

Production Functions:

A production function gives the technological relationship between quantities of inputs - labor, physical capital, technology and others (e.g. human capital) - and quantities of output produced:

$$Y_t = F(K_t, L_t, A_t)$$

where:

- Y_t is the total amount of production of the final good at time t ;
- K_t is the capital stock which corresponds to the quantity of "machines" (or more specifically, equipment and structures) used for production ;
- L_t is total employment which corresponds to the total number of hours worked or the number of employees, and ;
- A_t is the technology at time t , an abstract concept that captures the efficiency with which inputs like labor and capital are used in production. In practice A_t often acts as a multiplier or "shifter" in the production process.

The Cobb-Douglas Production Function: is one of the most widely used functions in macroeconomics to represent production. The Cobb-Douglas production function can be written as:

$$\begin{aligned} Y_t &= F(K_t, L_t, A_t) \\ &= A_t K_t^\alpha L_t^{1-\alpha} \quad \text{with } \alpha \in (0, 1). \end{aligned}$$

A) Conditions

Throughout the whole course, we are making a bunch of assumptions on our production function, $Y = F(K, L, A)$, that we call the "*Neoclassical Production Function*".

- F continuous and twice differentiable: [Add graphs + example bakery](#)
 - $\frac{\partial F}{\partial K} = F_K > 0$, $\frac{\partial F}{\partial L} = F_L > 0$ (positive marginal productivity);
 - $\frac{\partial^2 F}{\partial K^2} = F_{KK} < 0$, $\frac{\partial^2 F}{\partial L^2} = F_{LL} < 0$ (decreasing marginal productivity);

Intuitively, this assumption states that the level of production increases with the amount of inputs. But as the marginal products of both capital and labor are diminishing, this means that more capital for instance, holding labor and technology constant, increases output by less and less as capital grows.

- Inada conditions satisfied: [Add graphs](#)

$$\lim_{K \rightarrow +\infty} F_K = \lim_{L \rightarrow +\infty} F_L = 0, \quad \lim_{K \rightarrow 0} F_K = \lim_{L \rightarrow 0} F_L = +\infty$$

This implies that as capital or labor approaches zero, the marginal productivity becomes very high, encouraging more investment in that input. And when capital or labor becomes abundant, their marginal product approaches zero, preventing infinite accumulation. It will be clearer later but this assumption is made to ensure the existence of *interior equilibrium*.

- Constant returns to scale/homogeneous of degree one in K and L :

$$F(\lambda K, \lambda L) = \lambda \cdot F(K, L)$$

Intuitively, doubling the amounts of both capital and labor would result in doubling the level of output.

Cobb-Douglas case: The Cobb-Douglas production function is indeed continuous and twice differentiable, it has a positive marginal productivity, with:

$$F_K = \alpha A_t \left(\frac{L_t}{K_t} \right)^{1-\alpha} = \alpha \frac{Y_t}{K_t} \qquad F_L = (1-\alpha) A_t \left(\frac{K_t}{L_t} \right)^\alpha = (1-\alpha) \frac{Y_t}{L_t}$$

The Cobb-Douglas has decreasing marginal returns, with:

$$F_{KK} = \alpha(\alpha-1) A_t L_t^{1-\alpha} K_t^{\alpha-2} \qquad F_{LL} = -(1-\alpha)\alpha A_t K_t^\alpha L_t^{-\alpha-1}$$

The Cobb-Douglas also follows the Inada conditions and it has constant returns to scale:

$$F(\lambda K, \lambda L) = A_t (\lambda K_t)^\alpha (\lambda L_t)^{1-\alpha} = \lambda^{\alpha+1-\alpha} F(K, L) = \lambda F(K, L)$$

B) Returns to scale

- Returns to scale are *constant* if $\forall \lambda > 1$, scaling inputs leaves the production unchanged, i.e. $F(\lambda K, \lambda L) = \lambda F(K, L)$.
- Returns to scale are *increasing* if $\forall \lambda > 1$, scaling inputs by a factor of λ results in output growing by more than λ , i.e., $F(\lambda K, \lambda L) > \lambda F(K, L)$.
- Returns to scale are *decreasing* if $\forall \lambda > 1$, output increases by less than the proportional increase in inputs, i.e., $F(\lambda K, \lambda L) < \lambda F(K, L)$.