Optimization B – QEM1 and IMMAEF Slides 3 - One week: November 17 and November 20, 2025

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Outline

- Finite Horizon Dynamic Programming
- First order necessary and sufficient conditions
- Bellman Equations
- The consumption-savings problem : Euler Equations
- An application

Example (CS) : The consumption-savings problem. This example is in Sundaram, R.K. (1999), *A First Course in Optimization Theory*, Cambridge University Press.

A consumer faces a finite horizon planning. He has an initial wealth $w_t \geq 0$ at the beginning of each period t, and consumes $c_t \geq 0$ that period.

The wealth at the beginning of the next period is :

$$w_{t+1} = (1 + r)(w_t - c_t),$$

where $r \geq 0$ is the interest rate.

The initial wealth $w_0 > 0$ is **given** and the final wealth is $w_T \geq 0$.

The preferences at period t are represented by a utility function u over the consumption at period t.

The consumer maximizes the sum of the utility levels $u(c_t)$ over time $t = 0, \dots, T-1$. The finite horizon maximization problem is then :

$$(CS) \begin{cases} \max \sum_{t=0}^{T-1} u(c_t) \\ w_{t+1} = (1+r)(w_t - c_t), \text{ for } t = 0, \dots, T-1 \\ c_t \geq 0, \text{ for } t = 0, \dots, T-1 \\ w_t \geq 0, \text{ for } t = 1, \dots, T \end{cases}$$

The general model

*Periods $t = 0, 1, \ldots, T-1, T$, where T is the **finite horizon**.*

*S is the **state** space, and A is the **action** space.*

*$s_t \in S$ denotes a **state** at period t, and $a_t \in A$ denotes an **action** (or **control**) at period t.*

*The **initial state** $s_0 \in S$ is **given**. The new state at period $t+1$ is determined by a **transition equation** :*

$$s_{t+1} = g_t(a_t, s_t).$$

*$f_t(a_t, s_t) \in \mathbb{R}$ is the **payoff at period** t associated with the action-state pair (a_t, s_t), and $f_T(s_T)$ is the **payoff at period** T associated with the terminal state s_T. The **global payoff** is the discounted sum of the instantaneous payoffs with discount factors $\beta = (\beta_0, ..., \beta_t, ..., \beta_{T-1}, \beta_T)$.*

Dynamic optimization problem

The finite horizon maximization problem is problem $(\mathcal{D})$ below.

$$(\mathcal{D}) \begin{cases} \max \sum_{t=0}^{T-1} \beta_t f_t(a_t, s_t) + \beta_T f_T(s_T) \\ s_{t+1} = g_t(a_t, s_t),\ t = 0, \ldots, T-1 \\ a_t \in A,\ t = 0, \ldots, T-1 \\ s_{t+1} \in S,\ t = 0, \ldots, T-1 \end{cases}$$

Remark. In this formulation, **the payoff of the last period does not depend on actions**, i.e., $f_T(s_T)$. In this case, f_T is called **scrap value function**, and it can be interpreted as a measure of the net value associated with the terminal state.

Notice that, the case where the payoff of the last period also depends on the action of the last period is covered by our analysis. This is because one can obviously add a fictitious period as terminal period, and a scrap value function that is constantly equal to zero.

Feasible paths

We assume that :

- the state space S is a subset of $\mathbb{R}$, and the action space A is a subset of $\mathbb{R}$,
- for all $t = 0, \ldots, T-1$, the set $A \times S$ is a subset of the domains of f_t and g_t, and the state pace S is included in the domain of f_T.

*For a given initial state s_0, the set of **feasible paths** $U(s_0)$ is the set of all $\left((a_t)_{t=0}^{T-1}, (s_t)_{t=1}^{T}\right) \in \mathbb{R}^T \times \mathbb{R}^T$ satisfying :*

1. *$s_t \in S$ for all $t = 1, \ldots, T$, $a_t \in A$ for all $t = 0, \ldots, T-1$,*
2. *the transition equations*

$$s_{t+1} = g_t(a_t, s_t), \; t = 0, \ldots, T-1.$$

First order optimality conditions for "interior" solutions

Assume that for all $t = 0, \dots, T$, the state space S and the action space A are **open sets.**

The **objective function** f of problem $(\mathcal{D})$ is :

$$f\left((a_t)_{t=0}^{T-1}, (s_t)_{t=0}^{T}\right) = \sum_{t=0}^{T-1} \beta_t f_t(a_t, s_t) + \beta_T f_T(s_T).$$

For each $t = 0, \dots, T-1$, the **equality constraint** is :

$$\gamma_{t+1}(a_t, s_t, s_{t+1}) = s_{t+1} - g_t(a_t, s_t) = 0,$$

and λ_{t+1} is the Lagrange multiplier associated with the constraint function γ_{t+1}. The Lagrangian function of the problem is then :

$$\mathcal{L}(\left((a_t)_{t=0}^{T-1}, (s_t)_{t=0}^{T}, (\lambda_t)_{t=1}^{T}\right) =$$
$$\sum_{t=0}^{T-1} \beta_t f_t(a_t, s_t) + \beta_T f_T(s_T) - \sum_{t=0}^{T-1} \lambda_{t+1}[s_{t+1} - g_t(a_t, s_t)].$$

Assumptions

Assumption 1. *For all $t = 0, \ldots, T-1$, the functions f_t and g_t are $\mathcal{C}^1$ on the open set $A \times S$, and the function f_T is $\mathcal{C}^1$ on the open set S.*

Observe that the T gradients :

$$\left(\nabla_{((a_t)_{t=0}^{T-1},(s_t)_{t=1}^{T})}\gamma_{t+1}(a_t, s_t, s_{t+1})\right)_{t=0,\ldots,T-1}$$

are **linearly independent**. Indeed, it is enough to consider the derivatives of the constraint functions $(\gamma_{t+1})_{t=0,\ldots,T-1}$ with respect to the T state variables $(s_t)_{t=1}^{T}$. One gets a square matrix (with T rows and T columns) whose determinant is equal to 1. Consequently, one has the following proposition on first order necessary conditions.

Proposition

Let $(a_t^*)_{t=0}^{T-1}$ *be a solution of problem* $(\mathcal{D})$ *with initial state* $s_0 = s_0^*$. *Let* $(s_t^*)_{t=1}^{T}$ *be the associated sequence of states given by the transition equations :* $s_{t+1}^* = g_t(a_t^*, s_t^*), t = 0, \ldots, T-1$. *Under Assumption 1, there exists a vector of Lagrange multipliers* $\lambda = (\lambda_1, \ldots, \lambda_T) \in \mathbb{R}^T$ *such that :*

$$(FOC) \begin{cases} \beta_t \frac{\partial f_t}{\partial a_t}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial a_t}(a_t^*, s_t^*) = 0, \quad t = 0, \ldots, T-1 \\ \beta_t \frac{\partial f_t}{\partial s_t}(a_t^*, s_t^*) + \lambda_{t+1} \frac{\partial g_t}{\partial s_t}(a_t^*, s_t^*) = \lambda_t, \quad t = 1, \ldots, T-1 \\ \beta_T \frac{\partial f_T}{\partial s_T}(s_T^*) = \lambda_T, \\ s_{t+1}^* = g_t(a_t^*, s_t^*), \quad t = 0, \ldots, T-1 \end{cases}$$

Notice that :

a) The initial state $s_0 = s_0^*$ **is not an endogenous** variable of (FOC), but a **parameter.**

b) For all $t = 1, \ldots, T-1$, the state s_t appears in **both** constraint functions γ_t and γ_{t+1}.

Sufficient conditions for optimality

Assumption 2. *The Lagrangian function is concave in* $\left((a_t)_{t=0}^{T-1}, (s_t)_{t=1}^{T}\right)$.

Proposition

Under Assumptions 1 and 2, if $\left((a_t^*)_{t=0}^{T-1}, (s_t^*)_{t=0}^{T}\right)$ *satisfies the first order condition (FOC) given in the previous proposition, then* $\left((a_t^*)_{t=0}^{T-1}, (s_t^*)_{t=1}^{T}\right)$ *is a solution of problem* $(\mathcal{D})$ *with initial state* s_0^*.

Notice that the propositions above apply only if the solution of problem $(\mathcal{D})$ belongs to the **interior** of the sets of actions and states. Also remark that the number of equations of system (FOC) increases with the number of periods.

Introduction to the Bellman principle

In this section, the state space S and the action space A are **not necessarily open sets.** Further, the functions f_t and g_t are **not required to be $\mathcal{C}^1$.**

Fix a period $t \leq T - 1$ and a state $s_t \in S$.

Consider the following **truncated problem** $(\mathcal{D}_t)$ with **initial state** s_t and periods $\tau = t, \ldots, T$.

$$(\mathcal{D}_t) \begin{cases} \max \sum_{\tau=t}^{T-1} \beta_\tau f_\tau(a_\tau, s_\tau) + \beta_T f_T(s_T) \\ s_{\tau+1} = g_\tau(a_\tau, s_\tau), \tau = t, \ldots, T-1 \\ a_\tau \in A, \tau = t, \ldots, T-1 \\ s_{\tau+1} \in S, \tau = t, \ldots, T-1 \end{cases}$$

Let $(a_t^*)_{t=0}^{T-1}$ be a **solution of problem** $(\mathcal{D})$ with initial state $s_0 = s_0^* \in S$.

Let $(s_t^*)_{t=1}^{T}$ be the associated sequence of states given by the transition equations $s_{t+1}^* = g_t(a_t^*, s_t^*)$ for all $t = 0, \ldots, T-1$.

Then, $(a_\tau^*)_{\tau=t}^{T-1}$ is a **solution of the truncated problem** $(\mathcal{D}_t)$ with **initial state** $s_t = s_t^*$.

That is, the optimal path that starts at period t **only** depends on the state s_t^*, and it does not depend on the previous action-state pairs (a_τ^*, s_τ^*) with $\tau < t$.

The Bellman equations

For all $t = 0, \dots, T-1$ and for all $s_t \in S$, define the set $\Phi_t(s_t)$ of all the **feasible actions** at state s_t, i.e.,

$$\Phi_t(s_t) = \{a_t \in A \colon g_t(a_t, s_t) \in S\}.$$

For all $t = 0, \dots, T-1$, let $V_t(s_t)$ be the **value function** *of problem $(\mathcal{D}_t)$ with initial state $s_t \in S$.*

Then, one gets the **Bellman equations***, i.e., for all $t = 0, \dots, T-1$:*

- $V_t(s_t) = \max\{\beta_t f_t(a_t, s_t) + V_{t+1}(g_t(a_t, s_t)) \mid a_t \in \Phi_t(s_t)\}$,

  and a_t^* is a solution of the maximization problem above.

- $V_T(g_{T-1}(a_{T-1}, s_{T-1})) = \beta_T f_T(g_{T-1}(a_{T-1}, s_{T-1}))$, *with $s_T = g_{T-1}(a_{T-1}, s_{T-1}) \in S$.*

Dynamic programming algorithm

The Bellman equations are very useful to construct the solutions of problem $(\mathcal{D})$.

Step 1. The payoff at **period** T is $\beta_T f_T(s_T)$. Hence, $V_T(g_{T-1}(a_{T-1}, s_{T-1})) = \beta_T f_T(g_{T-1}(a_{T-1}, s_{T-1}))$, with $s_T = g_{T-1}(a_{T-1}, s_{T-1}) \in S$.

Step 2. At **period** $T-1$, we solve the following maximization problem with initial state $s_{T-1} \in S$:

$$\begin{cases} \max \beta_{T-1} f_{T-1}(a_{T-1}, s_{T-1}) + \beta_T f_T(g_{T-1}(a_{T-1}, s_{T-1})) \\ a_{T-1} \in \Phi_{T-1}(s_{T-1}) \end{cases}$$

This is a **one dimensional problem** where **the only variable is the action** a_{T-1}. We then determine the set of optimal actions $\alpha^*_{T-1}(s_{T-1})$ and the value function $V_{T-1}(s_{T-1})$ of the problem above.

Dynamic programming algorithm continued

Step 3. At **period** $T-2$, we solve the following maximization problem with initial state $s_{T-2} \in S$, where **the only variable is the action** a_{T-2} :

$$\begin{cases} \max \beta_{T-2} f_{T-2}(a_{T-2}, s_{T-2}) + V_{T-1}(g_{T-2}(a_{T-2}, s_{T-2})) \\ a_{T-2} \in \Phi_{T-2}(s_{T-2}) \end{cases}$$

to get the set of optimal actions $\alpha^*_{T-2}(s_{T-2})$ and the value function $V_{T-2}(s_{T-2})$ of the problem above.

We continue to **work backwards** until the final step that allows to compute the set of optimal actions $\alpha^*_0(s_0)$ and the value function $V_0(s_0)$.

Remark. At each step, the set of optimal solutions $\alpha^*_t(s_t)$ might be a **singleton**, meaning that the maximization problem $\max\{\beta_t f_t(a_t, s_t) + V_{t+1}(g_t(a_t, s_t)) \mid a_t \in \Phi_t(s_t)\}$ **has a unique solution.**

The algorithm provides then the solutions of problem $(\mathcal{D})$ with initial state s_0, by piking actions in the sets of optimal solutions and using the following recursive formula :

- $a_0^* \in \alpha_0^*(s_0)$,
- $s_1^* = g_0(a_0^*, s_0)$ and $a_1^* \in \alpha_1^*(g_0(a_0^*, s_0))$,
- $\forall t = 1, ..., T-2$:

$$s_{t+1}^* = g_t(a_t^*, s_t^*) \text{ and } a_{t+1}^* \in \alpha_{t+1}^*(g_t(a_t^*, s_t^*)),$$

- $s_T^* = g_{T-1}(a_{T-1}^*, s_{T-1}^*)$.

The consumption-savings problem

Consider the consumption-savings problem presented in **Example (CS)** :

$$(CS) \begin{cases} \max \sum_{t=0}^{T-1} u(c_t) \\ w_{t+1} = (1+r)(w_t - c_t), \text{ for } t = 0, \ldots, T-1 \\ c_t \geq 0, \text{ for } t = 0, \ldots, T-1 \\ w_{t+1} \geq 0, \text{ for } t = 0, \ldots, T-1 \end{cases}$$

Following the general formulation of problem $(\mathcal{D})$, the state at period t is the wealth w_t, the action at period t is the consumption c_t, and $\beta_t = 1$ for all $t = 0, \ldots, T-1$.

Karush-Kuhn-Tucker (KKT) conditions

Assume that the utility function u is differentiable and concave on the **open** set $\mathbb{R}_{++}$. Further, $u'(c) > 0$ for all $c > 0$ (this implies that u is strictly increasing on $\mathbb{R}_{++}$).

We now focus on solutions where for all $t \leq T - 1$, the action-state pairs are in the **interior** of the sets of actions and states. That is, $c_t^* > 0$ and $w_t^* > 0$ for all $t = 0, \ldots, T - 1$, and $w_0 = w_0^* > 0$.

For each $t = 0, \ldots, T - 1$, $\lambda_{t+1} \in \mathbb{R}$ is the Lagrange multiplier associated with the **equality constraint** :

$$w_{t+1} - (1 + r)(w_t - c_t) = 0,$$

and $\mu_T \geq 0$ is the Lagrange multiplier associated with the following **inequality constraint** :

$$-w_T \leq 0.$$

We get then the following Karush-Kuhn-Tucker necessary and sufficient conditions.

$$(KKT)_{CS}\left\{\begin{array}{l} u'(c_t^*) = \lambda_{t+1}(1+r), \text{ for } t = 0,\dots,T-1 \\ \lambda_t = \lambda_{t+1}(1+r), \text{ for } t = 1,\dots,T-1 \\ w_{t+1}^* = (1+r)(w_t^* - c_t^*), \text{ for } t = 0,\dots,T-1 \\ w_T^* \geq 0,\ \lambda_T \geq 0, \text{ and } \lambda_T w_T^* = 0 \end{array}\right.$$

Observe that the condition $\mu_T - \lambda_T = 0$ entails $\lambda_T = \mu_T \geq 0$.

Hence, the Lagrange multiplier μ_T (associated with the inequality constraint $-w_T \leq 0$) does not appear in the above conditions, but it provides important information on the sign of the Lagrange multiplier λ_T.

Analysis of the sign of the Lagrange multipliers

If $w_T^* > 0$, then $\lambda_T = 0$. Consequently, all the Lagrange multipliers are equal to 0, because :

$$\lambda_t = \lambda_{t+1}(1 + r), \ \forall t = 1, \ldots, T - 1.$$

But, this is impossible, because $u'(c_t^*) > 0$ for all $t \leq T - 1$.

Hence, **it must be** that $w_T^* = 0$ and $\lambda_T > 0$. Consequently, the above equalities imply that :

$$\lambda_t > 0, \ \forall t = 1, \ldots, T - 1.$$

Euler Equations

Hence we can eliminate the Lagrange multipliers to obtain the following **Euler equations** from system $(KKT)_{CS}$.

$$u'(c_t^*) = (1+r)u'(c_{t+1}^*), \ \forall t = 0, \dots, T-2$$

We have then $T-1$ equations and $T-1$ unknowns $(c_t^*)_{t=0}^{T-2}$, because $c_{T-1}^* = w_{T-1}^*$ is uniquely determined by the transition equation $0 = (1+r)(w_{T-1}^* - c_{T-1}^*)$, since $w_T^* = 0$.

The remaining consumption solutions $c_0^*, c_1^*, ..., c_{T-2}^*$ are completely determined by the Euler equations.

Using the transition equations, one obtains then all the optimal levels of wealth, provided that the initial wealth is sufficiently large to allow for strictly positive optimal consumptions.

An application

Consider the consumption-savings problem with $T = 2$, $r = 0$ and $u(c) = \sqrt{c}$.

The initial wealth $w_0 > 0$ is given.

$$(\mathcal{C}) \begin{cases} \max \sum_{t=0}^{1} \sqrt{c_t} \\ w_{t+1} = (w_t - c_t), \text{ for } t = 0, 1 \\ c_t \geq 0, \text{ for } t = 0, 1 \\ w_t \geq 0, \text{ for } t = 1, 2 \end{cases}$$

In order to determine the solution of problem $(\mathcal{C})$, we apply the two methods that we have studied above. We first use the Bellman principle. We then use the Euler equations. The solution is obviously the same.

First methodology : Bellman Equations

The state space is $S = \mathbb{R}_+$, and the action space is $A = \mathbb{R}_+$. For all $t = 0, 1$, and for every wealth $w_t \geq 0$, the set of **feasible actions** is $\Phi_t(w_t) = \{ c_t \in \mathbb{R}_+ : (w_t - c_t) \geq 0\} = [0, w_t]$.

Step 1. The payoff at period $T = 2$ is constantly equal to zero, i.e., $f_2(w_2) = 0$ for all w_2. Then, at period $T = 2$, the value at $w_2 = (w_1 - c_1)$ with $w_2 \in \mathbb{R}_+$ is given by $V_2((w_1 - c_1)) = 0$.

Step 2. At period $T - 1 = 1$, we solve the following problem :

$$\begin{cases} \max \sqrt{c_1} \\ c_1 \in [0, w_1] \end{cases}$$

We obtain this problem from Bellman equations, because $\beta_1 f_1(c_1, w_1) + V_2((w_1 - c_1)) = \sqrt{c_1}$, and the set of **feasible actions** is $\Phi_1(w_1) = [0, w_1]$.

The solution of the above problem is $c_1 = w_1$, and then the value function is $V_1(w_1) = \sqrt{w_1}$.

Step 3. At period $T - 2 = 0$, we solve the following problem :

$$\begin{cases} \max \sqrt{c_0} + \sqrt{(w_0 - c_0)} \\ c_0 \in [0, w_0] \end{cases}$$

We obtain this problem from Bellman equations, because $\beta_0 f_0(c_0, w_0) + V_1((w_0 - c_0)) = \sqrt{c_0} + \sqrt{(w_0 - c_0)}$, and the set of **feasible actions** is $\Phi_0(w_0) = [0, w_0]$.

The solution of the above problem is $c_0 = \dfrac{w_0}{2}$.

Thus, the solution of problem $(\mathcal{C})$ is $(c_0^*, c_1^*, w_1^*, w_2^*)$ where :

- $c_0^* = \dfrac{w_0}{2}$, and $w_1^* = \dfrac{w_0}{2}$ since $w_1^* = w_0 - c_0^* = w_0 - \dfrac{w_0}{2}$.
- $c_1^* = \dfrac{w_0}{2}$ since $c_1^* = w_1^*$, and $w_2^* = 0$, because $w_2^* = w_1^* - c_1^* = w_1^* - w_1^*$.

Second methodology : Euler Equations

Since $T = 2$, we have only one Euler equation, i.e., $u'(c_0^*) = u'(c_1^*)$, that entails the first equation of the system below.

Further, we have already proved that it must be $w_2^* = 0$.

Thus, $c_0^* > 0$, $c_1^* > 0$ and $w_1^* > 0$ are determined by the following system :

$$\begin{cases} \dfrac{1}{2\sqrt{c_0^*}} = \dfrac{1}{2\sqrt{c_1^*}} \\ w_1^* - c_1^* = 0 \\ w_0 - c_0^* = w_1^* \end{cases}$$

Hence, we have that $c_0^* = c_1^* = w_1^*$. From the last equation of the above system, one gets $w_1^* = \dfrac{w_0}{2}$, and then $c_0^* = c_1^* = \dfrac{w_0}{2}$.