The Solow Model with Technogical Progress:

In the Solow model, at some point an economy converges and remains for ever in a steady state, in which its economy doesn't grow anymore per capita. But in practice no country is stuck forever in a specific state and doesn't grow at all per capita for a long period of time. That is why, the main conclusion of the basic Solow Model is that capital accumulation alone is not enough to explain long-term economic growth. Something is missing in the equations, and this could be technological progress.

A) Types of Technological Progress:

So far, the models we've discussed have not accounted for technological progress. Now, we introduce changes in A_t which capture improvements in the technological capabilities of the economy. The key question is how to model the impact of changes in A_t on the aggregate production function. Recall that to produce we need labor and capital. The main idea here is that there exists different types of technological progress, some which will favor the use of labor over capital, others the opposite, or some do not favor more one input to another.

The first approach is *"Hicks-neutral"* technological progress, where A_t is simply a multiplicative factor applied to the production function:

$$\hat{F}(K_t, L_t, A_t) = A_t F(K_t, L_t)$$

Here technology A_t acts as a scaling factor for the entire production function. This form of progress is named after John Hicks and suggests that a change in technology affects productivity but doesn't alter the balance between capital and labor. The graph on the left shows how a reduction in productivity shifts the production function downward without changing the capital-labor ratio.

Another form is *"capital-augmented"* or *"Solow-neutral"* technological progress:

$$\hat{F}(K_t, L_t, A_t) = F(A_t K_t, L_t)$$

This type of progress implies that an increase in A_t enhances capital's effectiveness, essentially allowing the economy to produce more as if it had more capital. In the middle graph, a drop in technology leads to a larger reduction in the use of capital than labor.

Lastly, we have *"labor-augmenting"* or *"Harrod-neutral"* technological progress, named after Roy Harrod:

$$\hat{F}(K_t, L_t, A_t) = F(K_t, A_t L_t)$$

Here, an increase in A_t boosts labor's productivity, making it seem as if there is more labor available. The graph on the right shows that a drop in technology reduces by a lot labor's use while capital remains relatively less affected.

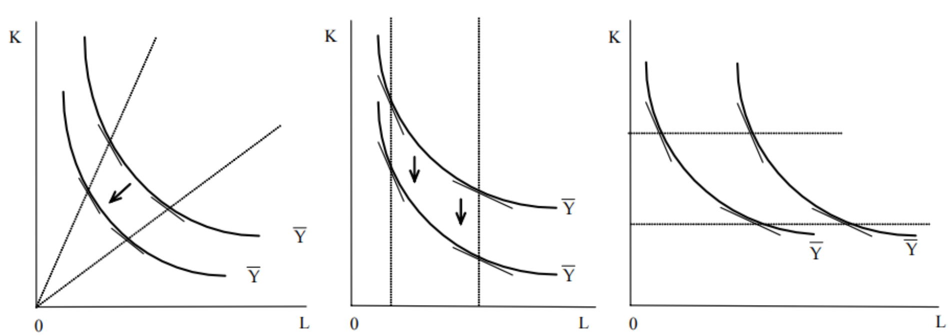

Figure: Hicks-neutral, Solow-neutral and Harrod-neutral productivity shifts

Of course, in practice, technological change can be a mixture of multiple forms of progress. For instance, A_t could be a vector containing different types of technological progress $A_t = \left(A_t^H, A_t^K, A_t^L\right)$ and the production would look like:

$$\hat{F}(K_t, L_t, A_t) = A_t^H F\left[A_t^K K_t, A_t^L L_t\right]$$

While all these types of technological progress are possible, in most cases we will deal with *"labor-augmenting"/"Harrod-neutral"* technological progress.

B) The Solow growth model with Technological Progress

Assuming purely *"labor augmenting"/"Harrod-neutral"* technological progress, that is:

$$Y_t = F(K_t, A_t L_t)$$

Similar to how we defined the capital per worker ratio earlier, $k = \frac{K_t}{L_t}$, we now introduce the concept of *effective capital-labor ratio*:

$$\hat{k}_t = \frac{K_t}{A_t L_t}$$

This leads to the *output per unit of effective labor* to be:

$$\hat{y}_t = \frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right) = f\left(\hat{k}_t\right)$$

We also assume that, similarly to population growth at rate n, technological progress occurs at a constant exogenous rate γ, such that:

$$A_{t+1} = (1+\gamma)A_t = (1+\gamma)^t A_0$$

It's important to note that if output per unit of effective labor, $\widehat{y}_t$, remains constant, total output per worker, y_t, will grow at rate γ since A_t is growing at this rate. This highlights that in models with technological progress, instead of seeking a steady state where income per capita remains constant, we should look for a *Balanced Growth Path (BGP)* where variables per unit of effective labor ($\widehat{k}_t$, $\widehat{y}_t$) stay constant.

We can now rewrite the capital accumulation equation in terms of per unit of effective labor:

$$K_{t+1} = (1-\delta)K_t + sY_t$$

Dividing by $A_t L_t$:

$$\iff \frac{K_{t+1}}{A_t L_t} = (1-\delta)\frac{K_t}{A_t L_t} + s\frac{Y_t}{A_t L_t}$$

Using that $A_{t+1} = (1+\gamma)A_t$ and $L_{t+1} = (1+n)L_t$:

$$\iff \frac{K_{t+1}}{A_{t+1}L_{t+1}}\frac{A_{t+1}L_{t+1}}{A_t L_t} = \widehat{k}_{t+1}(1+\gamma)(1+n) = (1-\delta)\widehat{k}_t + s\widehat{y}_t$$

$$\iff \widehat{k}_{t+1} = \frac{(1-\delta)\widehat{k}_t + sf\left(\widehat{k}_t\right)}{(1+\gamma)(1+n)}$$

At this point, to find the steady-state value of the capital per unit of effective labor $\widehat{k}^*$, we simply replace $\widehat{k}_t$ and $\widehat{k}_{t+1}$ by $\widehat{k}^*$ and solve for the steady state.

For exemple, take a labor augmenting production function, $Y_t = K_t^{\alpha}\left(A_t L_t\right)^{1-\alpha}$, such that $\hat{y}_t = \hat{k}_t$. We solve for the Steady State as follows:

$$
\begin{aligned}
&(1+\gamma)(1+n)\hat{k}^* - (1-\delta)\hat{k}^* = s(\hat{k}^*)^{\alpha} \\
\iff &(\hat{k}^*)^{1-\alpha} = \frac{s}{(1+\gamma)(1+n)-(1-\delta)} \\
\iff &\hat{k}^* = \left[\frac{s}{(1+\gamma)(1+n)-(1-\delta)}\right]^{\frac{1}{1-\alpha}}
\end{aligned}
$$