p\}$ is the set determined by the constraint functions of problem $(\mathcal{P}_r)$.

We are interested in studying the **marginal effects of changes in r on the value function v.**

Assumption

We make the following assumption.

Assumption (A). *There exist* $\mathcal{C}^1$ *mappings* $x(\cdot)$ *and* $\lambda(\cdot)$ *defined in the open neighborhood* B *of* $\bar{r}$*, i.e.,*

$$x : r \in B \to x(r) = (x_1(r), ..., x_n(r)) \in \mathbb{R}^n, \text{ and}$$
$$\lambda : r \in B \to \lambda(r) = (\lambda_1(r), ..., \lambda_p(r)) \in \mathbb{R}^p$$

such that for all $r \in B$ *:*

1. $x(r)$ *is the* ***unique solution*** *of problem* $(\mathcal{P}_r)$*, and*
2. $\nabla_x f(x(r), r) - \sum_{i=1}^{p} \lambda_i(r) \nabla_x g_i(x(r), r) = 0.$

A remark on the Implicit Function Theorem

Notice that Assumption (A) is an assumption on **endogenous** variables. i.e., $x \in U$ and $\lambda \in \mathbb{R}^p$.

Nevertheless, Assumption (A) can be obtained as a consequence of the **Implicit Function Theorem.**

Indeed, one can determine **appropriate assumptions** on the objective function f and on the constraints function g_i in such a way that one can apply the Implicit Function Theorem to the system of equations $F(x, \lambda, r) = 0$, where the mapping F is given by :

$$F : (x, \lambda, r) \in U \times \mathbb{R}^p \times B \to F(x, \lambda, r) \in \mathbb{R}^n \times \mathbb{R}^p,$$

with $F(x, \lambda, r) = (D(x, \lambda, r), G(x, \lambda, r))$ and

$$\begin{cases} D(x, \lambda, r) = \nabla_x f(x, r) - \sum_{i=1}^{p} \lambda_i \nabla_x g_i(x, r) \\ G(x, \lambda, r) = (g_1(x, r), ..., g_i(x, r), ..., g_p(x, r)) \end{cases}$$

The Envelope Theorem

For every $r \in B$, we have then $v(r) = f(x(r), r)$.

Under Assumption (A), one gets the following **Envelope Theorem** by using the chain rule.

Theorem

Assume that the objective function f and the constraint functions $g_1, \ldots, g_i, \ldots, g_p$ are $\mathcal{C}^2$ on U.

If Assumption (A) is satisfied, then :

$$\nabla_r v(\bar{r}) = \nabla_r f(x(\bar{r}), \bar{r}) - \sum_{i=1}^{p} \lambda_i(\bar{r}) \nabla_r g_i(x(\bar{r}), \bar{r}).$$

That is, for all $\ell = 1, ..., k$:

$$\frac{\partial v}{\partial r_\ell}(\bar{r}) = \frac{\partial f}{\partial r_\ell}(x(\bar{r}), \bar{r}) - \sum_{i=1}^{p} \lambda_i(\bar{r}) \frac{\partial g_i}{\partial r_\ell}(x(\bar{r}), \bar{r}).$$

Two important remarks on the Envelope Theorem

Remark (1)

*In the **unconstrained** case, i.e., we are in an **open set** with no constraints at all, one gets :*

$$\nabla_r v(\bar{r}) = \nabla_r f(x(\bar{r}), \bar{r})$$

That is, for all $\ell = 1, ..., k$ *:*

$$\frac{\partial v}{\partial r_\ell}(\bar{r}) = \frac{\partial f}{\partial r_\ell}(x(\bar{r}), \bar{r}).$$

In the second remark below, we consider the case where the **objective function f does not depend on the parameter r**, i.e., $f(x)$.

Remark (2)

*If the number k of parameters equals the number p of **equality constraints**, and for all $i = 1, ..., p$, the constraint functions are **additively separable** as follows :*

$$g_i(x, r) = \widetilde{g}_i(x) - r_i,$$

then one gets :

$$\nabla_r v(\bar{r}) = \lambda(\bar{r})$$

That is, for all $i = 1, ..., k$:

$$\frac{\partial v}{\partial r_i}(\bar{r}) = \lambda_i(\bar{r}).$$

Parameterized problems with inequality constraints

The previous analysis can be extended to the case of **inequality constraints.** However, its extension is not a trivial task.

Consider $r = (r_1, ..., r_k, ..., r_\ell) \in \mathbb{R}^\ell$ and the following maximization problem $(\mathcal{I}_r)$.

$$(\mathcal{I}_r) \begin{cases} \max\limits_{x \in U} f(x, r) \\ h_j(x, r) \leq 0, \, j = 1, \ldots, m \end{cases}$$

We write the Karush-Kuhn-Tucker conditions associated with problem $(\mathcal{I}_r)$.

$$(KKT)_r \begin{cases} \text{1) } \nabla_x f(x,r) = \sum_{j=1}^{m} \mu_j \nabla_x h_j(x,r), \\ \text{2) } \forall j = 1, ..., m,\ \mu_j \geq 0,\ h_j(x,r) \leq 0 \text{ and } \mu_j h_j(x,r) = 0 \end{cases}$$

For all $j = 1, ..., m$, the conditions in item 2) translate in the following equations :

$$\min\{\mu_j, -h_j(x,r)\} = 0.$$

Notice that the function $\min\{\mu_j, -h_j(x,r)\}$ is **not differentiable everywhere.**

This is because if x is on the **boundary** of constraint j, the changes in parameters r can cause x to **jump from the boundary to the interior** of constraint j.

Hence in the case of **inequality constraints**, Assumption (A) must be adapted as follows.

Assumption (B). *There exist $\mathcal{C}^1$ mappings $x(\cdot)$ and $\mu(\cdot)$ defined in an open neighborhood W of $\bar{r}$ such that for all $r \in W$:*

1. *$x(r)$ is the **unique** solution of problem $(\mathcal{I}_r)$.*
2. *$h_j(x(r), r) = 0$ for all $j \in J(x(\bar{r}))$ and $h_j(x(r), r) < 0$ for all $j \notin J(x(\bar{r}))$, i.e., $J(x(r)) = J(x(\bar{r}))$.*
3. $$\nabla_x f(x(r), r) - \sum_{j \in J(x(\bar{r}))} \mu_j(r) \nabla_x h_j(x(r), r) = 0.$$
4. *$\mu_j(r) > 0$ for all $j \in J(x(\bar{r}))$ and $\mu_j(r) = 0$ for all $j \notin J(x(\bar{r}))$.*

For all $r \in W$, the **value function** of problem $(\mathcal{P}_r)$ is then defined as :

$$v(r) = f(x(r), r)$$

Under Assumption (B) one gets an analogous result as the previous Envelope Theorem.

Theorem

Assume that the objective function f and the constraint functions $h_1, \dots, h_j, \dots, h_m$ are $\mathcal{C}^2$ on U. If Assumption (B) is satisfied, then :

$$\nabla_r v(\bar{r}) = \nabla_r f(x(\bar{r}), \bar{r}) - \sum_{j \in J(x(\bar{r}))} \mu_j(\bar{r}) \nabla_r h_j(x(\bar{r}), \bar{r}).$$