Chapter 3 Nash Equilibrium : Illustrations

1. EXERCISE 58.1 (Cournot's duopoly game with linear inverse demand and different unit costs) Find the Nash equilibrium of Cournot's game when there are two firms, the inverse demand function is given by :

$$P(Q)=\begin{cases}\alpha-Q & \text{si } Q\leq\alpha,\\ 0 & \text{si } Q\geq\alpha.\end{cases}$$

   the cost function of each firm i is $C_i(q_i)=c_iq_i$, where $c_1>c_2$, and $c_1<\alpha$. (There are two cases, depending on the size of c_1 relative to c_2.) Which firm produces more output in an equilibrium? What is the effect of technical change that lowers firm 2's unit cost c_2 (while not affecting firm 1 's unit cost c_1) on the firms' equilibrium outputs, the total output, and the price?

2. EXERCISE 59.1 (Cournot's duopoly game with linear inverse demand and a quadratic cost function) Find the Nash equilibrium of Cournot's game when there are two firms, the inverse demand function is given by

$$P(Q)=\begin{cases}\alpha-Q & \text{si } Q\leq\alpha,\\ 0 & \text{si } Q\geq\alpha.\end{cases}$$

   and the cost function of each firm i is $C_i(q_i)=q_i^2$.

   In the next exercise each firm's cost function has a component that is independent of output. You will find in this case that Cournot's game may have more than one Nash equilibrium.

3. EXERCISE 59.2 (Cournot's duopoly game with linear inverse demand and a fixed cost) Find the Nash equilibria of Cournot's game when there are two firms, the inverse demand function is given by

$$P(Q)=\begin{cases}\alpha-Q & \text{si } Q\leq\alpha,\\ 0 & \text{si } Q\geq\alpha.\end{cases}$$

   and the cost function of each firm i is given by

$$C_i(q_i)=\begin{cases}0 & \text{if } q_i=0\\ f+cq_i & \text{if } q_i>0\end{cases}$$

   where $c\geq 0, f>0$, and $c<\alpha$. (Note that the fixed cost f affects only the firm's decision of whether to operate; it does not affect the output a firm wishes to produce if it wishes to operate.)

4. EXERCISE 60.1 (Variant of Cournot's duopoly game with market-share-maximizing firms) Find the Nash equilibrium (equilibria?) of a variant of the example of Cournot's duopoly game that differs from the one in this section (linear inverse demand, constant unit cost) only in that one of the two firms chooses its output to maximize its market share subject to not making a loss, rather than to maximize its profit. What happens if each firm maximizes its market share?

5. EXERCISE 60.2 (Nash equilibrium of Cournot's duopoly game and collusive outcomes) Find the total output (call it Q^*) that maximizes the firms' total profit in Cournot's game when there are two firms and retain the assumptions that the inverse demand function takes the linear form and the cost function of each firm i is $C_i(q_i)=cq_i$ for all q_i, with $c<\alpha$. Compare $\frac{1}{2}Q^*$ with each firm's output in the Nash equilibrium, and show that each firm's equilibrium profit is less than its profit in the "collusive" outcome in which each firm produces $\frac{1}{2}Q^*$. Why is this collusive outcome not a Nash equilibrium?

6. EXERCISE 61.1 (Cournot's game with many firms) Consider Cournot's game in the case of an arbitrary number n of firms; retain the assumptions that the inverse demand function takes the linear form and the cost function of each firm i is $C_i(q_i) = cq_i$ for all q_i, with $c < \alpha$. Find the best response function of each firm and set up the conditions for $(q_1^*, \ldots, q_n^*)$ to be a Nash equilibrium (see (36.3)), assuming that there is a Nash equilibrium in which all firms' outputs are positive. Solve these equations to find the Nash equilibrium. (For $n = 2$ your answer should be $\left(\frac{1}{3}(\alpha - c), \frac{1}{3}(\alpha - c)\right)$, the equilibrium found in Section 3.1.3. First show that in an equilibrium all firms produce the same output, then solve for that output. (If you cannot show that all firms produce the same output, simply assume that they do.) Find the price at which output is sold in a Nash equilibrium and show that this price decreases as n increases, approaching c as the number of firms increases without bound.

7. EXERCISE 62.1 (Nash equilibrium of Cournot's game with small firms) Suppose that there are infinitely many firms, all of which have the same cost function C. Assume that $C(0) = 0$, and for $q > 0$ the function $C(q)/q$ has a unique minimizer q; denote the minimum of $C(q)/q$ by p. Assume that the inverse demand function $\bar{P}$ is decreasing. Show that in any Nash equilibrium the firms' total output $\dot{Q}^*$ satisfies

$$P\left(Q^* + \underline{q}\right) \leq \underline{p} \leq P\left(Q^*\right)$$

(That is, the price is at least the minimal value $\underline{p}$ of the average cost, but is close enough to this minimum that increasing the total output of the firms by q would reduce the price to at most $\underline{p}$.) To establish these inequalities, show that if $P(Q^*) < \underline{p}$. or $P\left(Q^* + \underline{q}\right) > \underline{p}$, then Q^* is not the total output of the firms in a Nash equilibrium, because in each case at least one firm can deviate and increase its profit.

8. EXERCISE 63.1 (Interaction among resource users) A group of n firms uses a common resource (a river or a forest, for example) to produce output. As more of the resource is used, any given firm can produce less output. Denote by x_i the amount of the resource used by firm $i (= 1, \ldots, n)$. Assume specifically that firm i 's output is $x_i(1 - (x_1 + \cdots + x_n))$ if $x_1 + \cdots + x_n \leq 1$, and zero otherwise. Each firm i chooses x_i to maximize its output. Formulate this situation as a strategic game. Find values of α and c such that the game is the same as the one studied in Exercise 61.1, and hence find its Nash equilibria. Find an action profile $(x_1, \ldots, x_n)$ at which each firm's output is higher than it is at the Nash equilibrium.

9. EXERCISE 67.1 (Bertrand's duopoly game with constant unit cost) Consider the extent to which the analysis depends upon the demand function D taking the specific form $D(p) = \alpha - p$. Suppose that D is any function for which $D(p) \geq 0$ for all p and there exists $\bar{p} > c$ such that $D(p) > 0$ for all $p \leq \bar{p}$. Is (c, c) still a Nash equilibrium? Is it still the only Nash equilibrium?

10. EXERCISE 67.2 (Bertrand's duopoly game with discrete prices) Consider the variant of the example of Bertrand's duopoly game in this section in which each firm is restricted to choose a price that is an integral number of cents. Take the monetary unit to be a cent, and assume that c is an integer and

$\alpha > c + 1$. Is (c, c) a Nash equilibrium of this game? Is there any other Nash equilibrium?

11. EXERCISE 68.1 (Bertrand's oligopoly game) Consider Bertrand's oligopoly game when the cost and demand functions satisfy the conditions in Section 3.2.2 and there are n firms, with $n \geq 3$. Show that the set of Nash equilibria is the set of profiles $(p_1, \ldots, p_n)$ of prices for which $p_i \geq c$ for all i and at least two prices are equal to c. (Show that any such profile is a Nash equilibrium, and that every other profile is not a Nash equilibrium.)

12. EXERCISE 68.2 (Bertrand's duopoly game with different unit costs) Consider Bertrand's duopoly game under a variant of the assumptions of Section 3.2.2 in which the firms' unit costs are different, equal to c_1 and c_2, where $c_1 < c_2$. Denote by p_1^m the price that maximizes $(p - c_1)(\alpha - p)$, and assume that $c_2 < p_1^m$ and that the function $(p - c_1)(\alpha - p)$ is increasing in p up to p_1^m.

13. a. Suppose that the rule for splitting up consumers when the prices are equal assigns all consumers to firm 1 when both firms charge the price c_2. Show that $(p_1, p_2) = (c_2, c_2)$ is a Nash equilibrium and that no other pair of prices is a Nash equilibrium. b. Show that no Nash equilibrium exists if the rule for splitting up consumers when the prices are equal assigns some consumers to firm 2 when both firms charge c_2.

    EXERCISE 69.1 (Bertrand's duopoly game with fixed costs) Consider Bertrand's game under a variant of the assumptions of Section 3.2 .2 in which the cost function of each firm i is given by $C_i(q_i) = f + cq_i$ for $q_i > 0$, and $C_i(0) = 0$, where f is positive and less than the maximum of $(p - c)(\alpha - p)$ with respect to p. Denote by $\bar{p}$ the price p that satisfies $(p - c)(\alpha - p) = f$ and is less than the maximizer of $(p - c)(\alpha - p)$ (see Figure 69.1). Show that if firm 1 gets all the demand when both firms charge the same price, then $(\bar{p}, \bar{p})$ is a Nash equilibrium. Show also that no other pair of prices is a Nash equilibrium. (First consider cases in which the firms charge the same price, then cases in which they charge different prices.)

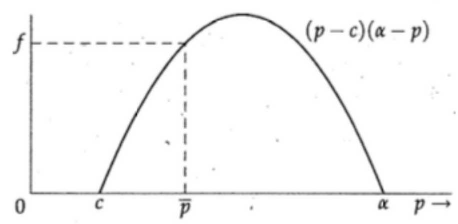

Figure 69.1 The determination of the price $\bar{p}$ in Exercise 69.1.

14. EXERCISE 73.1 (Electoral competition with asymmetric voters' preferences) Consider a variant of Hotelling's model in which voters' preferences are asymmetric. Specifically, suppose that each voter cares twice as much about policy differences to the left of her favorite position than about policy differences to the right of her favorite position. How does this affect the Nash equilibrium?

15. EXERCISE 74.1 (Electoral competition with three candidates) Consider a variant of Hotelling's model in which there are three candidates and each candidate has the option of staying out of the race, which she regards as better than losing and worse than tying for first place. Show that if less than one-third of the citizens' favorite positions are equal to the median favorite position (m), then the game has no

Nash equilibrium. Argue as follows. First, show that the game has no Nash equilibrium in which a single candidate enters the race. Second, show that in any Nash equilibrium in which more than one candidate enters, all candidates that enter tie for first place. Third, show that there is no Nash equilibrium in which two candidates enter the race. Fourth, show that there is no Nash equilibrium in which all three candidates enter the race and choose the same position. Finally, show that there is no Nash equilibrium in which all three candidates enter the race and do not all choose the same position.

16. EXERCISE 74.2 (U.S. presidential election) Consider a variant of Hotelling's model that captures features of a U.S. presidential election. Voters are divided between two states. . State 1 has more electoral college votes than does state 2. The winner is the candidate who obtains the most electoral college votes. Denote by m_i the median favorite position among the citizens of state i, for $i = 1, 2$; assume that $m_2 < m_1$. Each of two candidates chooses a single position. Each citizen votes (nonstrategically) for the candidate whose position is closest to her favorite position. The candidate who wins a majority of the votes in a state obtains all the electoral college votes of that state; if for some state the candidates obtain the same number of votes, they each obtain half of the electoral college votes of that state. Find the Nash equilibrium (equilibria?) of the strategic game that models this situation.

17. EXERCISE 75.1 (Electoral competition between candidates who care only about the winning position) Consider the variant of Hotelling's model in which the candidates (like the citizens) care about the winner's position, and not at all about winning per se. There are two candidates. Each candidate has a favorite position; her dislike for other positions increases with their distance from her favorite position. Assume that the favorite position of one candidate is less than m and the favorite position of the other candidate is greater than m. Assume also that if the candidates tie when they take the positions x_1 and x_2, then the outcome is the compromise policy $\frac{1}{2}(x_1 + x_2)$. Find the set of Nash equilibria of the strategic game that models this situation. (First consider pairs (x_1, x_2) of positions for which either $x_1 < m$ and $x_2 < m$, or $x_1 > m$ and $x_2 > m$. Next consider pairs (x_1, x_2) for which either $x_1 < m < x_2$, or $x_2 < m < x_1$, then those for which $x_1 = m$ and $x_2 \neq m$, or $x_1 \neq m$ and $x_2 = m$. Finally consider the pair (m, m).)

18. EXERCISE 75.2 (Citizen-candidates) Consider a game in which the players are the citizens. Any citizen may, at some cost $c > 0$, become a candidate. Assume that the only position a citizen can espouse is her favorite position, so that a citizen's only decision is whether to stand as a candidate. After all citizens have (simultaneously) decided whether to become candidates, each citizen votes for her favorite candidate, as in Hotelling's model. Citizens care about the position of the winning candidate; a citizen whose favorite position is x loses $|x - x^*|$ if the winning candidate's position is x^*. (For any number $z, |z|$ denotes the absolute value of z : $|z| = z$ if $z > 0$ and $|z| = -z$ if $z < 0$.) Winning confers the benefit b. Thus a citizen who becomes a candidate and ties with $k-1$ other candidates for first place obtains the payoff

$b/k - c$; a citizen with favorite position x who becomes a candidate and is not one of the candidates tied for first place obtains the payoff $-|x - x^*| - c$, where x^* is the winner's position; and a citizen with favorite position x who does not become a candidate obtains the payoff $-|x - x^*|$, where x^* is the winner's position. Assume that for every position x there is a citizen for whom x is the favorite position. Show that if $b \leq 2c$, then the game has a Nash equilibrium in which one citizen becomes a candidate. Is there an equilibrium (for any values of b and c) in which two citizens, each with favorite position m, become candidates? Is there an equilibrium in which two citizens with favorite positions different from m become candidates?

19. EXERCISE 75.3 (Electoral competition for more general preferences) Suppose that there is a finite number of positions and a finite, odd, number of voters. For any positions x and y, each voter either prefers x to y or prefers y to x. (That is, no voter regards any two positions as equally desirable:) We say that a position x^* is a Condorcet winner if for every position y different from x^*, a majority of voters prefer x^* to y.

    (a) Show that for any configuration of preferences there is at most one Condorcet winner.

    (b) Give an example in which no Condorcet winner exists. (Suppose that there are three positions (x, y, and z) and three voters. Assume that voter 1 prefers x to y to z. Construct preferences for the other two voters with the properties that one voter prefers x to y and the other prefers y to x, one prefers x to z and the other prefers z to x, and one prefers y to z and the other prefers z to y. The preferences you construct must, of course, satisfy the condition that a voter who prefers a to b, and b to c also prefers a to c, where a, b, and c are any positions.)

    (c) Consider the strategic game in which two candidates simultaneously choose positions, as in Hotelling's model. If the candidates choose different positions, each voter endorses the candidate whose position she prefers, and the candidate who receives the most votes wins. If the candidates choose the same position, they tie. Show that this game has a unique Nash equilibrium if the voters' preferences are such that there is a Condorcet winner, and has no Nash equilibrium if the voters' preferences are such that there is no Condorcet winner.

20. EXERCISE 76.1 (Competition in product characteristics) In the variant of Hotelling's model that captures competing firms' choices of product characteristics, show that when there are two firms the unique Nash equilibrium is (m, m) (both firms offer the consumers' median favorite product), and when there are three firms there is no Nash equilibrium. (Start by arguing that when there are two firms whose products differ, either firm is better off making its product more similar to that of its rival.)