Exercises for Tutorials
"Macroeconomics: Economic Growth"

Université Paris 1 Panthéon-Sorbonne - Licence 3

Academic Year 2024/2025 - First semester

Dossier 1 :
Computing growth rates
Vademecum (with exercises)

Here below you can find some useful formulas.

Power functions

$x^0 = 1$ $\qquad$ $x^1 = x$
$x^a x^b = x^{a+b}$ $\qquad$ $(x^a)^b = x^{ab}$
$x^{-a} = \frac{1}{x^a}$ $\qquad$ $x^{1/n} = \sqrt[n]{x}$

Natural logarithms and log-transformations

$\ln xy = \ln x + \ln y$ $\qquad$ $\ln \frac{x}{z} = \ln x - \ln z$
$\ln 1 = 0$ $\qquad$ $\ln e = 1 \quad (e \approx 2.72)$
$\ln x^a = a \ln x$ $\qquad$ $\ln e^x = x$

Derivatives to know:

- Derivative of a power function respect to x: $(x^a)' = ax^{a-1}$
- Derivative of a log function: $\frac{d(\ln x)}{dx} = (\ln x)' = \frac{1}{x}$
- Derivative of a composite function: $(g[f(x)])' = g'[f(x)]f'(x)$
- Derivative of an exponential function: $(e^x)' = e^x \qquad$ *and* $\qquad (e^u)' = e^u u'$

This part briefly reviews how to compute growth rates in continuous and discrete time.

Growth rates in continuous time

Consider a given variable y. Its evolution over time is given by the function $y(t)$, which can be also written for simplicity as y_t.
The derivative of y_t respect to time is written as $\dot{y}_t = \frac{dy_t}{dt}$ and it indicates the instantaneous variation of y respect to t.
The growth rate of this function is g_y and it is defined as this instantaneous variation over the value of the function y_t at the instant t, i.e.:

$$g_y = \frac{\dot{y}_t}{y_t} = \frac{\frac{dy_t}{dt}}{y_t} \tag{1}$$

The derivative of the logarithm of the variable y respect to time is its growth rate. In fact, we take the derivative of a composed function (see formulas above):

$$\frac{d \ln y_t}{dt} = \frac{1}{y_t}\frac{dy_t}{dt} = \frac{\dot{y}_t}{y_t} = g_y \tag{2}$$

Growth rate of the product of variables

Suppose that

$$y_t = x_t z_t \tag{3}$$

Using the formulas concerning the logarithms, we can write:

$$\ln y_t = \ln x_t + \ln z_t \tag{4}$$

Taking the derivative respect to time of both sides of the equation, we obtain:

$$\frac{d \ln y_t}{dt} = \frac{d \ln x_t}{dt} + \frac{d \ln z_t}{dt} \tag{5}$$

That is:

$$g_y = g_x + g_z \tag{6}$$

The growth rate of the product of two variables is the sum of the growth rates of the two variables.

Constant growth rates

If a variable X grows at a constant rate g and we are in the context of continuous time, we can write:

$$X(t) = X_0 e^{gt} \tag{7}$$

We can verify that

$$\frac{\frac{dX}{dt}}{X} = g \tag{8}$$

Growth rate in discrete time

In some models, we do not consider the evolution of a variable at each point in time. In fact, we can consider the evolution of a variable at regular intervals: t, $t+1$, $t+2$, etc. (for instance, each unit is an year). In this latter case, the model is written in discrete time and not in continuous time.
In this case, the growth rate of a variable y_t between $t-1$ and t is:

$$g_y = \frac{y_t - y_{t-1}}{y_{t-1}} \approx \ln y_t - \ln y_{t-1} \tag{9}$$

when the difference between y_t and y_{t-1} is small. This approximation is related to the property: $\ln x \approx x - 1$ when $x \to 1$.
Using this approximation and indicating with g_x and g_z the growth rates of x_t and z_t of the equation (3), we have:

$$g_y \approx g_x + g_z \tag{10}$$

Finally, in discrete time, if a variable y grows at a constant rate a, then its value at time t is given by:

$$y_t = y_0(1+a)^t. \tag{11}$$

Exercise 1.

(From Jones and Vollrath, 2013). Suppose $x(t) = e^{.05t}$ and $z(t) = e^{.01t}$. Calculate the growth rate of $y(t)$ for each of the following cases:

a) $y = x$

b) $y = z$

c) $y = xz$

d) $y = x/z$

e) $y = x^{\beta} z^{1-\beta}$, where $\beta = 1/2$

f) $y = (x/z)^{\beta}$, where $\beta = 1/3$

Exercise 2.

(From Jones and Vollrath, 2013). Express the growth rate of y in terms of the growth rates of k, l and m for the following cases. Assume β is some arbitrary constant.

a) $y = k^{\beta}$

b) $y = k/m$

c) $y = (k/m)^{\beta}$

Exercise 3.

(From Jones and Vollrath, 2013). Assume $\frac{\dot{x}}{x} = .10$ and $\frac{\dot{z}}{z} = .20$, and suppose that $x(0) = 2$ and $z(0) = 1$. Calculate the numerical values of $y(t)$ for $t = 0, t = 1, t = 2$ and $t = 10$ for the following cases:

a) $y = xz$

b) $y = x/z$

c) $y = x^{\beta} z^{1-\beta}$, where $\beta = 1/3$

Exercise 4.

(From Romer, 2001). Suppose that the growth rate of a variable, X, is constant and equal to $a > 0$ from time 0 to time t_1; drops to 0 at time t_1; rises gradually from 0 to a from time t_1 to time t_2; and is constant and equal to a after time t_2.

1. Sketch a graph of the growth rate of X as a function of time.
2. Sketch a graph of $\ln X$ as a function of time.

Dossier 2:
The Solow model and its applications

Exercise 1. Review of Cobb-Douglas production function

A Cobb-Douglas function is written as:

$$Y = AK^{\alpha}L^{\beta}, \qquad 0 < \alpha, \beta < 1 \tag{12}$$

Compute the marginal productivity of capital and labor, the marginal rate of substitution between capital and labor, and the elasticity of substitution between capital and labor. What are the conditions on α and β to have constant return to scale?

Exercise 2. The Solow model without technological progress

We assume an economy à la Solow without technological progress. The economy is characterized by the following aggregate production function:

$$Y_t = {K_t}^{\alpha}{L_t}^{1-\alpha}, \qquad \textit{with} \quad 0 < \alpha < 1 \qquad \textit{and} \qquad y = Y/L; \quad k = K/L \tag{13}$$

where K_t is the stock of capital, L_t the level of employment. Active population grows at an exogenous rate $n \geq 0$. We also assume that households save a constant fraction of their income: $0 < s < 1$. The depreciation rate of capital is δ and it is also between 0 and 1.

1. Write the production function in terms of output and capital per worker: $y = f(k)$ (intensive form).
2. We assume that $\dot{K} = sY - \delta K$. Find the differential equation that describes the evolution of k (also called ≪fundamental dynamic equation≫ of the model).
3. Sketch a graph showing that, independently from the initial level of capital per capita, this economy converges towards the long-run steady state.
4. Explain in economic terms the dynamics that you showed in the graph.
5. What is the long-run growth rate of Y in this economy?
6. Suppose that the government introduces a law reducing the incentives to invest and save, so that the saving rate s reduces. Show the effects of this policy, both in the short and long run, using the same framework as above.
7. Graphically show the effects over time of this policy on the logarithm of income per worker and on the growth rate of income per worker.
8. Suppose that the government decides to put in place a tax on labor and capital income. Instead of receiving $wL + rK = Y$, consumers receive $wL(1-\tau) + rK(1-\tau) = (1-\tau)Y$. Suppose the economy is at the steady-state and find the effects of this tax on the output per worker, both in the short and long run. What would it happen if the government invests these fiscal revenues instead of consuming them?

Exercise 3. The Solow model with technological progress

Consider an economy that can be modelled using a Solow model with technological progress. The production function is $Y = K^{\alpha}(AL)^{1-\alpha}$, where $0 < \alpha < 1$. Y is the aggregate output, K is the stock of capital, L is the number of workers and A is labor-augmenting technology. Capital depreciates at a rate equal to δ (with $0 < \delta < 1$). A grows at an exogenous and constant rate γ, and L grows at an exogenous and constant rate n. Denote the output per worker with $y = Y/L$, the capital per worker with $k = K/L$, the output per unit of effective labor with $\hat{y} = Y/(AL)$, the capital per unit of effective labor with $\hat{k} = K/(AL)$, and the exogenous saving rate with s (with $0 < s < 1$). Factor markets are perfectly competitive.

1. Show how the production function can be written as a relationship between $\hat{y}$ and $\hat{k}$.

2. Find the equation that expresses the growth rate of the capital per unit of effective labor as a function of $\alpha, \gamma, n, \delta$ and $\hat{k}$.

3. Using graphs, explain the dynamics of the capital and output per unit of effective labour during the transition towards the balance growth path.

4. Find the production per worker y^* and the consumption per worker c^* on the balance growth path as functions of $s, \alpha, \delta, \gamma, n$ and A.

5. Find the growth rates, in the long run, of the production per worker y and aggregate production Y. Why can we say that in this model the technological progress is the engine of long run growth?

6. Sketch the income trajectory from a situation of disequilibrium with respect to the balance growth path BGP (capital per unit of efficient worker is lower than its than its steady-state value at the initial point in time).

7. Suppose that a reform of the research sector increases the growth rate of A, and this moves from γ to γ'. The aim is to analyze the effects of this shock on impact, in transition and in the long term, with the help of graphs and by explaining the mechanisms at work. Suppose that the economy is in a steady state at the time of the shock.

   (a) Graphically analyze the effect of the change in the pace of technical progress using the Solow diagram. What can you conclude at this stage?

   (b) Draw the new long-term trajectory of income per worker.

   (c) Using the growth rate equation for income per worker, study the trajectory of income per worker in transition.

8. Suppose now that a reform of the research sector leads to a one-time increase in A (but the growth rate γ of A remains unchanged). Do the same exercise as 7.

9. Is it better to have a one-time increase in A or an increase of the growth rate of A?

Exercise 4. Golden rule of capital accumulation

Suppose a Solow economy with technological progress, with the same characteristics than the economy described in Exercise 3.

1. Find the equation for the consumption on the BGP ($\hat{c}^*$) as a function of BGP capital per unit of effective labor.

2. Show that the maximization of $\hat{c}^*$ respect to $\hat{k}^*$ implies a condition on the marginal productivity of capital per unit of effective labor. We refer to this as the "golden rule of capital accumulation".

3. Use this rule to find the value of the capital per unit of effective labor that maximizes consumption, and graphically show this value in a Solow diagram.
4. Which value of saving rate do we need to obtain the golden rule value of the stock of capital?
5. Is it possible to save too much in the Solow model?

Exercise 5. Education and growth

The objective of this exercise is to show that it is important to take into account the accumulation of human capital for explaining the disparities between countries. We augment the Solow model by introducing human capital. The individual level of human capital h is the result of educational choices of the society. It is assumed that it is the same for all workers and that it is exogenous and constant. K, L and Y denote the aggregate levels of capital, labor and output, respectively. We use the following Cobb-Douglas production function: $Y = K^{\alpha}(hL)^{1-\alpha}$, where $0 < \alpha < 1$. We also use the usual assumptions of the Solow model, such as a constant saving rate and a constant population growth rate. Moreover, we assume that the rate of capital depreciation is zero. We also define: $\tilde{k} = K/(hL)$ and $y = Y/L$.

1. Show that the model is reduced, as in Solow, to a differential equation in $\tilde{k}$. Define the long run growth path.
2. Find the level y^* of per capita income in the long run, and interpret the effect of the different parameters on this level.
3. We are now outside the long run growth path. Express the growth rate of per capita income as a function of h, y and the parameters of the model. Interpret the influence of h and y on this growth rate. Explain why the usual convergence tests are poorly specified.

Exercise 6. Growth accounting

In the Solow model with technological progress, consider an economy on its balance growth path (BGP), with a rate of technological progress (γ) of 2%. Suppose that γ increases from 2% to 3%. Moreover, suppose that α is equal to 1/3.

1. What is the growth rate of output per worker before the change? How does this growth rate evolve in the long run?
2. Use the equation $\frac{\dot{y}}{y} = \alpha\frac{\dot{k}}{k} + \frac{\dot{B}}{B}$, where B is the total factor productivity ($B = A^{1-\alpha}$), and perform the growth accounting exercise for this economy, both before the change and after the economy has reached its new BGP.
3. How much do the change in the growth rate of capital per capita and the increase in the total factor productivity B contribute to the increase in the growth rate of GDP per capita?
4. How can you reconcile the results of question 2 with the fact that the Solow model predicts that the growth rate of output per worker on the BGP is determined by rate of technological progress only?

Exercise 7. (Important) questions you should be able to answer following the lectures

1. According to the Solow model, can a government achieve higher long run growth through policies that increase the savings rate of the population?
2. According to the Solow model, what are the factors that explain the differences in growth rates and level of GDP per capita between countries?
3. Does the Solow model predict absolute convergence, or conditional convergence only? Are these predictions empirically verified?
4. Does the Solow model explain long run growth?
5. Is Solow's model Malthusian?

Dossier 3:
Economics of ideas and endogenous growth

Exercise 1. Population growth in AK model

(From Jones and Vollrath, 2013). Consider the AK model in which we do not normalize the size of the labor force to one.

1. Using the production function $Y = AKL^{1-\alpha}$ and the standard capital accumulation equation, show that the growth rate of output depends on L.
2. Suppose $n = 0$ and $sL^{1-\alpha} > \delta$. Analyze the dynamics of the economy and show its economic implications with Solow's diagram.
3. Based on last question, study the consequences of a one-time increase of population L.
4. What happens if L îs growing at some constant rate n?
5. The production function in question 1 can be interpreted as the consequence of of an externality of type $B = AK^{1-\alpha}$ affecting a standard production function of type $Y = BK^{\alpha}L^{1-\alpha}$, where the stock of ideas depends positively on capital accumulation.

   How can this externality be modified to eliminate the scale effect (How to move to $Y = AK$)
6. Analyse graphically the case where $Y = AK$.
7. Does labor affect production?

Exercise 2. Growth and researchers' share of the population

(From Jones and Vollrath, 2013). Consider a Romer model with the following production function: $Y = K^{\alpha}(AL_Y)^{1-\alpha}$, where Y stands for output, K for capital, L_Y for labour used in the production of goods and A for technology. The stock of capital K evolves in the same way as in the Solow model: $\dot{K} = s_K Y - \delta K$, where $s_K Y$ is the investment and δ is the rate of depreciation of the stock of capital. As in the Solow model, savings are assumed to be a constant part of the income: $S = I = s_K Y$. Population grows at a constant rate n. The evolution of technology is given by $\dot{A} = \theta L_A^{\lambda} A^{\phi}$, where $L_A/L = S_R$ is the constant share of researchers in the total population and $\frac{\dot{L}}{L} = \frac{\dot{L_A}}{L_A} = n$. The impact of the number of researchers on the dynamics of new ideas is nonlinear, summarized by λ. The rate θ at which new ideas are discovered by researchers could be constant, or it can be assumed that it depends on the existing stock of ideas; such a dependence (ϕ) could be either positive or negative.

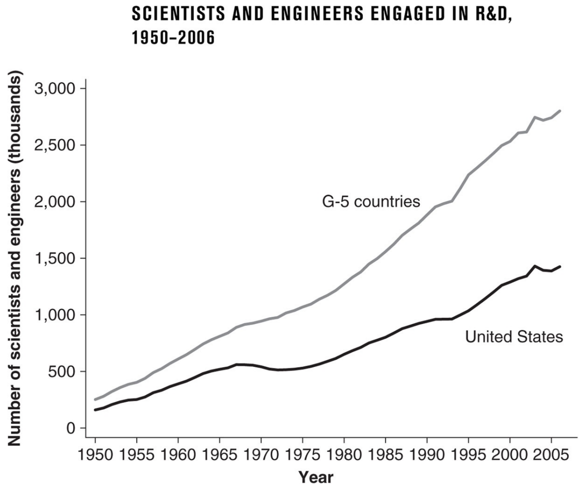
A) Assume the case in which $\lambda = 1$ and $\phi = 0$. Suppose there is a one-time increase in the productivity of research θ. Graphically show the effects of this one-time increase on the growth rate and the level of technology A over-time.

B) Consider the level of per capita income along a BGP given by the following equation:

$$y_t^* = \Big(\frac{s_k}{\delta + n + g_A}\Big)^{\frac{\alpha}{1-\alpha}}(1 - S_R)\frac{\theta S_R}{g_A}L_t \tag{14}$$

Find the value of S_R that maximizes output per worker along a BGP for this example. Why is it possible to do too much R&D according to this criterion?

C) Consider the following figure and the discussion about the fact that the number of scientists and engineers engaged in R&D has been growing faster than the rate of population growth in the advanced economies of the world. To take some plausible numbers, assume population growth is 1 percent and the growth rate of researchers is 3 percent per year. Assume that $\frac{\dot{A}}{A}$ has been constant and about 2 percent per year.



(a) Using the equation $0 = \lambda\frac{\dot{L_A}}{L_A} - (1-\phi)\frac{\dot{L_A}}{L_A}$, calculate an estimate of $\frac{\lambda}{1-\phi}$.

(b) Using the equation $g_A = \frac{\lambda n}{1-\phi}$, calculate an estimate of the long-run steady-state growth rate of the world economy.

(c) Why does your estimate of long-run steady-state growth differ from the 2 percent of growth of A observed historically?

(d) Does the fact that many developing countries are starting to engage in R&D change this calculation?

D) Find the ratio of the profit captured by the monopolist to the total potential available consumer surplus if the good were priced at marginal cost. Assume that marginal cost is constant at c and the demand curve is linear: $Q = a - bP$, where a, b, and c are constants with $a - bc > 0$.

Dossier 4:
Overlapping generation model

Exercise.

Consider an economy populated by individuals living for two periods, where in each period there are two generations. Population grows at a constant rate: $\frac{N_{t+1}}{N_t} = 1 + n$. The depreciation rate of capital is $\delta = 1$. Factor markets are perfectly competitive and the production function is $Y = K^{\alpha} L^{1-\alpha}$. Each generation works when it is young and consumes all its income when old. The utility function is

$$U(c_t, c_{t+1}) = c_t c_{t+1}^{\beta}$$

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1. Find the condition that defines the relationship between consumption in the two periods for the representative agent.

2. Find the dynamic equation of the model using the saving function of the young agents.

3. Find the value of the steady-state capital per worker.

4. Find the value of the golden-rule capital per worker.

5. Find the condition for which the steady-state capital per worker is greater than the corresponding value of the golden rule.

Dossier 5:
Population and the origin of sustained growth

Exercise. Malthusian economy and the origin of sustained growth

(From Jones and Vollrath, 2013). Consider a Malthusian economy, with a fixed factor replacing physical capital (assume it as land, and indicate it with X). The production function looks as follows: $Y = BX^{L^{(1-\beta)}}$, where Y is the aggregate output, L is labour, B is technology (total factor productivity), β is the share of land in income. In per capita terms, this can be re-written as $y = B(\frac{X}{L})^{\beta}$. Population grows according to the following equation: $\frac{\dot{L}}{L} = \theta(B(\frac{X}{L})^{\beta} - \underline{c})$, where $\underline{c}$ is the subsistence level of consumption and θ a parameter that mediates the response of population growth to income.

1. Consider the economy at its steady-state. What are the effects of a fall in the size of population on the dynamics of income per capita? Show the effects in the short and long-run.

2. Consider the economy at its steady-state. What are the effects of a permanent fall in productivity on the dynamics of income per capita? Show the effects in the short and long-run.

3. Consider the economy at its steady-state. What are the effects of temporary fall in productivity (i.e. the productivity temporary falls but then it goes back to its original level) on the dynamics of income per capita? Show the effects in the short and long-run.

4. How can the interaction between population growth and technical progress allow to move from the Malthusian economy? Is this explanation of the origins of economic growth sufficient?

Exercices supplementaires ou bonus

Exercise 1. Population in the Solow model without technological progress

Consider a Solow economy like that described in exercise 2. Suppose that the economy is at the steady-state.

1. What are the effects of a fall in the growth rate of the population on the steady-state values of capital per worker, output per worker, and consumption per worker?
2. Sketch the dynamics of these variables as the economy moves to its new steady-state and discuss the mechanisms.
3. Describe the effect of the fall in population growth on the dynamics of aggregate output (Y).
4. Suppose the economy is at the steady-state when there is a one-time inflow of migrant workers (i.e., L increases in a permanent way). Using graphs and relevant equations, explain the dynamics of capital per worker and output per worker, both in the short and long run, after this shock.

Exercise 2. Where are these economies headed? (Jones, 2012)

Consider the following data:

	(i)	(ii)	(iii)	(iv)	(v)
	$\tilde{y}_{90}$	s	u	n	$\tilde{A}_{90}$
US	1	0.210	11.8	0.009	1
Canada	0.93	0.253	10.4	0.010	1.05
Brazil	0.30	0.169	3.7	0.021	0.77
China	0.06	0.222	7.6	0.014	0.11
Kenya	0.05	0.126	4.5	0.037	0.16

We use a Solow model with the following production function: $Y = K^{\alpha}(AhL)^{1-\alpha}$, with $h = e^{\varphi u}$. h is the human capital per capita and it is a function of u (the average time devoted to training in a given country), and φ (the gain in wage associated with an year of additional training). It can be shown that, on the balance growth path (BGP) of this model, per capita income is $y^* = \left(\frac{s}{n+\gamma+\delta}\right)^{\frac{\alpha}{1-\alpha}} hA$.

1. Find the relative income of country i with respect to a reference country j in the BGP as predicted by this Solow model. Denote it as $\tilde{y}^*$.
2. The table above reports data for five countries. The first column shows the per capita income of each country respect to the US in 1990. The last column shows the ratio A_i/A_j for each country. Use these data to compute the relative income of BGP of these five countries. Assume that $\gamma + \delta = 0.75$, $\alpha = 1/3$ and $\varphi = 0.10$. Consider two extreme cases: (a) labor productivity ratios A_i/A_j remain constant; (b) there is complete convergence in the technological level.
3. On the basis of this analysis of the data following the Solow model, what are the countries that should experience the fastest and the slowest growth? Why ?

4. What are the limitations of such a forecasting exercise?

Exercise 3. Production function of ideas

(From Jones et Vollrath, 2013). What is the economic justification for thinking that the production function for new ideas takes the following form: $\dot{A} = \theta L_A A^{\phi}$? In particular, why might this production function exhibit increasing returns to scale?