Course #3.

◆ EXAMPLE 39.1 (A synergistic relationship) Two individuals are involved in a synergistic relationship. If both individuals devote more effort to the relationship, they are both better off. For any given effort of individual j, the return to individual i's effort first increases, then decreases. Specifically, an effort level is a nonnegative number, and individual i's preferences (for $i = 1, 2$) are represented by the payoff function $a_i(c + a_j - a_i)$, where a_i is i's effort level, a_j is the other individual's effort level, and $c > 0$ is a constant.

The following strategic game models this situation.

Players The two individuals.

Actions Each player's set of actions is the set of effort levels (nonnegative numbers).

Preferences Player i's preferences are represented by the payoff function $a_i(c + a_j - a_i)$, for $i = 1, 2$.

- $u_1(a_1, a_2) = a_1(c + a_2 - a_1)$
  $= ca_1 + a_2a_1 - a_1^2$
- $u_2(a_1, a_2) = a_2(c + a_1 - a_2)$
  $= \boxed{ca_2 + a_1a_2 - a_2^2}$

How to construct the best response function?

$a_1 = ?$ $\qquad$ a_2 is unknown.

a_2 will be an input to the best response function

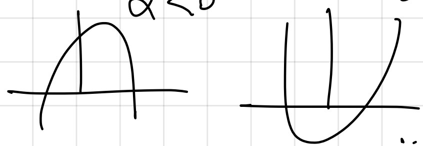
$BR_1(a_2) = a_1^*$. $\qquad$ objective = maximize u_1.

$f(x) = y$

$$\underline{u_1(a_1, a_2)} = (c + a_2)a_1 - 1a_1^2 + 0$$

$$= -1a_1^2 + (c + a_2)a_1 + 0$$

$$f(x) = \alpha x^2 + \beta x + \gamma$$



$$f'(x) = 2\alpha x + \beta = 0$$

$$\Leftrightarrow x = -\frac{\beta}{2\alpha}.$$ $\rightarrow$ max if $\underline{\alpha < 0}$.

$$f''(x) = 2\alpha < 0.$$

$$u_1' = -2Q_1 + (c + Q_2) = 0$$

$$\Leftrightarrow Q_1 = \frac{c+Q_2}{2}$$

$$BR_1(Q_2) = \frac{c+Q_2}{2} = Q_1^*$$

For player 2: $u_2(Q_1, Q_2) = -Q_2^2 + (c+Q_1)Q_2$

$$u_2' = \frac{du_2}{dQ_2} = -2Q_2 + c + Q_1 = 0$$

$$\Leftrightarrow Q_2 = \frac{c+Q_1}{2}$$

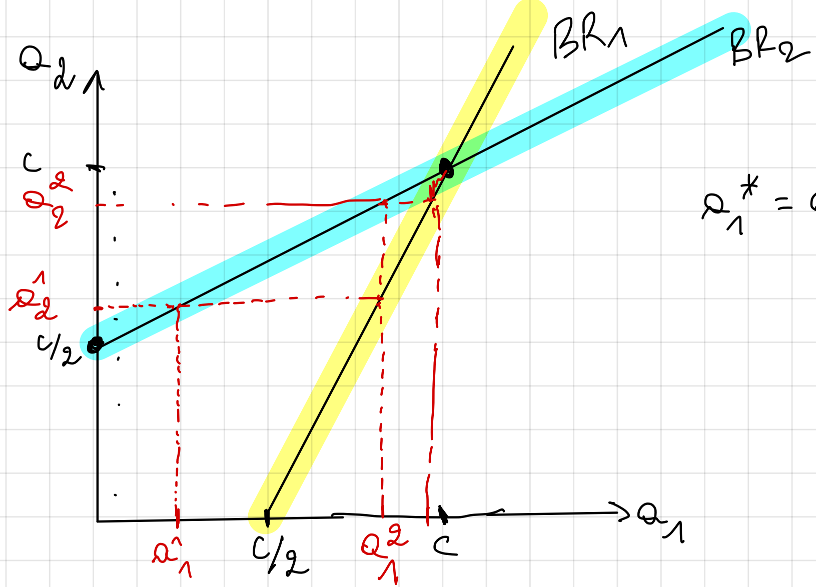
$$BR_2(Q_1) = \frac{c+Q_1}{2} = Q_2^*$$

$$\begin{cases} BR_1(Q_2^*) = Q_1^* \\ BR_2(Q_1^*) = Q_2^* \end{cases}$$

line.

$$Q_1 = BR_1 = \frac{c}{2} + \frac{Q_2}{2}$$

$$Q_2 = BR_2 = \frac{c}{2} + \frac{Q_1}{2}$$



$$Q_1^* = Q_2^* = c.$$

Analytically :

$$\begin{cases} Q_1 = \frac{c}{2} + \frac{1}{2} Q_2 & (1) \\ Q_2 = \frac{c}{2} + \frac{1}{2} Q_1 & (2) \end{cases}$$

system of linear equations

Substitute (2) in (1) : $Q_1 = \frac{c}{2} + \frac{1}{2}\left(\frac{c}{2} + \frac{1}{2} Q_1\right)$

$$\Leftrightarrow \frac{3}{4} Q_1 = \frac{3}{4} c \Leftrightarrow \boxed{Q_1 = c}.$$

$(2): \quad x_2 = \frac{c}{2} + \frac{c}{2} \iff \boxed{x_2 = c}$.

The Nash equilibrium is $(x_1^*, x_2^*) = (c, c)$.

CH 3 : Nash equilibrium - applications

- OLIGOPOLY: few firms.
- Cournot, Bertrand, Stackelberg.

competition : à la Cournot, decision = quantities
à la Bertrand, decision = price.
Stackelberg : first mover, second mover.

Cournot Model of oligopoly

Game
- players : n firms with $n \geq 2$.
  duopoly : $n = 2$
- Actions : the levels of production : $q_i \in \mathbb{R}^+$.
- payoffs : The profits π_i. $i = 1, \ldots, n$.

$\Pi_i = TR - TC$

$\underbrace{\text{total revenue}} - \underbrace{\text{total cost}}$.

We suppose identical and constant marginal cost $= c$ for $i = 1, 2$.

$TC = C(q_i) = c q_i$ for $i = 1, 2$.

if $q_i = 1$ then $C(1) = c$

if $q_i = 2$ then $C(2) = 2c$

if $q_i = 3$ then $C(3) = 3c$

marginal cost $: \Delta C(q_i) = C(2) - C(1) = 2c - c = c$

or $C(3) - C(2) = 3c - 2c = c$.

Constant marginal cost.

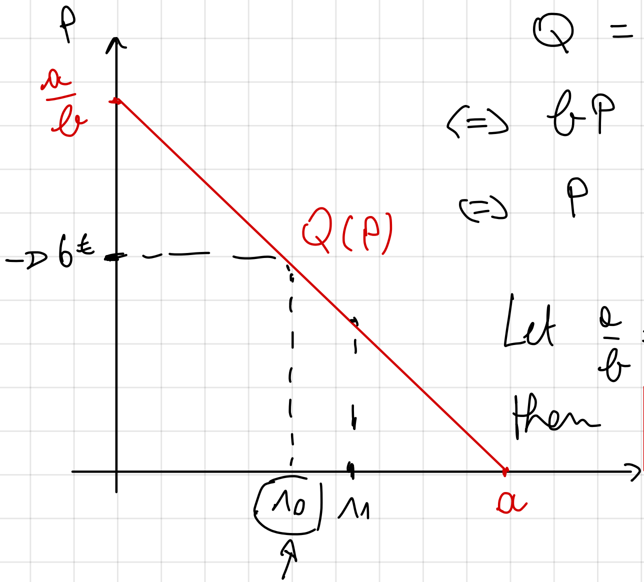
$TR = P \times Q$ Usually, consumers : demand function.

Classical demand function $Q(P) = a - bP$.

linear demand function

P is the unit price

Q is the quantity demanded.



$$Q = a - bP$$

$$\Leftrightarrow bP = a - Q$$

$$\Leftrightarrow P = \frac{a}{b} - \frac{Q}{b}$$

Let $\frac{a}{b} = \alpha$ and $\frac{1}{b} = \beta$

then $\boxed{P = \alpha - \beta Q}$ inverse demand function

$P(Q)$.

Consumers: are passive.

Players = the 2 firms.   Maximize the profit π_i

$\pi_i(q_i, q_j)$

$$\pi_i = TR - TC$$

$$= P(Q) \times q_i - c q_i \qquad \text{with } Q = q_1 + q_2$$

P56. 3.1.3. Duopoly, constant marginal cost, P(Q) linear

Nash equilibrium $i = 1,2$ $Q = q_1 + q_2$

$$P(Q) = \alpha - Q$$

$$= \alpha - q_1 - q_2.$$

$$\pi_1(q_1, q_2) = P(Q) q_1 - c q_1$$

$$= (\alpha - q_1 - q_2) q_1 - c q_1 \quad \text{if } q_1 + q_2 \leq \alpha.$$

If $q_1 + q_2 > \alpha$, Firm 1 produces nothing.

$$\pi_1(q_1, q_2) = \alpha q_1 - q_1^2 - q_2 q_1 - c q_1$$

$$= -q_1^2 + (\alpha - q_2 - c) q_1$$

α, q_2 and c are constant.

$$\frac{d\pi_1}{dq_1} = -2q_1 + (\alpha - q_2 - c) = 0$$

$$\Leftrightarrow q_1 = \frac{\alpha - q_2 - c}{2} = BR_1(q_2).$$

$$BR_1(q_2) = \begin{cases} \frac{\alpha - q_2 - c}{2} & \text{if } q_2 \leq \alpha - c. \\ 0 & \text{if } q_2 > \alpha - c \end{cases}$$

$$BR_1(q_2) = q_1^*.$$

Firm 2: $\pi_2(q_1, q_2) = P(Q) q_2 - c q_2$

$$= (\alpha - q_1 - q_2) q_2 - c q_2$$

$$= \alpha q_2 - q_1 q_2 - q_2^2 - c q_2$$

$$= -q_2^2 + (\alpha - q_1 - c) q_2$$

$$\frac{d\pi_2}{dq_2} = -2q_2 + \alpha - q_1 - c = 0$$

$$\Leftrightarrow q_2 = \frac{\alpha - q_1 - c}{2} = BR_2(q_1).$$

$$BR_2(q_1) = \begin{cases} \frac{\alpha - q_1 - c}{2} & \text{if } q_1 \leq \alpha - c \\ 0 & \text{otherwise.} \end{cases}$$

At equilibrium, we must observe

$$\boxed{q_1^* = BR_1(q_2^*) \qquad q_2^* = BR_2(q_1^*).}$$

$$\begin{cases} q_1 = \frac{1}{2}(\alpha - c) - \frac{1}{2} q_2 & (1) \\ q_2 = \frac{1}{2}(\alpha - c) - \frac{1}{2} q_1 & (2) \end{cases}$$

(2) in (1): $q_1 = \frac{1}{2}(\alpha - c) - \frac{1}{2}\left(\frac{1}{2}(\alpha - c) - \frac{1}{2} q_1\right)$

$$\Leftrightarrow \frac{3}{4} q_1 = \frac{1}{4}(\alpha - c).$$

$$\Leftrightarrow q_1^* = \frac{\alpha - c}{3}.$$

Hence $q_2^* = \frac{\alpha - c}{3}$.

Exercise 58.1 : $c_1 > c_2$.

From duopoly to oligopoly.

n firms, all identicals. Symetric game.

$$Q = \sum_{i=1}^{n} q_i \qquad P(Q) = \alpha - Q$$

$$= \alpha - q_1 - q_2 - \cdots - q_n.$$

$$\pi_1(Q) = P(Q)q_1 - cq_1.$$

$$= \alpha q_1 - q_1^2 - q_1 q_2 - q_1 q_3 \cdots - q_1 q_n - cq_1$$

α, q_{-i}, c are constant.

$$\frac{d\pi_1}{dq_1} = \boxed{\alpha - 2q_1 - q_2 - q_3 - \cdots - q_n - c = 0} \quad (1).$$

We know that the game is symetric, at equilibrium

$$q_1^* = q_2^* = \dots = q_n^* = q^*$$

Hence $Q^* = nq^*$

$$(1): \alpha - q_1 - (q_1 + q_2 + \dots + q_n) - c = 0.$$

at equilibrium :

$$\alpha - q^* - nq^* - c = 0$$

$$\Leftrightarrow \alpha - c = q^*(n+1)$$

$$\Leftrightarrow q^* = \frac{\alpha - c}{n+1}.$$

At equilibrium : $Q^* = nq^* = \dfrac{n(\alpha - c)}{n+1}$

$$\alpha - Q^* = \frac{(n+1)\alpha - n(\alpha - c)}{n+1} \qquad P(Q^*) = \frac{\alpha + nc}{n+1} = \frac{\alpha}{n+1} + \frac{nc}{n+1}$$

$$\lim_{n \to +\infty} P(Q^*) = \lim_{n \to +\infty} \frac{\alpha}{n+1} + \lim_{n \to +\infty} \frac{n}{n+1} \times c$$

$= \quad O \quad + \quad C$

course #4

Feb 19th. 2026.

$\boxed{\text{Next week}}$

Wednesday 10 - 12 AM.

BROCA Room B01.

Thursday 2-4 PM

Cencier.

3.2. Bertrand's model of oligopoly.

Duopoly $n=2$ homogeneous good.

Competition with prices

$Q_i(p_i, p_j)$ $i \neq j$ the classical demand function to firm $i = 1,2$.

$$Q_1(p_1, p_2) = \begin{cases} 0 & \text{if } p_1 > p_2 \\ \frac{1}{2} Q(p) & \text{if } p_1 = p_2 = p \\ Q(p_1) & \text{if } p_1 < p_2 . \end{cases}$$

marginal cost is constant $= c$.

Claim: only one Nash equilibrium : $p_1^* = p_2^* = c$.

- If $p_i > p_j > c$: i can increase π_i by choosing $p_i' \in (c; p_j)$ i.e : $p_j > p_i' > c$.

- If $p_i = p_j > c$: i can increase π_i by choosing $p_i' \in (c; p_j)$.

$$\pi_i = 1/2\, Q(p) p_i - c \tfrac{1}{2} Q(p).$$

$$\pi_i' = Q(p_i) p_i - c Q(p_i).$$

$$\overset{12}{p_1} > \overset{10}{p_2} = \overset{10}{c}$$

- If $p_i > p_j = c$: j can increase its $p_j' > c$ with $p_j' < p_i$ and $\pi_j' > 0$..

$\pi_i = 0$. $\pi_j = 0$

Conclusion: As soon as a price p_i is greater than c then the other firm has an incentive to underprice $(p_j < p_i)$ to increase π_j.

Let's construct the best response function for firm 1,

we need to imagine all possible p_2 :

	$p_2 < c$	$p_2 = c$	$p_2 > c$
$BR_1(p_2)$:	$p_1^* = c$ or $c > p_1^* > p_2$	$p_1^* = c$	$p_1^* = p_2 - \varepsilon$ $\varepsilon = 0{,}01$ € .

When p_2 is very large, what is the best response for firm 1.

p_1^* is the monopoly price.

Suppose that the inverse demand function is $P(Q) = a - Q$

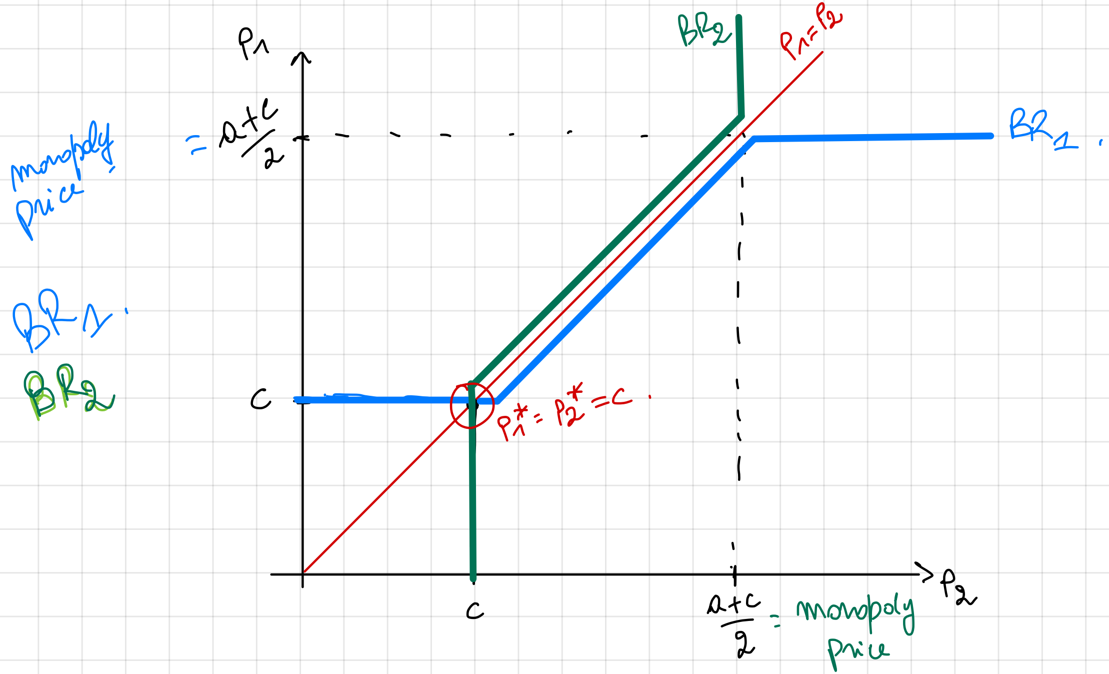
$$TR = P(Q) \times Q = (a - Q)Q = aQ - Q^2$$

$$TC = cQ$$

$$\pi = aQ - Q^2 - cQ$$

$$\frac{d\pi}{dQ} = a - 2Q - c = 0 \iff Q^* = \frac{a-c}{2}.$$

$$P(Q^*) = a - Q^* = a - \frac{a-c}{2} = \frac{a+c}{2}.$$



Cournot or Bertrand ?

Cournot : competition with quantities

depict better situations where quantities are difficult to adjust

ex: oil, agricultural. / Bertrand if quantities are easy to adjust.
→ digital.

3.3. Hotelling's model

1929

~ taste.

Horizontal differenciation.

vs. vertical " .

→ beach
→ 3 ice-cream sellers x_i.

Consumers $\sim U[0;1]$.

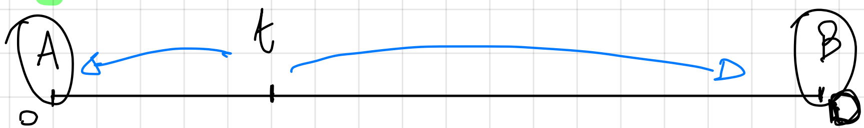
- each consumer want to buy one and only one ice cream.
- each citizen votes for only one candidate.

① Competition on location

- duopoly : A and B.
- citizen $\sim U[0;D]$ D = distance

- cost of travelling $= c$ ex: going from 0 to t will cost $c \cdot t$, with $0 \leq t \leq D$.
- consumers buy from the nearest firm.
- $p_A = p_B$.

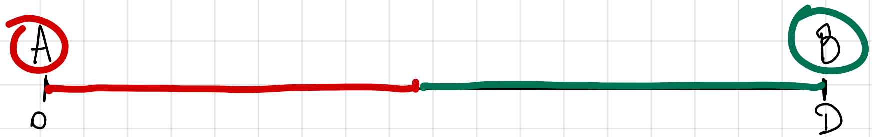
$V - ct - p_A$

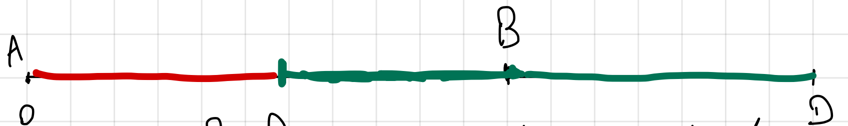


travel cost: • to A $= c \cdot t$

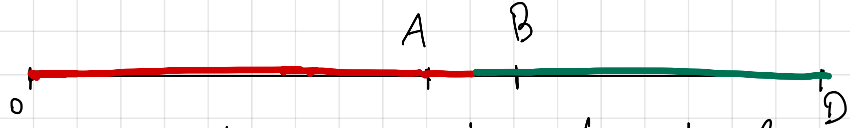
• to B $= c(D - t)$.

Because $ct < c(D-t) \rightarrow t$ buys from A.



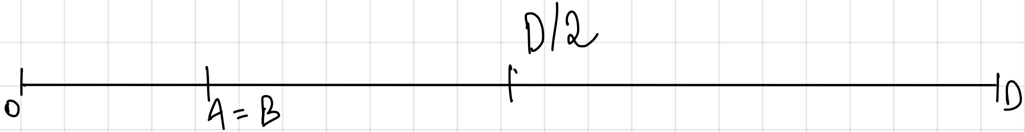


B has an incentive to get closer to A

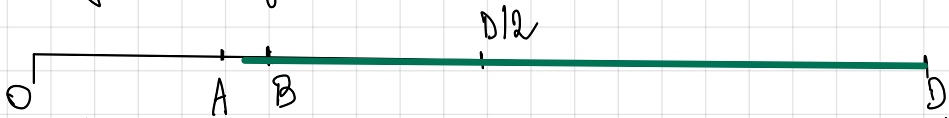


A gets an incentive to get closer to B.

- There is no Nash equilibrium where firms are located at different places.
- Is there any NE where both firms are located elsewhere than $D/2$?



They both get $D/2$ clients. It's not a NE:



By deviating, B gets more than $D/2$ clients.

The only NE is $A = B = D/2$. They both locate at $D/2$.

2 Competition on prices (fixed location)