Optimization B:
Convex Analysis and Dynamics

Problem Sets*

QEM1 (First Year of the QEM Program)
M1 IMMAEF
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1 Some one-variable optimization problems

Exercise 1 Find the solution(s) of the following maximization problems, when it exists, and compute the values of these three problems.

$$(\mathcal{P}_1)\begin{cases} \max \sqrt{x} + 2\sqrt{c-x} \\ x \in [0,c] \end{cases}$$

where c is a positive real number.

$$(\mathcal{P}_2)\begin{cases} \max x^2 + 2(c-x) \\ x \in [0,c] \end{cases}$$

where c is a positive real number.

$$(\mathcal{P}_3)\begin{cases} \max ax - e^x \\ x \in \mathbb{R} \end{cases}$$

where a is a positive real number.

*The following exercises are borrowed from the textbook: *Further Mathematics for Economic Analysis*, by Sydsaeter K., Hammmond P., Seierstadt A., Strom A. (2005), hereafter **SHSS**. Exercises 1, 35, 38, 40, and 41 are borrowed from the previous teacher.

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2 Unconstrained optimization

Exercise 2 (**SHSS 3.1, 4**) Find the functions $x^*(r)$ and $y^*(r)$ such that $x = x^*(r)$ and $y = y^*(r)$ solve the problem:

$$\max_{x,y} f(x,y,r) = -x^2 - xy - 2y^2 + 2rx + 2ry$$

where r is a parameter.

Exercise 3 (**SHSS 3.1, 5**) Find the solutions $x^*(r,s)$ and $y^*(r,s)$ of the problem

$$\max_{x,y} f(x,y,r,s) = r^2x + 3s^2y - x^2 - 8y^2$$

where r and s are parameters.

Exercise 4 (**SHSS 3.2, 1**) The function

$$f(x_1,x_2,x_3) = x_1^2 + x_2^2 + 3x_3^2 - x_1x_2 + 2x_1x_3 + x_2x_3$$

defined on $\mathbb{R}^3$ has only one stationary point. Show that it is a local minimum point.

Exercise 5 (**SHSS 3.2, 2**) Let f be defined for all $(x,y) \in \mathbb{R}^2$ by

$$f(x,y) = x^3 + y^3 - 3xy$$

1. Show that $(0,0)$ and $(1,1)$ are the only stationary points of f, and compute the quadratic form associated with the Hessian matrix of f at the stationary points.
2. Check the definiteness of this quadratic form at the stationary points.
3. Classify the stationary points, local minimum, local maximum, saddle point.

Exercise 6 (**SHSS 3.2, 3**) Classify the stationary points of

(a) $f(x,y,z) = x^2 + x^2y + y^2z + z^2 - 4z$

(b) $f(x_1,x_2,x_3,x_4) = 20x_2 + 48x_3 + 6x_4 + 8x_1x_2 - 4x_1^2 - 12x_3^2 - x_4^2 - 4x_2^3$

Exercise 7 (**SHSS 3.2, 4**) Suppose $f(x,y)$ has only one stationary point (x^*,y^*) which is a local minimum point. Is (x^*,y^*) necessarily a global minimum point? It may be surprising that the answer is no. Prove this by examining the function defined for all $(x,y) \in \mathbb{R}^2$ by $f(x,y) = (1+y)^3x^2 + y^2$. (*Hint:* Look at $f(x,-2)$ as $x \to \infty$.)

3 Optimization with equality constraints

Exercise 8 (**SHSS 3.3, 1**)

1. Solve the problem: $\max -x^2 - y^2 - z^2$ subject to $x + 2y + z = a$, where a is a parameter.

2. Compute the value function $f^*(a)$ and verify that the derivative of the value function is equal to the Lagrange multiplier.

Exercise 9 (**SHSS 3.3, 2**)

1. Solve the problem:

$$\max x + 4y + z \text{ subject to } x^2 + y^2 + z^2 = 216 \text{ and } x + 2y + 3z = 0$$

2. Change the first constraint to $x^2 + y^2 + z^2 = 215$ and the second to $x + 2y + 3z = 0.1$. Estimate the corresponding change in the maximum value by using that the partial derivatives of the value function equal the Lagrange multipliers.

Exercise 10 (**SHSS 3.3, 3**)

1. Solve the problem:

$$\max e^x + y + z \text{ subject to } \begin{cases} x + y + z = 1 \\ x^2 + y^2 + z^2 = 1 \end{cases}$$

2. Replace the constraints by $x + y + z = 1.02$ and $x^2 + y^2 + z^2 = 0.98$. What is the approximate change in optimal value of the objective function?

Exercise 11 (**SHSS 3.3, 4**)

1. Solve the following utility maximizing problem, assuming $m \geq 4$.

$$\max U(x_1, x_2) = \tfrac{1}{2}\ln(1 + x_1) + \tfrac{1}{4}\ln(1 + x_2) \text{ subject to } 2x_1 + 3x_2 = m$$

2. With $U^*(m)$ as indirect utility function, show that $dU^*/dm = \lambda$.

Exercise 12 (**SHSS 3.3, 5**)

1. Solve the problem: $\max 1 - rx^2 - y^2$ subject to $x + y = m$, with $r > 0$.

2. Find the value function $f^*(r, m)$. Compute $\partial f^*/\partial r$ and $\partial f^*/\partial m$ and verify that they are equal to the partial derivatives of the Lagrangian $\partial \mathcal{L}/\partial r$ and $\partial \mathcal{L}/\partial m$ (respectively) computed at the solution.

Exercise 13 (**SHSS 3.3, 6**)

1. Solve the problem:

$$\max x^2 + y^2 + z^2 \text{ subject to } x^2 + y^2 + 4z^2 = 1 \text{ and } x + 3y + 2z = 0$$

2. Suppose we change the first constraint to $x^2 + y^2 + 4z^2 = 1.05$ and the second constraint to $x + 3y + 2z = 0.05$. Estimate the corresponding change in the value function.

Exercise 14 (**SHSS 3.3, 7**) Let $x = (x_1, ..., x_j, ..., x_n) \in \mathbb{R}^n$.

1. Let $U(x) = \sum_{j=1}^{n} \alpha_j \ln(x_j - a_j)$, where α_j, a_j, p_j, and m are all positive constants with $\sum_{j=1}^{n} \alpha_j = 1$, and with $m > \sum_{i=1}^{n} p_i a_i$. Consider the price system $p = (p_1, ..., p_j, ..., p_n)$ and show that if x^* solves:

   $\max U(x)$ subject to $p \cdot x = m$, $x_j \geq 0$, $j = 1, 2, \ldots, n$,

   then the expenditure on good j is the following linear function of prices and income

   $p_j x_j^* = \alpha_j m + p_j a_j - \alpha_j \sum_{i=1}^{n} p_i a_i$, $j = 1, 2, \ldots, n$.

2. Let $U^*(p, m) = U(x^*)$ denote the indirect utility function. Verify **Roy's identity**, i.e.,

$$\frac{\partial U^*}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = -\lambda x_i^*, \quad i = 1, \ldots, n$$

Exercise 15 (**SHSS 3.3, 8**)

1. Find the solution of the following problem by solving the constraints for x and y:

   minimize $x^2 + (y-1)^2 + z^2$ subject to $x + y = \sqrt{2}$ and $x^2 + y^2 = 1$

2. Note that there are three variables and two constraints (the variable z does not appear in the constraints). Show that the **constraint qualification condition** is not satisfied, and that there are no Lagrange multipliers for which the Lagrangian is stationary at the solution point.

Exercise 16 (**SHSS 3.3, 10**) Consider the problem:

$$\max_{x,r} f(x, r) \text{ subject to } \begin{cases} g_j(x, r) = 0, \ j = 1, \ldots, m \\ r_i = b_{m+i}, \ i = 1, \ldots, k \end{cases}$$

where f and $g_1, \ldots, g_m$ are given functions and $b_{m+1}, \ldots, b_{m+k}$ are fixed parameters. (We maximize f w.r.t. both $x = (x_1, \ldots, x_n)$ and $r = (r_1, \ldots, r_k)$, but with $r_1, \ldots, r_k$ completely fixed.) Define $\tilde{b} = (0, \ldots, 0, b_{m+1}, \ldots, b_{m+k})$ (there are m zeros). Prove that the partial derivative of the value function with respect to r_i is equal to the partial derivative of the Lagrangian with respect to r_i computed at the solution for $i = m+1, \ldots, m+k$ by using the fact that the multiplier equals the partial derivative of the value function and those first order condition for the optimization problem that refer to the variables r_i.

Exercise 17 (**SHSS 3.4, 1**)

1. Find the four points that satisfy the first order conditions for the problem:

   $\max(\min) x^2 + y^2$ subject to $4x^2 + 2y^2 = 4$

2. Compute the bordered Hessian determinant $B_2(x, y)$ of order 2 at the four points found in (1). What can you conclude?

3. Can you give a geometric interpretation of the problem?

Exercise 18 (**SHSS 3.4, 2**) Compute the bordered Hessian determinants B_2 and B_3 of order 2 and 3 for the problem:

$$\max\ (\min)\ x^2 + y^2 + z^2 \text{ subject to } x + y + z = 1.$$

Show that the second order conditions for a local minimum are satisfied.

Exercise 19 (**SHSS 3.4, 3**) Use the sufficient conditions on the bordered Hessian determinants to classify the candidates for optimality in the problem:

$$\text{local max (min) } x + y + z \text{ subject to } x^2 + y^2 + z^2 = 1 \text{ and } x - y - z = 1$$

4 Optimization with inequality constraints

Exercise 20 (**SHSS 3.5, 2**)

1. Consider the nonlinear programming problem (where c is a positive constant):

   $$\max \ln(x+1) + \ln(y+1) \text{ subject to } \begin{cases} x + 2y \leq c \\ x + y \leq 2 \end{cases}$$

   Write down the necessary Kuhn-Tucker conditions for a point (x, y) to be a solution of the problem.

2. Solve the problem for $c = 5/2$.

3. Let $V(c)$ denote the value function. Find the value of $V'(5/2)$.

Exercise 21 (**SHSS 3.5, 3**) Solve the following problem (assuming it has a solution): $\min 4\ln(x^2 + 2) + y^2$ subject to $x^2 + y \geq 2$, $x \geq 1$.
(*Hint:* Reformulate it as a standard Kuhn-Tucker maximization problem.)

Exercise 22 (**SHSS 3.5, 4**) Solve the problem: $\max -(x-a)^2 - (y-b)^2$ subject to $x \leq 1$, $y \leq 2$, for all possible values of the constants a and b. (A good check of the results is to use a geometric interpretation of the problem).

Exercise 23 (**SHSS 3.5, 6**)

(a) Find the only possible solution to the nonlinear programming problem:

$$\max x^5 - y^3 \text{ subject to } x \leq 1,\ x \leq y$$

(b) Solve the problem by using iterated optimization. That is, find first the maximum value $f(x)$ in the problem of maximizing $x^5 - y^3$ subject to $x \leq y$, where x is fixed and y varies. Then maximize $f(x)$ subject to $x \leq 1$.

Exercise 24 (**SHSS 3.6, 2**) Solve the problem:

$\max xy + x + y$ subject to $x^2 + y^2 \leq 2$, $x + y \leq 1$.

Exercise 25 (**SHSS 3.7, 1**)
(a) Solve the nonlinear programming problem (a and b are constants):

$\max 100 - e^{-x} - e^{-y} - e^{-z}$ subject to $x + y + z \leq a$, $x \leq b$

(b) Let $f^*(a, b)$ be the (optimal) value function. Compute the partial derivatives of f^* with respect to a and b, and relate them to the Lagrange multipliers.

(c) Put $b = 0$, and show that $F^*(a) = f^*(a, 0)$ is concave in a.

Exercise 26 (**SHSS 3.7, 3**)

1. Consider the problem:

$$\max(\min) x^2 + y^2 \text{ subject to } r^2 \leq 2x^2 + 4y^2 \leq s^2$$

   where $0 < r < s$. Solve the maximization problem and verify the Envelope Theorem in this case.

2. Can you give a geometric interpretation of the problem and its solution?

Exercise 27 (**SHSS 3.8, 2**) Solve the following nonlinear programming problems:

(a) $\max xy$ subject to $x + 2y \leq 2$, $x \geq 0$, $y \geq 0$

(b) $\max x^\alpha y^\beta$ subject to $x + 2y \leq 2$, $x > 0$, $y > 0$, where $\alpha > 0$ and $\beta > 0$, and $\alpha + \beta \leq 1$.

Exercise 28 (**SHSS 3.8, 3**)
(a) Solve the following problem for all values of the constant c:

$\max f(x, y) = cx + y$ subject to $g(x, y) = x^2 + 3y^2 \leq 2 \leq 2$, $x \geq 0$, $y \geq 0$

(b) Let $f^*(c)$ denote the value function. Verify that it is continuous. Check if the Envelope Theorem holds true.

Exercise 29 (**SHSS 3.8, 5**) A model for studying the export of gas from Russia to the rest of Europe involves the following optimization problem:

$$\max[x+y-\frac{1}{2}(x+y)^2-\frac{1}{4}x-\frac{1}{3}y] \text{ subject to } x \leq 5, y \leq 3, -x+2y \leq 2, x \geq 0, y \geq 0$$

Sketch the admissible set S in the xy-plane, and show that the maximum cannot occur at an interior point of S. Solve the problem.

5 Finite horizon dynamic programming

Exercise 30 (**SHSS 12.1, 1**)
(a) Solve the problem:

$$\max \sum_{t=0}^{2} [1 - (x_t^2 + 2u_t^2)], \ x_{t+1} = x_t - u_t, \ t = 0, 1 \tag{1}$$

where $x_0 = 5$ and $u_t \in \mathbb{R}$. (Compute $J_s(x)$ and $u_s^*(x)$ for $s = 2, 1, 0$).

(b) Use the difference equation $x_{t+1} = x_t - u_t$ to compute x_1 and x_2 in terms of u_0 and u_1 (with $x_0 = 5$), and find the sum in (1) as a function S of u_0, u_1, and u_2. Next, maximize this function as in Example 2, Section 12.1 of SHSS.

Exercise 31 (**SHSS 12.1, 2**) Consider the problem:

$$\max_{u_t \in [0,1]} \sum_{t=0}^{T} \left(\frac{1}{1+r} \right)^t \sqrt{u_t x_t}, \ x_{t+1} = \rho(1 - u_t) x_t, \ t = 0, 1, \ldots, T-1, \ x_0 > 0$$

where r is the rate of discount. Compute $J_s(x)$ and $u_s^*(x)$ for $s = T, T-1, T-2$.

Exercise 32 (**SHSS 12.1, 4**) Consider the problem:

$$\max_{u_t \in [0,1]} \sum_{t=0}^{T} (3 - u_t) x_t^2, \ x_{t+1} = u_t x_t, \ t = 0, 1, \ldots, T-1, \ x_0 \text{ is given}$$

(a) Compute the value functions $J_T(x)$, $J_{T-1}(x)$, $J_{T-2}(x)$, and the corresponding control function $u_T^*(x)$, $u_{T-1}^*(x)$ and $u_{T-2}^*(x)$.

(b) Find an expression for $J_{T-n}(x)$ for $n = 0, 1, \ldots, T$, and the corresponding optimal controls.

Exercise 33 (**SHSS 12.1, 5**) Solve the problem:

$$\max_{u_t \in [0,1]} \sum_{t=0}^{T-1} \left(-\frac{2}{3} u_t \right) + \ln x_T, \ x_{t+1} = x_t(1 + u_t), \ t = 0, 1, \ldots, T-1, \ x_0 > 0 \text{ given}$$

Exercise 34 (**SHSS 12.1, 7**)
(a) Consider the problem:

$$\max_{u_t \in \mathbb{R}} \sum_{t=0}^{T-1} \left(-e^{-\gamma u_t} \right) - \alpha e^{-\gamma x_T}, \ x_{t+1} = 2x_t - u_t, \ t = 0, 1, \ldots, T-1, \ x_0 \text{ given}$$

where α and γ are positive constants. Compute $J_T(x)$, $J_{T-1}(x)$, and $J_{T-2}(x)$.

(b) Prove that $J_t(x)$ written in the form $J_t(x) = -\alpha_t e^{-\gamma x}$, and find a difference equation for α_t.

Exercise 35 Compute the solution of the following problem when $u(c) = \sqrt{c}$ and $u(c) = \ln(c)$.

$$\begin{cases} \max\ u(c_1) + \beta u(c_2) \\ c_2 = (1+r)(w_0 - c_1) \\ c_1 \geq 0,\ c_2 \geq 0 \end{cases}$$

where $w_0 > 0$ is given.

6 Stationary dynamic programming

Exercise 36 (**SHSS 12.3, 1**) Consider the problem:

$$\max_{u_t \in \mathbb{R}} \sum_{t=0}^{\infty} \beta^t \left(-e^{-u_t} - \frac{1}{2} e^{-x_t} \right),\ x_{t+1} = 2x_t - u_t,\ t = 0, 1, \ldots,\ x_0 \text{ given}$$

where $\beta \in\ ]0, 1[$. Find a constant $\alpha > 0$ such that $J(x) = -\alpha e^{-x}$ solves the Bellman equation, and show that α is unique.

Exercise 37 (**SHSS 12.3, 2**)

(a) Consider the following problem with $\beta \in\ ]0, 1[$:

$$\max_{u_t \in \mathbb{R}} \sum_{t=0}^{\infty} \beta^t \left(-\frac{2}{3} x_t^2 - u_t^2 \right),\ x_{t+1} = x_t + u_t,\ t = 0, 1, \ldots,\ x_0 \text{ given}$$

Suppose that $J(x) = -\alpha x^2$ solves the Bellman equation. Find a quadratic equation for α. Then find the associated value of u^*.

(b) By looking at the objective function, show that, given any starting value x_0, it is reasonable to ignore any policy that fails to satisfy both $|x_t| \leq |x_{t-1}|$ and $|u_t| \leq |x_{t-1}|$ for $t = 1, 2, \ldots$

Exercise 38 Consider the following macroeconomic model.[1]

$$(R) \begin{cases} \max \sum_{t=0}^{\infty} \beta^t u(c_t) \\ k_{t+1} = F(k_t) - c_t,\ t = 0, 1, \ldots, +\infty \\ c_t \geq 0,\ k_t \geq 0,\ t = 0, 1, \ldots, +\infty \end{cases}$$

with $k_0 > 0$. The basic assumptions are as follows.

1. $\beta \in\ ]0, 1[$, u is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $u(0) = 0$, u is $\mathcal{C}^2$ on $\mathbb{R}_{++}$, strictly concave and strictly increasing on $\mathbb{R}_+$ with $u'(c) > 0$ for all $c > 0$, and u satisfies the following Inada condition:

$$\lim_{c \to 0} u'(c) = +\infty$$

[1]This example can be found in Le Van, C., and Dana, R.-A. (2003), *Dynamic Programming in Economics*, Kluwer Academic Publishers.

2. The production function F is a continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ with $F(0) = 0$, F is $\mathcal{C}^2$ on $\mathbb{R}_{++}$, strictly concave and strictly increasing on $\mathbb{R}_+$ with $F'(k) > 0$ for all $k > 0$, and F satisfies the following conditions:

$$\lim_{k \to 0} F'(k) = M > 0 \text{ or } +\infty, \text{ and } \lim_{k \to +\infty} F'(k) < 1$$

(a) Write the first order necessary conditions for any interior solution (c_t^*, k_t^*), that is $c_t^* > 0$ and $k_t^* > 0$ for all t.

(b) Deduce the following Euler equations from the first order conditions above.

$$\beta u'(c_{t+1}^*) F'(k_{t+1}^*) = u'(c_t^*)$$

(c) Show that an optimal solution is always an interior solution as a consequence of the Inada condition $u'(0) = +\infty$.

Exercise 39 Consider the stationary optimization problem:

$$(D) \begin{cases} \max \sum_{t=0}^{\infty} \beta^t f(a_t, s_t) \\ s_{t+1} = g(a_t, s_t),\ t = 0, 1, \ldots, +\infty \\ a_t \in A,\ t = 0, 1, \ldots, +\infty \\ s_t \in S,\ t = 0, 1, \ldots, +\infty \end{cases}$$

Assume that the value function V of problem (D) is a well defined function on some interval $\mathcal{I} \subseteq S$. Prove that V is concave on $\mathcal{I}$ if the following three properties holds true.

(i) The sets $A \subseteq \mathbb{R}$ and $S \subseteq \mathbb{R}$ are convex.

(ii) If $s \in S$ and $\widetilde{s} \geq s$, then $\widetilde{s} \in S$.

(iii) The functions f and g are concave on $A \times S$ and increasing with respect to the state variable s.

Exercise 40 Consider the macroeconomic model and the maximization problem (R) given in Exercise 38.

(a) Verify that the above three properties (i), (ii), and (iii) are satisfied.

(b) Show that, at an interior solution, one gets:

$$V'(k_0^*) = u'(c_0^*) F'(k_0^*)$$

Exercise 41 (*Steady state*) We consider the Bellman equation and we denote $\alpha(s_0)$ the optimal solution given s_0. A fixed point s^* of $g(\alpha(\cdot), \cdot)$ is called a *steady state*. Show that if $s_0 = s^*$, then the optimal solution of the problem is the constant sequence $(\alpha(s^*), s^*)_{t \in \mathbb{N}}$.