The Solow Growth Model

In this chapter, we introduce a simple framework that helps explain the key factors driving economic growth and differences in growth rates between countries. This model, called the Solow Growth Model, was developed by Robert Solow in 1956 and has since become foundational in understanding growth. Solow later received the Nobel Prize in Economics for his contributions.

A) The Economic Environment

We consider a closed economy with a single final good. The economy operates in discrete time over an infinite horizon, meaning time progresses in steps indexed by $t = 0, 1, 2, \ldots$. Each period can represent days, weeks, or years, depending on the context.

Agents:

There are two agents in the model: households and firms.

- **Households:** We assume that all households are identical, meaning the economy can be simplified by focusing on a *representative household*. While we don't go into too much detail about household behavior, we assume that households save a fixed proportion $s \in (0,1)$ of their income in each period.[1]
- **Firms:** Similarly, we assume all firms are identical and produce the final good using the same technology. Thus, we can analyze the economy as if there is a *single representative firm*. [2]

Factors of production:

Households own all the factors of production, meaning they supply inlastically labor to firms in exchange for wages, w_t, and rent out capital to firms at a rental rate r_t. The eocnomy begins with an initial capital stock K_0, whcih is always positive and given.

A common assumption in this model is that population grows at a constant rate n, sot that the labor force working at time $t+1$ is given by:

$$L_{t+1} = L_t(1+n) = L_0(1+n)^t$$

For now, we assume that there is no technological progress, in such a way that A_t doesn't grow over time: $A_t = A_{t+1} = A$

Regarding capital, we assume that as time passes capital depreciates, meaning that machines that are used in production lose some of their value. More specifically, at each period, a constant fraction $\delta \in (0,1)$ of the capital stock depreciates, so that out of 1 unit of capital this period, only $1-\delta$ is left in the next period. It thus follows that the law of motion of the capital stock is given by:

$$K_{t+1} = (1-\delta)K_t + I_t,$$

where I_t stands for investment at time t.

From national income accounting, in this closed economy (no international trade), total output is either consumed or invested, thus:

$$Y_t = F(K_t, L_t, A) = C_t + I_t$$

[1] Of course this assumption of constant saving rate is very questionable. In practice the saving rates of households changes with the business cycle and with the policies put in place by the government. In practice households save more during economic booms and desave during recessions. Typically, Aiyagari (95) estimates that 25% of savings in the US are for precautionary motives. Nevertheless, the exogenous saving rate is a convenient starting point, and we will later relax this assumption, especially in the Ramsey Growth model.

[2] Of course, in practice in the economy, firms are highly heterogeneous. Even within a narrowly defined sector of an economy, not two firms are identical. But making the assumption of a representative firm simplifies a lot computations.

Since this is a closed economy, we assume that the market for physical capital is in equilibrium, meaning that the supply of investment capital equals the demand for it. This means that total investment equals total savings, which is a constant fraction s of the income that households save. Therefore, we have:

$$I_t = S_t = sY_t$$

From this, we derive the *fundamental law of motion of capital* of the Solow Growth Model:

$$K_{t+1} = sF(K_t, L_t, A) + (1-\delta)K_t$$

B) Solving the model:

To move forward, we define a crucial object in our analysis: the **capital-labor ratio**, denoted as k_t, which is the amount of capital per worker:

$$k_t = \frac{K_t}{L_t}$$

Given that the production function exhibits constant returns to scale, we can express output per worker (output-labor ratio) as:

$$y_t = \frac{Y_t}{L_t} = F\left(\frac{K_t}{L_t}, 1, A\right) = f(k_t)$$

We can now rearrange the capital accumulation equation in terms of the capital-labor ratio to write the law of motion for k_t, which describes how capital per worker evolves over time and should be a constant at Steady State:

$$\begin{aligned}
& K_{t+1} = sF(K_t, L_t, A) + (1-\delta)K_t \\
\iff & \frac{K_{t+1}}{L_t} = s\frac{F(K_t, L_t, A)}{L_t} + (1-\delta)\frac{K_t}{L_t} \\
\iff & \frac{K_{t+1}}{L_{t+1}}\frac{L_{t+1}}{L_t} = k_{t+1}(1+n) = sf(k_t) + (1-\delta)k_t \\
\iff & k_{t+1} = \frac{sf(k_t) + k_t(1-\delta)}{(1+n)} \qquad (1)
\end{aligned}$$

This is the **fundamental equation of the Solow model**. It includes three main components:

- $sf(k_t)$: Represents new investment per worker.
- δ: Reflects the depreciation of capital per worker.
- n: Represents the population growth rate, or "dilution effect" — as population increases, the amount of capital available per worker decreases unless there's sufficient investment.

An equilibrium, also called steady state, occurs when $k_{t+1} = k_t = k^*$, meaning the capital-labor ratio remains constant over time. The investment required to maintain this constant ratio is known as the break-even investment and equals $(n+\delta)k_t$.

We can graph k_{t+1} as a function of k_t. For this, we just need to compute the first and second order derivative of k_{t+1} with respect to k_t to see the form of the function.

$$\frac{\partial k_{t+1}}{k_t} = \frac{sf'(k_t) + (1-\delta)}{(1+n)} > 0 \qquad\qquad \frac{\partial^2 k_{t+1}}{\partial k_t^2} = \frac{sf''(k_t)}{(1+n)} < 0$$

So k_{t+1} is increase and concave in k_t:

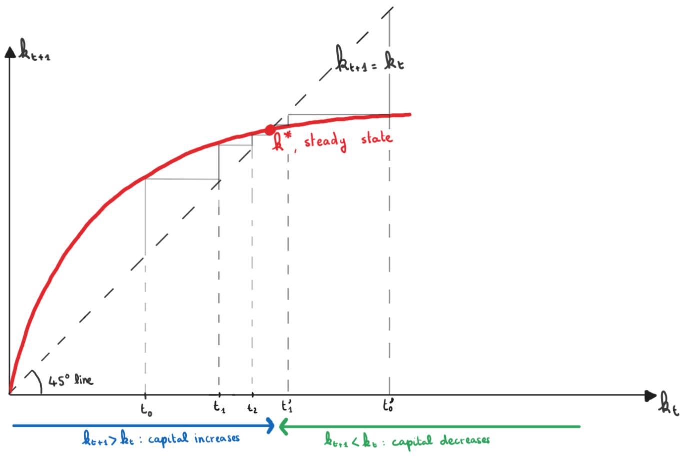

Figure 1: Dynamics of k_t

To solve for the equilibrium, we need to find the **steady-state** value of the capital per worker (or capital-labor ratio), where $k_{t+1} = k_t = k^*$. You can do this by substituting k_t and k_{t+1} with k^* in equation (1) and then isolating k^*.

In the long run, when capital per worker reaches the steady state, both capital and labor grow at the same rate n[1], and the economy follows a *Balanced Growth Path (BGP)*. From, k^*, the output per worker is given by: $y^* = f(k^*)$ and consimption per worker is: $c^* = (1 - s)f(k^*)$.

Existence and Uniqueness of the Steady State:

The *Inada Conditions* ensure that the steady-state equilibrium exists. Looking at the graph, you can see that when $k_0 < k^*$, $k_{t+1} > k_t$ caital per worker increases and is moving towards k^* from the left. Conversely, when $k_0 > k^*$, capital per worker decreases, converging to k^* from the right.

But why are the Inada conditions so crucial? These conditions guarantee that the slope of k_{t+1} is sufficiently large when k_t is near zero and gradually flattens out as k_t increases and goes to infinity. This ensures that at some point, the graph of k_{t+1} crosses the 45-degree line, allowing the economy to converge to the steady state. More specifically:

1. When k_t is very very small, $f'(k_t)$ is big, so capital per worker increases rapidly, moving towards k^*.
2. As k_t approaches k^*, the growth slows as $k_{t+1}(k_t)$ flattens out.
3. If $k_t > k^*$, the capital per worker decreases, pulling the economy back towards k^*.

In addition, the fact that $k_{t+1}(k_t)$ is concave, ensures the uniqueness of the solution, that is that the graph of k_{t+1} intersects the 45-degree line at only one point (excluding $k = 0$, which is a trivial solution).

Global Staility:

The steady state k^* is also said to be *gloablly stable*, meaning that if you shift slightly to the left or to the right of k^*, the economy converges back to k^* irrespective of the size of the shift. The economy is also said to have *no hysteresis*, meaning regardless of the initial capital per worker, k_0, the economy will always converge to k^*, except when $k_0 = 0$.

[1]Indeed, at the steady state, k^* is constant while labor grows at rate n. Since total capital $K^* = k^* \times L$, it also grows at rate n. With both capital and labor growing at the same rate n, and given the assumption of constant returns to scale in the production function, total output also grows at rate n.