Macroeconomics: Economic Growth (Licence 3)
Lesson 4: The Solow Model (Part 3)

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Acknowledges: some slides and figures are taken or adapted from the supplemental ressources to the textbook "Introduction to Economic Growth" by Charles I. Jones and Dietrich Vollrath, Third Edition, Norton W.W. Company Inc.

Long run levels

Long run levels: variables per unit of effective labor and per capita

- We can give a precise answer to what determines the capital and income
- in terms of units of effective labor and in per capita terms

$$\hat{k}^* = \frac{k^*}{A} = \left(\frac{s}{\delta + \gamma + n}\right)^{\frac{1}{1-\alpha}} \tag{1}$$

$$\hat{y}^* = \frac{y^*}{A} = \left(\frac{s}{\delta + \gamma + n}\right)^{\frac{\alpha}{1-\alpha}} \tag{2}$$

Long run growth (variables per unit of effective labor)

- We have seen that in the long run $\dot{\hat{k}} = 0$, thus:

$$\frac{\dot{\hat{k}}}{\hat{k}} = 0 \tag{3}$$

- Since $\frac{\dot{\hat{y}}}{\hat{y}} = \alpha \frac{\dot{\hat{k}}}{\hat{k}}$

$$\frac{\dot{\hat{y}}}{\hat{y}} = 0 \tag{4}$$

Long run growth (per worker variables) Trend growth

- We have defined: $\hat{k} = \frac{k}{A} \rightarrow \frac{\dot{\hat{k}}}{\hat{k}} = \frac{\dot{k}}{k} - \frac{\dot{A}}{A}$
- Since in the log run $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ and $\frac{\dot{\hat{A}}}{\hat{A}} = \gamma$
- In the long run the growth rate of capital per worker is

$$\frac{\dot{k}}{k} = \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A} = \gamma \tag{5}$$

Long run growth (per worker variables)

- Since $\hat{y} = \frac{y}{A} \rightarrow y = \hat{y}A = \hat{k}^{\alpha}A$
- Thus $\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A}$
- Thereby the growth rate of income per worker in the long run is:

$$\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A} = \gamma \tag{6}$$

- Since in the log run $\frac{\dot{\hat{k}}}{\hat{k}} = 0$ and $\frac{\dot{\hat{A}}}{\hat{A}} = \gamma$
- In the Solow model with technological progress in the long run we have:
- $g_{\hat{k}} = g_{\hat{y}} = 0$ and $g_k = g_y = g_A = \gamma$

Long run growth (per worker variables)

- The Solow model with technological progress
- produces a balanced growth path (BGP): where everything's growth rate remains constant
- Respect to the version of the model without technological progress,
- here the growth in A allows trend growth in output per worker in the long-run!

Long run growth (aggregate variables)

- Since $\hat{k} = \frac{K}{AL}$ thus:

$$\frac{\dot{K}}{K} = \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{L}}{L} + \frac{\dot{\hat{A}}}{\hat{A}} = n + \gamma \tag{7}$$

- Since $\hat{y} = \frac{Y}{AL}$ thus:

$$\frac{\dot{Y}}{Y} = \frac{\dot{\hat{y}}}{\hat{y}} + \frac{\dot{L}}{L} + \frac{\dot{\hat{A}}}{\hat{A}} = n + \gamma \tag{8}$$

Comparative statics: a change in investment or saving s

- We replicate the analysis of a change in investment (or saving) rate s on the dynamics of the economy
- Results are similar to the model without technological progress
- Following an exogenous increase in s:
  - Capital grows faster than the grows rate of AL
  - Until the new long run equilibrium is reached
  - The effects on transition dynamics are similar to the model without technological progress

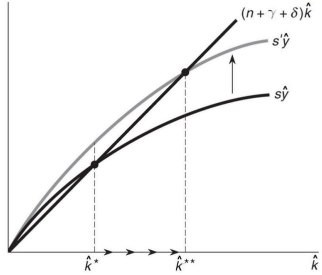
Comparative statics: a change in investment or saving s

- Suppose the economy is at the steady-state, what happens if s changes?
- No impact on long-run growth
- Since $g_y = g_k = \gamma$
- $\rightarrow$ A change in s has NO impact on the long-run growth of y and k
- Since $g_Y = g_K = n + \gamma$
- $\rightarrow$ A change in s has NO impact on the long-run growth of Y and K

Comparative statics: a change in investment or saving s

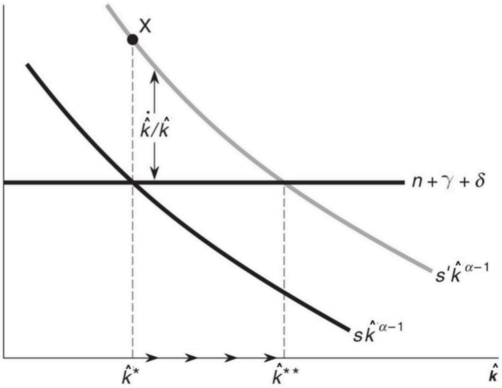
- Suppose the economy is at the steady-state, what happens if s changes?
- Only impacts on levels of y, k
- Since $k^* = A(\frac{s}{\delta+\gamma+n})^{\frac{1}{1-\alpha}}$ and $y^* = A(\frac{s}{\delta+\gamma+n})^{\frac{\alpha}{1-\alpha}}$
- $\rightarrow$ A change in s has an impact on the long-run levels of y and k
- Higher s increases k^* and y^*

Comparative statics: a change in investment or saving s

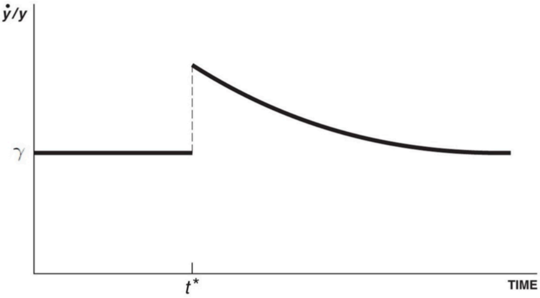


- Suppose the economy is at the steady-state, what happens if s changes?
- Since $\frac{\dot{\hat{k}}}{\hat{k}} = s\hat{k}^{\alpha-1} - (n+\delta+\gamma)$, $\frac{\dot{k}}{k} = \frac{\dot{\hat{k}}}{\hat{k}} + \gamma$ and $\frac{\dot{y}}{y} = \alpha\frac{\dot{\hat{k}}}{\hat{k}} + \gamma$
- $\rightarrow$ A change in s has an impact on the growth rates during transition dynamics of y and k
- A change in s has only temporary effects on the growth rates of k and y and permanent effects on the long run levels of k and y

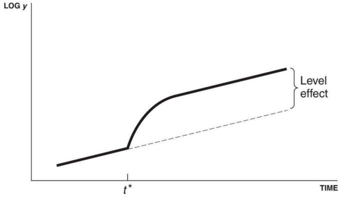
Comparative statics: a change in investment or saving s



Comparative statics: a change in investment or saving s



Comparative statics: a change in investment or saving s



- Only impacts on levels of y, k
- Since $k^* = A(\frac{s}{\delta+\gamma+n})^{\frac{1}{1-\alpha}}$ and $y^* = A(\frac{s}{\delta+\gamma+n})^{\frac{\alpha}{1-\alpha}}$
- $\rightarrow$ A change in s has an impact on the long-run levels of y and k
- Higher s increases k^* and y^*

Comparative statics: a change in investment or saving s

- What is remarkable is that, despite in this version of the Solow model we have the presence of technological change and permanent growth of per capita output,
- changes in investment rates have permanent effects on the levels of per capita output only
- and temporary effects on the transition growth (no effect on the long run growth)
- Following a permanent increase in investments, growth of per worker output increases on impact to subsequently converge to the old rate of growth
- This implies that y grows after the increase in s but then converges on a higher trajectory that is parallel to the previous one.

Comparative statics: an increase in growth rate of A γ

- Suppose that the economy is on a BGP, and we have a permanent improvement in the growth rate of technological progress: $\gamma' > \gamma$
- We want to understand the dynamics of the output per worker
- In the old BGP, we know that $\frac{\dot{y}}{y} = \gamma$ and
- In the new BGP, we know that $\frac{\dot{y}}{y} = \gamma'$
- To understand what happens to y during the transition, recall that:
    - $\frac{\dot{y}}{y} = \frac{\dot{\hat{y}}}{\hat{y}} + \frac{\dot{A}}{A} = \alpha \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A}$
    - So, to understand what happens to y we need to look at what happens to $\hat{k}$ and A.

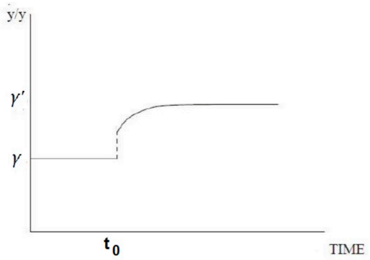
Comparative statics: an increase in growth rate of A γ

- At time $t0$ when we have the change from γ to γ':
    - $\frac{\dot{\hat{k}}}{\hat{k}} = \gamma - \gamma' < 0$
    - While $\frac{\dot{A}}{A} = \gamma' > \gamma > 0$
    - And: $\frac{\dot{y}}{y} = \alpha \frac{\dot{\hat{k}}}{\hat{k}} + \frac{\dot{A}}{A}$
    - Thus, $\frac{\dot{y}}{y} = \alpha(\gamma - \gamma') + \gamma' = \gamma'(1-\alpha) + \alpha\gamma$
- As a consequence: $\frac{\dot{y}}{y}$ is a weighted average of γ and $\gamma' \rightarrow \gamma < \frac{\dot{y}}{y} < \gamma'$

Comparative statics: an increase in growth rate of A γ

- $\frac{\dot{y}}{y} = \alpha(\gamma - \gamma') + \gamma' = \gamma'(1-\alpha) + \alpha\gamma$
- Note that the jump of $\frac{\dot{y}}{y}$ at $t0$ depends on the size of α
- For a very large α : we have a very small jump
- We can see this in a plot.

Comparative statics: an increase in growth rate of A γ

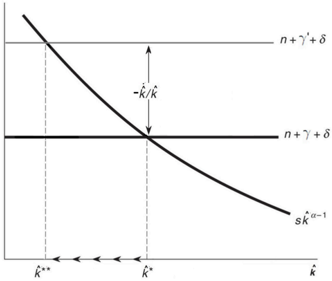


- Note that the jump of $\frac{\dot{y}}{y}$ at $t0$ depends on the size of α
- For a very large α : we have a very small jump
- The jump in the graph is the dashed vertical line at $t0$

Comparative statics: an increase in growth rate of A γ

- After $t0$, $\frac{\dot{y}}{y}$ will tend to γ'
- Why $\frac{\dot{y}}{y}$ tend to γ' in a non-linearly way?
- During the transition, we have that $\frac{\dot{\hat{k}}}{\hat{k}} < 0$ non-linearly trends towards 0
- While $\frac{\dot{A}}{A} = \gamma'$
- When the new BGP is reached, we know that $\frac{\dot{\hat{k}}}{\hat{k}} = 0$
- thus y will depend only on A that is $\frac{\dot{y}}{y} = \frac{\dot{A}}{A} = \gamma'$
- We can see this in a plot.

Comparative statics: an increase in growth rate of A γ



- During the transition, we have that $\frac{\dot{\hat{k}}}{\hat{k}} < 0$ non-linearly trends towards 0
- While $\frac{\dot{A}}{A} = \gamma'$
- When the new BGP is reached, we know that $\frac{\dot{\hat{k}}}{\hat{k}} = 0$
- thus y will depend only on A that is $\frac{\dot{y}}{y} = \frac{\dot{A}}{A} = \gamma'$

Growth vs. level effects

- In the version of the Solow model with technological progress (like in the simple version), policies affecting s and n:
  - Do NOT have an impact on the long-run growth rate of income (and capital) per worker
  - Have an impact on the long-run level of income (and capital) per worker
  - Have an impact on the growth rates of income (and capital) per worker during the transition to the new long-run equilibrium
- Thus, changes in s and n have temporary effects only on the growth rates of k and y and
- permanent effects on the long-run levels of k and y.
- BUT a change to the exogenous rate of technological progress will have permanent effects on the growth rates of k and y.

Factor payments in a Solow Model

- What are the growth rates of w and r on a BGP?
- If the economy starts with $k < k*$, in the transition towards a BGP, w and r grow less, more or equal to their growth rates on the BGP?

Factor payments in a Solow Model

- If the economy starts with $k < k*$, in the transition towards a BGP, w and r grow less, more or equal to their growth rates on the BGP?
- Since: $Y = K^{\alpha}(AL)^{1-\alpha}$

$$r = \frac{dY}{dK} - \delta = \alpha \frac{Y}{K} - \delta = \alpha(\hat{k})^{\alpha - 1} - \delta$$

- Since α and δ are constants and since $\hat{k}$ is constant on the BGP
- r is also constant: $\frac{\dot{r}}{r} = (\alpha - 1)\frac{\dot{\hat{k}}}{\hat{k}} = 0$

Factor payments in a Solow Model

- Since α and δ are constants and since $\hat{k}$ is constant on the BGP
- r is also constant: $\frac{\dot{r}}{r} = (\alpha - 1)\frac{\dot{\hat{k}}}{\hat{k}} = 0$
- Since capital is paid its marginal product, the share of output going to capital is rK/Y
- On the BGP:

$$\frac{r\dot{K}/Y}{rK/Y} = \frac{\dot{r}}{r} + \frac{\dot{K}}{K} - \frac{\dot{Y}}{Y} = 0 + \gamma + n - \gamma - n = 0$$

- Thus, on the BGP, the share of output going to capital is constant.
- Since the shares of output going to capital and labor sum to one,
- $\rightarrow$ the share of output going to labor is also constant on the BGP.

Factor payments in a Solow Model

- We need to determine the growth rate of the marginal product of labor, w on the BGP
- Since: $Y = K^{\alpha}(AL)^{1-\alpha}$
- Taking the partial derivative of output with respect to L yields:

$$w = \frac{dY}{dL} = (1-\alpha)\frac{Y}{L} = A(1-\alpha)(\hat{k})^{\alpha}$$

- Taking the time derivative of the log of this expression yields the growth rate of the marginal product of labor:

$$\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \alpha\frac{\dot{\hat{k}}}{\hat{k}} = \gamma + 0 = \gamma$$

- On the BGP, the marginal product of labor rises at the rate of growth of the effectiveness of labor

Factor payments in a Solow Model

- if $\hat{k} < \hat{k}^*$, then in the adjustment path $\frac{\dot{\hat{k}}}{\hat{k}} > 0$, thus
  - w will grow at a rate larger than its BGP growth rate of γ
  - Since $\frac{\dot{w}}{w} = \frac{\dot{A}}{A} + \alpha \frac{\dot{\hat{k}}}{\hat{k}} = \gamma + \alpha \frac{\dot{\hat{k}}}{\hat{k}}$
  - r will grow at a rate smaller than its BGP growth rate of 0
  - Since $\frac{\dot{r}}{r} = (\alpha - 1)\frac{\dot{\hat{k}}}{k}$

Empirical evidence of the Solow model

- Evaluating the Solow model
- How the Solow model answers key questions of economic growth and development?
- (1) Why some countries are so rich and other so poor?

Empirical evidence of the Solow model

- Evaluating the Solow model
- (1) Why some countries are so rich and other so poor?
- The Solow model predicts that countries that are rich are those that invest more and have lower population growth
- $\rightarrow$ Allowing rich countries to accumulate more capital per worker
- and increase labor productivity.
- $\rightarrow$ It is supported by empirical analysis

Empirical evidence of the Solow model

- Evaluating the Solow model
- How the Solow model answers key questions of economic growth and development?
- (2) Why economies exhibit sustained growth in the Solow model?

Empirical evidence of the Solow model

- Evaluating the Solow model
- (2) Why economies exhibit sustained growth in the Solow model?
- Due to technological progress
- Technological progress offset the tendency for the marginal product of capital to fall (due to diminishing return to capital accumulation)
- In the long run countries exhibit per capita growth at the rate of technological progress
- Labor productivity grows because of investment in technology and indirectly due to the additional capital accumulation made possible by technological progress

Empirical evidence of the Solow model

- Evaluating the Solow model
- (3) How does the Solow model account for differences in growth rates across countries?

Empirical evidence of the Solow model

- Evaluating the Solow model
- (3) How does the Solow model account for differences in growth rates across countries?
- Based on the transition dynamics of the model
- countries can grow at rates different from their long run growth rates
- Example: an economy capital-technology ratio below its long run level will grow rapidly until the capital technology ratio reaches its steady-sate level.
- This explain why the rapid growth of Germany and Japan after the WWII when they had lower levels of capital stocks .

Empirical evidence of the Solow model

- Growth rates of output per worker (PIB per capita) are roughly constant over long periods of time
- The real interest rate or rate of return to capital is roughly constant over long periods of time

Empirical evidence of the Solow model

- The capital to output ratio is roughly constant over long periods of time
- Shares of capital and labor in national income are roughly stable over long periods of time
- But growth rates of productivity vary among countries.

Economic Growth

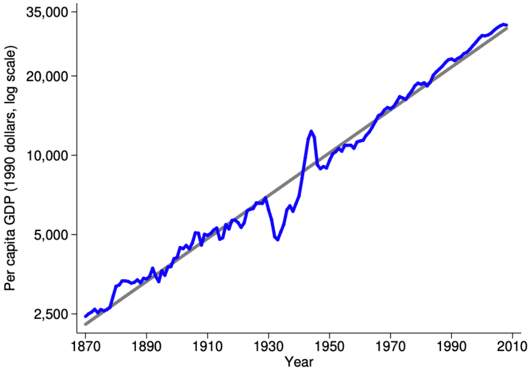
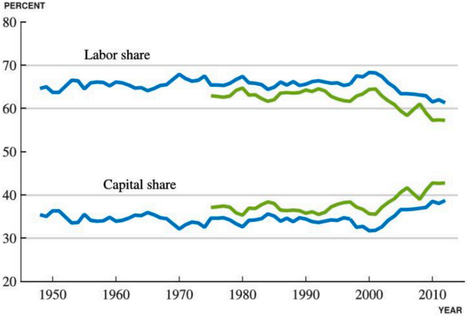


Figure 6: Capital and Labor Shares of Factor Payments, United States



Note: The series starting in 1975 are from Karabarbounis and Neiman (2014) and measure the factor shares for the corporate sector, which the authors argue is helpful in eliminating issues related to self-employment. The series starting in 1948 is from the Bureau of Labor Statistics *Multifactor Productivity Trends*, August 21, 2014, for the private business sector. The factor shares add to 100 percent.

Growth accounting

- The simple Cobb-Douglas version of the Solow model with exogenous technological change lends itself to a simple accounting of the contribution to output growth due to:
  - Factor accumulation (capital and labor) vs.
  - Technological change.

Growth accounting

- Solow published a second paper in 1957 "Technical Change and the Aggregate Production Function"
- He did a simple accounting exercise breaking down the growth of output into
- (1) Growth in capital
- (2) Growth in labor
- (3) Growth in technological change

Growth accounting

- Consider a Cobb-Douglas production function:

$$Y = BK^{\alpha}L^{\alpha-1}$$

- Where B is the Hicks-neutral Multi-factor Productivity

- Log-differentiating the above function, yields to:

$$\frac{\dot{Y}}{Y} = \frac{\dot{B}}{B} + \alpha\frac{\dot{K}}{K} + (\alpha-1)\frac{\dot{L}}{L}$$

- $\rightarrow$ Output growth is equal to a weighted average of capital and labor growth plus the growth of TFP.

- This equation has been used to understand the sources of growth in output.

Growth accounting

- Usually the interest is on output per worker rather than total output, thus dividing by L all variables we get:

$$\frac{\dot{y}}{y} = \frac{\dot{B}}{B} + \alpha \frac{\dot{k}}{k}$$

- $\rightarrow$ The growth rate of output per worker is decomposed into the contribution of physical capital per worker and TFP growth

Growth accounting

- The table in the next slide contains the growth accounting for the US from 1948 to 2010
- such calculation is officially done by the US Bureau of Labor Statistics (BLS).
- Labor input is computed as total number of hours rather than total employees.
- For the US α is found to be around $1/3$ thus $1-\alpha$ is around $2/3$
- this can be measured from national accounts data looking at wL/Y
- Where wL is total income for wage and salary earners and thus the complement to capital income.

Economic Growth

Annual average growth rates

	1948–2010	1948–73	1973–95	1995–2000	2000–2010
Output per hour	2.6	3.3	1.5	2.9	2.7
Contributions from:					
Capital per hour worked	1.0	1.0	0.7	1.2	1.2
Information technology	0.2	0.1	0.4	0.9	0.5
Other capital services	0.8	0.9	0.3	0.3	0.7
Labor composition	0.2	0.2	0.2	0.2	0.3
Multifactor productivity	1.4	2.1	0.6	1.5	1.3

- Output per hour grew at average annual rate of 2.6 between 1948-2010.
- The contribution from capital per hour was 1 p.p.
- The contribution of changing labor composition was 0.2 p.p
- The difference between output growth and traditional factors is attributed to the contribution of technical progress in TFP: 1.4 p.p. $(= 2.6 - 1.2)$

- The table also shows that GDP growth and its determinants (capital, labor and TFP growth) have change over the periods in the US
- Output growth slowdown determined by productivity growth slowdown in the period 1973-1995
- A rise in output growth as well as productivity growth in the period 1995-2010.
- How can we explain this growth?

Economic Growth

- How can we explain this growth?
- Some economists highlight the role of information technology
- Information-technology revolution (adoption of computers) can explain both the productivity slowdown as well as recent productivity growth:
- Growth slow temporarily while the factories were adapting to new production techniques and workers learnt to use the new information technology
- The recent productivity boom is then associated with successful widespread adoption of this new technology

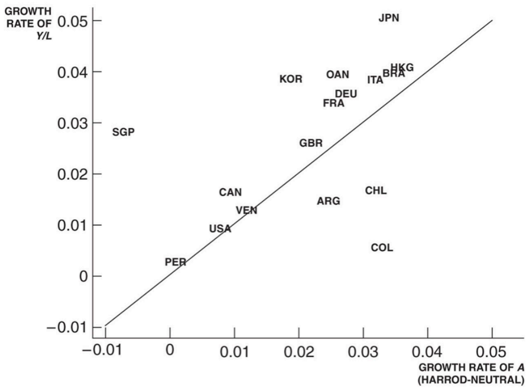
Economic Growth

- Because of the way in which TFP is computed it has been called "the Solow residual"
- It captures the growth in output that can not be explained by capital and labor growth
- Ina. sense it is a black box
- But what determines TFP growth and technological progress?

Growth accounting

- Figure in the next slide uses the growth accounting framework to identify the relative role of:
- factor accumulation vs technical change across countries over the period 1960-1990.
- Labor input is computed as total number of hours rather than total employees.
- Assume that $B = A^{1-\alpha}$
- In the graph, on a BGP output per worker and labor-augmenting technical change would grow at the same rate
- and thus observations would lie on the 45 degree line

Economic Growth



- If the actual observation lies above the line
- it means that factor accumulation contributed to growth beyond technical progress

Growth accounting

- In many fast growing Asian countries: factor accumulation contributed to growth beyond technical progress
- Singapore (SGP) is an extreme example, as growth of A is negative
- Which are the main determinants of growth in South Asian countries?