Chapter 1 Introduction

1. EXERCISE 5.3 (Altruistic preferences) Person 1 cares about both her income and person 2's income. Precisely, the value she attaches to each unit of her own income is the same as the value she attaches to any two units of person 2 's income. For example, she is indifferent between a situation in which her income is 1 and person 2's is 0 , and one in which her income is 0 and person 2 's is 2 . How do her preferences order the outcomes (1,4),(2,1), and (3,0), where the first component in each case is her income and the second component is person 2's income? Give a payoff function consistent with these preferences.

2. EXERCISE 6.1 (Alternative representations of preferences) A decision-maker's preferences over the set $A = a, b, c$ are represented by the payoff function u for which $u(a) = 0$, $u(b) = 1$, and $u(c) = 4$. Are they also represented by the function v for which $v(a) = -1, v(b) = 0$, and $v(c) = 2$? How about the function w for which $w(a) = w(b) = 0$ and $w(c) = 8$?

Chapter 2 Nash Equilibrium

1. EXERCISE 16.1 (Working on a joint project) Formulate a strategic game that models a situation in which two people work on a joint project in the case that their preferences are the same as those in the game in Figure 16.1 except that each person prefers to work hard than to goof off when the other person works hard. Present your game in a table like the one in Figure 16.1.

	Work hard	Goof off
Work hard	$2, 2$	$0, 3$
Goof off	$3, 0$	$1, 1$

Figure 16.1 Working on a joint project

2. EXERCISE 17.1 (Games equivalent to the Prisoner's Dilemma) Determine whether each of the games in Figure 17.1 differs from the Prisoner's Dilemma only in the names of the players' actions, or whether it differs also in one or both of the players' preferences.

	X	Y
X	$3, 3$	$1, 5$
Y	$5, 1$	$0, 0$

	X	Y
X	$2, 1$	$0, 5$
Y	$3, -2$	$1, -1$

3. EXERCISE 18.1 (Hermaphroditic fish) Members of some species of hermaphroditic fish choose, in each mating encounter, whether to play the role of a male or a female. Each fish has a preferred role, which uses up fewer resources and hence allows more future mating. A fish obtains a payoff of H if it mates in its preferred role and L if it mates in the other role, where $H > L$. (Payoffs are measured in terms of number of offspring, which fish are evolved to maximize.) Consider an encounter between two fish whose preferred roles are the same. Each fish has two possible actions: mate in either role or insist on its preferred role. If both fish offer to mate in either role, the roles are assigned randomly, and each fish's payoff is $1/2(H + L)$ (the average of H and L). If each fish insists on its preferred role, the fish do not mate; each goes off in search of another partner, and obtains the payoff S. The higher the chance of meeting another partner, the larger is S. Formulate this, situation as a strategic game and determine the

range of values of S, for any given values of H and L, for which the game differs from the Prisoner's Dilemma only in the names of the actions.

4. EXERCISE 20.1 (Games without conflict) Give some examples of two-player strategic games in which each player has two actions and the players have the same preferences, so that there is no conflict between their interests. (Present your games as tables like the one in Figure 19.2.)

	Head	Tail
Head	$1, -1$	$-1, 1$
Tail	$-1, 1$	$1, -1$

Figure 19.2 Matching Pennies

5. EXERCICE 27.1 (Variant of Prisoner's Dilemma with altruistic preferences) Each of two players has two possible actions, *Quiet* and *Fink*; each action pair results in the players' receiving amounts of *money* equal to the numbers corresponding to that action pair in Figure 26.1. (For example, if player 1 chooses *Quiet* and player 2 chooses *Fink*, then player 1 receives nothing, whereas player 2 receives \$3.) The players are not "selfish"; rather, the preferences of each player i are represented by the payoff function $m_i(a) + \alpha m_j(a)$, where $m_i(a)$ is the amount of money received by player i when the action profile is a, j is the other player, and α is a given nonnegative number. Player 1's payoff to the action pair (Quiet, Quiet), for example, is $2 + 2\alpha$.
   (a) Formulate a strategic game that models this situation in the case $\alpha = 1$. Is this game the Prisoner's Dilemma?
   (b) Find the range of values of α for which the resulting game is the Prisoner's Dilemma. For values of α for which the game is not the Prisoner's Dilemma, find the Nash equilibria.

6. EXERCISE 27.2 (Selfish anh altruistic social behavior) Two people enter a bus. Two adjacent cramped seats are free. Each person must decide whether to sit or stand. Sitting alone is more comfortable than sitting next to the other person, which is more comfortable than standing.
   (a) Suppose that each person cares only about her own comfort. Model the situation as a strategic game. Is this game the Prisoner's Dilemma? Find its Nash equilibrium (equilibria?).
   (b) Suppose that each person is altruistic, ranking the outcomes according to the other person's comfort, but, out of politeness, prefers to stand than to sit if the other person stands. Model the situation as a strategic game. Is this game the Prisoner's Dilemma? Find its Nash equilibrium (equilibria?).
   (c) Compare the people's comfort in the equilibria of the two games.

7. EXERCICE 30.1 (Variants of the Stag Hunt) Consider variants of the n-hunter Stag Hunt in which only m hunters, with $2 \leq m < n$, need to pursue the stag in order to catch it. (Continue to assume that there is a single stag.) Assume that a captured stag is shared only by the hunters who catch it. Under each of the following assumptions on the hunters' preferences, find the Nash equilibria of the strategic game that models the situation.

(a) As before, each hunter prefers the fraction $1/n$ of the stag to a hare.

(b) Each hunter prefers the fraction $1/k$ of the stag to a hare, but prefers a hare to any smaller fraction of the stag, where k is an integer with $m \leq k \leq n$.

8. EXERCICE 31.1 (Extension of the Stag Hunt) Extend the n-hunter Stag Hunt by giving each hunter K (a positive integer) units of effort, which she can allocate between pursuing the stag and catching hares. Denote the effort hunter i devotes to pursuing the stag by e_i, a nonnegative integer equal to at most K. The chance that the stag is caught depends on the smallest of all the hunters' efforts, denoted $\min_j e_j$. ("A chain is as strong as its weakest link.") Hunter i 's payoff to the action profile $(e_1, \ldots, e_n)$ is $2\min_j e_j - e_i$. (She is better off the more likely the stag is caught, and worse off the more effort she devotes to pursuing the stag, which means she catches fewer hares.) Is the action profile ($e,, \ldots, e$), in which every hunter devotes the same effort to pursuing the stag, a Nash equilibrium for any value of e ? (What is a player's payoff to this profile? What is her payoff if she deviates to a lower or higher effort level?) Is any action profile in which not all the players' effort levels are the same a Nash equilibrium? (Consider a player whose effort exceeds the minimum effort level of all players. What happens to her payoff if she reduces her effort level to the minimum?)

9. EXERCICE 31.2 (Hawk-Dove) Two animals are fighting over some prey. Each can be passive or aggressive. Each prefers to be aggressive if its opponent is passive, and passive if its opponent is aggressive; given its own stance, it prefers the outcome in which its opponent is passive to that in which its opponent is aggressive. Formulate this situation as a strategic game and find its Nash equilibria.

10. EXERCISE 33.1 (Contributing to a public good) Each of n people chooses whether to contribute a fixed amount toward the provision of a public good. The good is provided if and only if at least k people contribute, where $2 \leq k \leq n$; if it is not provided, contributions are not refunded. Each person ranks outcomes from best to worst as follows: (i) any outcome in which the good is provided and she does not contribute, (ii) any outcome in which the good is provided and she contributes, (iii) any outcome in which the good is not provided and she does not contribute, (iv) any outcome in which the good is not provided and she contributes. Formulate this situation as a strategic game and find its Nash equilibria. (Is there a Nash equilibrium in which more than k people contribute? One in which k people contribute? One in which fewer than k people contribute? (Be careful!))

11. EXERCISE 34.1 (Guessing two-thirds of the average) Each of three people announces an integer from 1 to K. If the three integers are different, the person whose integer is closest to $\frac{2}{3}$ of the average of the three integers wins \$1. If two or more integers are the same, \$1 is split equally between the people whose integer is closest to $\frac{2}{3}$ of the average integer. Is there any integer k such that the action profile (k, k, k), in which every person announces the same integer k, is a Nash equilibrium? (If $k \geq 2$, what happens if a person announces a smaller number?) Is any other action profile a Nash equilibrium? (What is the payoff of a person whose number is

the highest of the three? Can she increase this payoff by announcing a different number?)

12. EXERCISE 34.2 (Voter participation) Two candidates, A and B, compete in an election. Of the n citizens, k support candidate A and $m(= n - k)$ support candidate B. Each citizen decides whether to vote, at a cost, for the candidate she supports, or to abstain. A citizen who abstains receives the payoff of 2 if the candidate she supports wins, 1 if this candidate ties for first place, and 0 if this candidate loses. A citizen who votes receives the payoffs $2-c, 1-c$, and $-c$ in these three cases, where $0 < c < 1$.

    (a) For $k = m = 1$, is the game the same (except for the names of the actions) as any considered so far in this chapter?

    (b) For $k = m$, find the set of Nash equilibria. (Is the action profile in which everyone votes a Nash equilibrium? Is there any Nash equilibrium in which the candidates tie and not everyone votes? Is there any Nash equilibrium in which one of the candidates wins by one vote? Is there any Nash equilibrium in which one of the candidates wins by two or more votes?)

    (c) What is the set of Nash equilibria for $k < m$?

13. EXERCISE 34.3 (Choosing a route) Four people must drive from A to B at the same time. Each of them must choose a route. Two routes are available, one via X and one via Y. (Refer to the left panel of Figure 35.1.) The roads from A to X, and from Y to B are both short and narrow; in each case, one car takes 6 minutes, and each additional car increases the travel time per car by 3 minutes. (If two cars drive from A to X, for example, each car takes 9 minutes.) The roads from A to Y, and from X to B are long and wide; on A to Y one çar takes 20 minutes, and each additional car increases the travel time per car by 1 minute; on X to B one car takes 20 minutes, and each additional car increases the travel time per car by 0.9 minutes. Formulate this situation as a strategic game and find the Nash equilibria. (If all four people take one of the routes, can any of them do better by taking the other route? What if three take one route and one takes the other route, or if two take each route?)

    Now suppose that a relatively short, wide road is built from X to Y, giving each person four options for travel from A to $B: A-X-B, A-Y-B, A-X-Y-B$, and $A-Y-$ X-B. Assume that a person who takes $A-X-Y-B$ travels the $A-X$ portion at the same time as someone who takes $A-X-B$, and the $Y-B$ portion at the same time as someone who takes $A-Y-B$. (Think of there being constant flows of traffic.) On the road between X and Y, one car takes 7 minutes and each additional car increases the travel time per car by 1 minute. Find the Nash equilibria in this new situation. Compare each person's travel time with her travel time in the equilibrium before the road from X to Y was built.

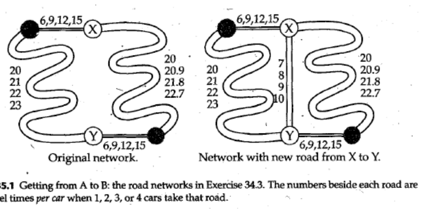

Figure 35.1 Getting from A to B: the road networks in Exercise 34.3. The numbers beside each road are the travel times *per car* when 1, 2, 3, or 4 cars take that road.

14. EXERCISE 37.1 (Finding Nash equi-

libria using best response functions)

(a) Find the players' best response functions in the Prisoner's Dilemma (Figure 15.1), BoS (Figure 19.1), Matching Pennies (Figure 19.2), and the two-player Stag Hunt (left panel of Figure 21.1) (and verify the Nash equilibria of each game).

(b) Find the Nash equilibria of the game in Figure 38.1 by finding the players' best response functions.

15. EXERCISE 38.2 (Dividing money) Two people have \$10 to divide between themselves. They use the following procedure. Each person names a number of dollars (a nonnegative integer), at most equal to 10 . If the sum of the amounts that the people name is at most 10 , then each person receives the amount of money she named (and the remainder is destroyed). If the sum of the amounts that the people name exceeds 10 and the amounts named are different, then the person who named the smaller amount receives that amount and the other person receives the remaining money. If the sum of the amounts that the people name exceeds 10 and the amounts named are the same, then each person receives \$5. Determine the best response of each player to each of the other player's actions, plot them in a diagram like Figure 38.2, and thus find the Nash equilibria of the game.

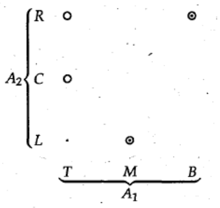

Figure 38.2 The players' best response functions for the game in Figure 37.1. Player 1's best responses are indicated by circles, and player 2's by dots. The action pairs for which there is both a circle and a dot are the Nash equilibria.

16. EXERCISE 41.1 (Strict and nonstrict Nash equilibria) Which of the Nash equilibria of the game whose best response functions are given in Figure 41.1 are strict (see the definition on page 33)?

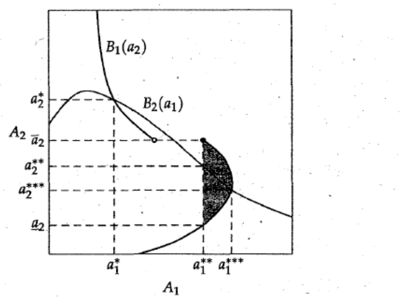

Figure 41.1 An example of the best response functions of a two-player game in which each player's set of actions is an interval of numbers. The set of Nash equilibria of the game consists of the pairs of actions (a_1^*, a_2^*) and (a_1^{***}, a_2^{***}), and all the pairs of actions on player 2's best response function between (a_1^{**}, a_2^{**}) and (a_1^{***}, a_2^{***}).

17. EXERCISE 42.1 (Finding Nash equilibria using best response functions) Find the Nash equilibria of the two-player strategic game in which each player's set of actions is the set of nonnegative numbers and the players' payoff functions are $u_1(a_1, a_2) = a_1(a_2 - a_1)$ and $u_2(a_1, a_2) = a_2(1 - a_1 - a_2)$.

18. EXERCISE 42.2 (A joint project) Two people are engaged in a joint project. If each person i puts in the effort x_i, a nonnegative number equal to at most 1 , which costs her $c(x_i)$, the outcome of the project is worth $f(x_1, x_2)$. The worth of the project is split equally between the two people, regardless of their effort levels. Formulate this situation as a strategic game. Find the Nash equilibria of the game when (a) $f(x_1, x_2) = 3x_1x_2$ and $c(x_i) = x_i^2$ for $i = 1, 2$, and (b) $f(x_1, x_2) = 4x_1x_2$ and $c(x_i) = x_i$ for $i = 1, 2$. In each case, is there a pair of effort levels that yields higher payoffs for both players than do the Nash equilibrium effort levels?

19. EXERCISE 44.1 (Contributing to a public good) Consider the model in

this section when $u_i(c_1, c_2)$ is the sum of three parts: the amount $c_1 + c_2$ of the public good provided, the amount $w_i - c_i$ person i spends on private goods, and a term $(w_i - c_i)(c_1 + c_2)$ that reflects an interaction between the amount of the public good and her private consumption-the greater the amount of the public good, the more she values her private consumption. In summary, suppose that person i 's payoff is $c_1+c_2+w_i-c_i+(w_i - c_i)(c_1 + c_2)$, or

$$w_i + c_j + (w_i - c_i)(c_1 + c_2)$$

where j is the other person. Assume that $w_1 = w_2 = w$, and that each player i'**s** contribution c_i may be any number (positive or negative, possibly larger than w). Find the Nash equilibrium of the game that models this situation. (You can calculate the best responses explicitly. Imposing the sensible restriction that c_i lie between 0 and w complicates the analysis but does not change the answer.) Show that in the Nash equilibrium both players are worse off than they are when both contribute half of their wealth to the public good. If you can, extend the analysis to the case of n people. As the number of people increases, how does the total amount contributed in a Nash equilibrium change? Compare the players' equilibrium payoffs with their payoffs when each contributes half her wealth to the public good, as n increases without bound. (The game is studied further in Exercise 388.1.)

20. EXERCISE 47.1 (Strict equilibria and dominated actions) For the game in Figure 48.1, determine, for each player, whether any action is strictly dominated or weakly dominated. Find the Nash equilibria of the game; determine whether any equilibrium is strict.

	L	C	R
L	0, 0	1, 0	1, 1
C	1, 1	1, 1	3, 0
R	1, 1	2, 1	2, 2

Figure 48.1

21. EXERCISE 47.2 (Nash equilibrium and weakly dominated actions) Give an example of a two-player strategic game in which each player has finitely many actions and in the only Nash equilibrium both players' actions are weakly dominated.

22. EXERCISE 49.1 (Voting between three candidates) Suppose there are three candidates, A, B, and C, and no citizen is indifferent between any two of them. A tie for first place is possible in this case; assume that a citizen who prefers a win by x to a win by y ranks a tie between x and y between an outright win for x and an outright win for y. Show that a citizen's only weakly dominated action is a vote for her least preferred candidate. Find a Nash equilibrium in which some citizen does not vote for her favorite candidate, but the action she takes is not weakly dominated.

23. EXERCISE 49.2 (Approval voting) In the system of "approval voting", a citizen" may vote for as many candidates as she wishes. If there are two candidates, say A and B, for example, a citizen may vote for neither candidate, for A, for B, or for both A and B. As before, the candidate who obtains the most votes wins. Show that any action that includes a vote for a citizen's least preferred candidate is weakly dominated, as is any action that does not

include a vote for her most preferred candidate. More difficult: show that if there are k candidates, then for a citizen who prefers candidate 1 to candidate 2 to ... to candidate k, the action that consists of votes for candidates 1 and $k-1$ is not weakly dominated.

24. EXERCISE 52.2 (Equilibrium for pairwise interactions in a single population) Find all the Nash equilibria of the game in Figure 53.1. Which of the equilibria, if any, correspond to a steady state if the game models pairwise interactions between the members of a single population?

	A	B	C
A	$1, 1$	$2, 1$	$4, 1$
B	$1, 2$	$5, 5$	$3, 6$
C	$1, 4$	$6, 3$	$0, 0$

Figure 53.1