Optimization B – QEM1 and IMMAEF Slides 4 - One week: November 24 and November 27, 2025

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Outline

- Infinite Horizon Dynamic Programming
- Macroeconomics : An example
- Stationary optimization problem
- Bellman Equation
- Euler Equations : Truncated problem at period T
- Two applications
- Basic properties of the value function

Example (M) : Macroeconomics. The following example is a discrete time version of the well-known Ramsey model, which is usually formalized in continuous time (see, for instance, Cass, D., *Review of Economic Studies* 32, 1965).

The following example can be found in Le Van, C., and Dana, R.-A. (2003), *Dynamic Programming in Economics*, Kluwer Academic Publishers.

At each period $t = 0, 1, \ldots, +\infty$, $k_t \in \mathbb{R}_+$ is the per capita capital stock, and $c_t \in \mathbb{R}_+$ is the per capita consumption.

The initial per capita stock $k_0 > 0$ is **given**. F is a production function, and the allocation between consumption at period t and investment for the next period $t + 1$ is given by :

$$k_{t+1} = F(k_t) - c_t.$$

The agent (for instance, a social planner) maximizes the intertemporal welfare, which is represented by a discounted sum of utility levels $u(c_t)$.

β^t is the discount factor at period t, and it measures the rate at which the agent discounts time t's preferences for the present.

The **infinite horizon** maximization problem is :

$$(R)\begin{cases} \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ k_{t+1} = F(k_t) - c_t,\ t = 0, 1, \ldots, +\infty \\ c_t \geq 0,\ k_t \geq 0,\ t = 0, 1, \ldots, +\infty \end{cases}$$

The **basic assumptions** on the above model are as follows.

1. $\beta \in]0,1[$. *The utility function u is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $u(0) = 0$. The utility function u is $\mathcal{C}^2$ on $\mathbb{R}_{++}$,* ***strictly concave*** *and* ***strictly increasing*** *on $\mathbb{R}_+$ with $u'(c) > 0$ for all $c > 0$, and it satisfies the* ***Inada condition*** *:*

$$\lim_{c \to 0} u'(c) = +\infty$$

2. *The production function F is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $F(0) = 0$. The production function F is $\mathcal{C}^2$ on $\mathbb{R}_{++}$,* ***strictly concave*** *and* ***strictly increasing*** *on $\mathbb{R}_+$ with $F'(k) > 0$ for all $k > 0$, and it satisfies the following conditions :*

$$\lim_{k \to 0} F'(k) = M > 0 \text{ or } +\infty, \text{ and } \lim_{k \to +\infty} F'(k) < 1$$

Stationary optimization problem

We study dynamic optimization problems with discrete time over an **infinite horizon** :

$$t = 0, 1, \ldots, +\infty.$$

We focus on **stationary optimization problems.** That is, infinite horizon dynamic optimization problems, where the payoff functions f_t and the functions g_t that determine the transition equations are **independent of time**, i.e.,

$$f_t = f \text{ and } g_t = g, \ \forall \ t = 0, 1, \ldots, +\infty$$

For all $t = 0, 1, \ldots, +\infty$, the **state** space is S, and the **action** space is A.

The discount factor at period t is β^t where $\beta \in]0,1[$.

Then, the **stationary optimization problem** is :

$$(D)\begin{cases} \max \sum_{t=0}^{\infty} \beta^t f(a_t, s_t) \\ s_{t+1} = g(a_t, s_t),\ t = 0, 1, \ldots, +\infty \\ a_t \in A,\ t = 0, 1, \ldots, +\infty \\ s_t \in S,\ t = 0, 1, \ldots, +\infty \end{cases}$$

The initial state s_0 is **given**, and the set of feasible paths is :

$U(s_0) = \{(a_t, s_t)_{t\in\mathbb{N}} \mid \forall t \in \mathbb{N}, s_{t+1} = g(a_t, s_t) \text{ and } (a_t, s_t) \in A \times S\}.$

$V(s_0)$ denotes the **value** of the above problem for the initial state s_0, i.e.,

$$V(s_0) = \sup\{\textstyle\sum_{t=0}^{\infty} \beta^t f(a_t, s_t) \colon (a_t, s_t)_{t\in\mathbb{N}} \in U(s_0)\}.$$

Boundedness Condition

Assumption S (Boundedness Condition) *There exists a real number K such that $|f(a,s)| \leq K$ for all $(a,s) \in A \times S$.*

Notice that if the set $A \times S$ is **compact** and the function f is **continuous** on $A \times S$, then Assumption S is satisfied.

Let $\mathcal{I} \subseteq \mathbb{R}$ be the set of feasible initial states.

Assumption S implies that the value $V(s_0)$ of problem (D) is finite for all feasible initial states $s_0 \in \mathcal{I}$, since $\beta \in]0,1[$.

That is, V is a well defined function from $\mathcal{I}$ to $\mathbb{R}$, even if an optimal solution of problem (D) might not exist.

A stronger assumption on the problem

Let $A(s_0)$ be the set of feasible actions a_0 at state s_0, i.e.,

$$A(s_0) = \{a_0 \in A \colon g(a_0, s_0) \in S\}.$$

The following conditions are sufficient for ensuring that Assumption S holds true **locally** around feasible paths.

Assumption B *There exist an interval* $\mathcal{I} \subseteq \mathbb{R}$ *and* $\varepsilon > 0$*, such that for all* $s_0 \in \mathcal{I}$ *:*

a) For all $a_0 \in A(s_0)$*,* $g(a_0, s_0) \in \mathcal{I}$*.*

b) The set $A(s_0)$ *is compact.*

c) $U(s_0) \subseteq \prod_{t \in \mathbb{N}} \overline{B}(0, \varepsilon)$*.*

d) The functions f and g are continuous.

Assumption B also ensures that :

1. the set $U(s_0)$ is compact,
2. the function $\sum_{t=0}^{\infty} \beta^t f(a_t, s_t)$ is well defined and continuous on $U(s_0)$.

Hence, by **Weierstrass Theorem**, problem (D) has at least a solution (we come back on this topic in Slides 5).

From now on, without loss of generality, V is **well defined** on some interval $\mathcal{I} \subseteq \mathbb{R}$ and it is the value function of problem (D).

That is, for all $s_0 \in \mathcal{I}$, $V(s_0) \in \mathbb{R}$ and

$$V(s_0) = \max\{\sum_{t=0}^{\infty} \beta^t f(a_t, s_t) : (a_t, s_t)_{t \in \mathbb{N}} \in U(s_0)\}.$$

Bellman Equation with infinite horizon

Proposition

*The value function V satisfies the **Bellman equation**. That is, for all $s_0 \in \mathcal{I}$:*

$$V(s_0) = \max\{f(a_0, s_0) + \beta V(g(a_0, s_0)) \mid a_0 \in A(s_0)\},$$

where $A(s_0)$ is the set of feasible actions a_0 at state s_0.

The Bellman equation is a functional equation, where the value function V is the unknown. We come back on this issue in Slides 5.

From the Bellman equation, one checks that the optimal action at period t can be computed as a solution of the following maximization problem :

$$\max\{f(a_t, s_t) + \beta V(g(a_t, s_t)) \mid a_t \in A(s_t)\},$$

where $A(s_t)$ is the set of feasible actions a_t at state s_t, i.e.,

$$A(s_t) = \{a_t \in A \colon g(a_t, s_t) \in S\}.$$

Since f and g are independent of time, the Bellman equation can be written without the time subscript. That is, for all $s \in \mathcal{I}$:

$$V(s) = \max\{f(a, s) + \beta V(g(a, s)) \mid a \in A(s)\}. \tag{1}$$

Remark that it does not mean that the solution of problem (D) is independent of time, because the maximization problem in (1) might have several solutions.

Stationary optimal solution

Assume that for all $s \in \mathcal{I}$, the maximization problem :

$$\max\{f(a,s) + \beta V(g(a,s)) \mid a \in A(s)\}$$

has a unique solution $\alpha^*(s)$. Define the following function :

$$\sigma : s \in \mathcal{I} \to \sigma(s) = g(\alpha^*(s), s) \in S$$

If s^* is a **fixed point** of the function σ, i.e., $\sigma(s^*) = s^*$, then the sequence $(\alpha^*(s^*), s^*)$ is a **stationary optimal solution** of problem (D) with initial state $s_0 = s^*$, that is :

$$\alpha^*(s_t^*) = \alpha^*(s_\tau^*), \forall (t,\tau) \in \mathbb{N} \times \mathbb{N}, \text{ and } s_t^* = s^*, \forall t \in \mathbb{N}.$$

The truncated problem at period T

Let $(a_t^*, s_t^*)_{t\in\mathbb{N}}$ be an optimal solution of problem (D) for the initial state $s_0 = s_0^*$.

Consider $T \geq 2$ and the following **truncated problem** of problem (D), where the variable are $(a_t)_{t=0}^{T-1}$ and $(s_t)_{t=1}^{T-1}$.

$$(P_T) \begin{cases} \max \sum_{t=0}^{T-1} \beta^t f(a_t, s_t) + \sum_{t=T}^{\infty} \beta^t f(a_t^*, s_t^*) \\ s_{t+1} = g(a_t, s_t), \quad t = 0, \ldots, T-2 \\ s_T^* = g(a_{T-1}, s_{T-1}), \\ (a_t, s_t) \in A \times S, \quad t = 0, \ldots, T-1 \end{cases}$$

Then, the **truncated sequence** $((a_t^*)_{t=0}^{T-1}, (s_t^*)_{t=1}^{T-1})$ is a solution of problem (P_T) with initial state $s_0 = s_0^*$, for all $T = 2, \ldots, +\infty$.

First order conditions (FOC) for "interior" solutions

The **open** set $\mathrm{Int}(A \times S)$ denotes the **interior** of the set $A \times S$.

Let $(a_t^*, s_t^*)_{t \in \mathbb{N}}$ be an optimal solution of problem (D) for the initial state $s_0 = s_0^*$, such that $(a_t^*, s_t^*) \in \mathrm{Int}(A \times S)$ and $s_{t+1}^* = g\left(a_t^*, s_t^*\right)$ for all $t \in \mathbb{N}$.

We make the following assumptions to use first order conditions (FOC), that are necessary and sufficient to solve the truncated problems (P_T).

Assumption A.

1) The functions f and g are $\mathcal{C}^1$ on $\mathrm{Int}(A \times S)$.

2) The functions f and g are concave on $\mathrm{Int}(A \times S)$.

Euler equations with infinite horizon

Under **Assumption A**, we obtain then the following **Euler equations.**

Assume that the partial derivatives of f and g with respect to the action are different from 0 on $\mathrm{Int}(A \times S)$, i.e. :

(E) $\frac{\partial f}{\partial a_t}(a_t^*, s_t^*) \neq 0$ *and* $\frac{\partial g}{\partial a_t}(a_t^*, s_t^*) \neq 0, \ \forall \ t = 0, 1, \ldots, +\infty.$

Then, the following Euler equation holds true for all $t = 0, 1, \ldots, +\infty$.

$$\beta^{t+1} \frac{\partial f}{\partial s_{t+1}}(a_{t+1}^*, s_{t+1}^*) =$$

$$\beta^{t+1} \frac{\frac{\partial f}{\partial a_{t+1}}(a_{t+1}^*, s_{t+1}^*)}{\frac{\partial g}{\partial a_{t+1}}(a_{t+1}^*, s_{t+1}^*)} \frac{\partial g}{\partial s_{t+1}}(a_{t+1}^*, s_{t+1}^*) - \beta^t \frac{\frac{\partial f}{\partial a_t}(a_t^*, s_t^*)}{\frac{\partial g}{\partial a_t}(a_t^*, s_t^*)}$$

Further, $s_{t+1}^* = g\,(a_t^*, s_t^*)$, for all $t = 0, 1, \ldots, +\infty$.

The Euler equations can be proved using first order conditions (FOC) associated with problems (P_T) with $T \geq 2$.

We illustrate the proof for $T = 2$. It is an easy matter to use the same arguments for every $T = n + 2$ with $n \in \mathbb{N}$.

Consider FOC associated with problem (P_2). That is :

$$\begin{cases} (\text{t=0}) & \frac{\partial f}{\partial a_0}(a_0^*, s_0^*) = -\,\lambda_1 \frac{\partial g}{\partial a_0}(a_0^*, s_0^*) \\ (\text{t=1}) & \beta \frac{\partial f}{\partial a_1}(a_1^*, s_1^*) = -\,\lambda_2 \frac{\partial g}{\partial a_1}(a_1^*, s_1^*) \\ (\text{t=1}) & \beta \frac{\partial f}{\partial s_1}(a_1^*, s_1^*) = \lambda_1 - \lambda_2 \frac{\partial g}{\partial s_1}(a_1^*, s_1^*) \end{cases}$$

From **(E)**, the Lagrange multipliers $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ are completely determined by the first two equations. Hence, it is enough to replace λ_1 and λ_2 in the third equation to obtain :

$$\beta \frac{\partial f}{\partial s_1}(a_1^*, s_1^*) = \beta \frac{\frac{\partial f}{\partial a_1}(a_1^*, s_1^*)}{\frac{\partial g}{\partial a_1}(a_1^*, s_1^*)} \frac{\partial g}{\partial s_1}(a_1^*, s_1^*) - \frac{\frac{\partial f}{\partial a_0}(a_0^*, s_0^*)}{\frac{\partial g}{\partial a_0}(a_0^*, s_0^*)}$$

Then, the Euler equation holds true for $t = 0$.

Two applications

1. Consider the macroeconomic model given in **Example (M)**.

The Euler equations for interior solutions become :

$$\beta u'(c_{t+1}^*)F'(k_{t+1}^*) = u'(c_t^*),\ \forall\ t = 0, 1, \ldots, +\infty.$$

Further, $k_{t+1}^* = F(k_t^*) - c_t^*$ for all $t = 0, 1, \ldots, +\infty$.

Remark that in this example, the utility function u is independent of the state variable. That is, the derivative of u with respect to the state variable is equal to zero :

$$\frac{\partial u}{\partial k_{t+1}}(c_{t+1}^*, k_{t+1}^*) = 0,\ \forall\ t = 0, 1, \ldots, +\infty$$

2. We consider the consumption-savings model in Slides 3, and we extend its analysis to an infinite horizon.

$$(CS) \begin{cases} \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ w_{t+1} = (w_t - c_t), \text{ for } t = 0, \dots, \infty \\ c_t \geq 0, \text{ for } t = 0, \dots, \infty \\ w_t \geq 0, \text{ for } t = 1, \dots, \infty \end{cases}$$

In this framework, $\beta \in]0, 1[$, u is $\mathcal{C}^1$ on $\mathbb{R}_{++}$, strictly increasing, strictly concave, and u satisfies the Inada condition.

Euler equations translate in :

$$\beta u'(c^*_{t+1}) = u'(c^*_t), \ \forall t = 0, 1, \dots, +\infty.$$

Further, $w^*_{t+1} = (w^*_t - c^*_t)$ for all $t = 0, 1, \dots, +\infty$.

Properties of the value function

We now come back to some important properties of the value function. We assume that it is well defined on some interval $\mathcal{I} \subseteq \mathbb{R}$, that is :

$$V : s_0 \in \mathcal{I} \rightarrow V(s_0) \in \mathbb{R}$$

Proposition

Assume that :

1) The sets A and S are convex.

2) If $s \in S$ and $\tilde{s} \geq s$, then $\tilde{s} \in S$.

3) The functions f and g are concave on $A \times S$ and increasing with respect to the state variable.

*Then the value function V is **concave** on its interval $\mathcal{I}$ of definition, and consequently, the value function V is **continuous** on the interior of this interval.*

Differentiability of the value function

Theorem

*Assume the same assumptions that ensure the **concavity** of the value function. Further, assume that the functions f and g are $\mathcal{C}^1$ on a neighborhood of (a_0^*, s_0^*) and $\frac{\partial g}{\partial a}(a_0^*, s_0^*) \neq 0$.*

*Then, the value function V is **differentiable** at s_0^* and :*

$$V'(s_0^*) = -\frac{\partial f}{\partial a}(a_0^*, s_0^*)\frac{\frac{\partial g}{\partial s}(a_0^*, s_0^*)}{\frac{\partial g}{\partial a}(a_0^*, s_0^*)} + \frac{\partial f}{\partial s}(a_0^*, s_0^*)$$