The Ramsey Growth Model:

In the Solow model, consumers had not much choice, the saving rate was given, although this assumption makes no sense. In reality, consumers face a trade-off between consumption today and consumption tomorrow (i.e. saving) (*for instance, if the Central Bank changes the interest rate, it will impact people's savings and firms' investment*, also we have seen previously that there exists precautionary motives in savings, meaning that consumers tend to save more during economic booms, to build-up savings in case of a bad shock, and use their savings during recessions, in order to smooth their consumption).

The conclusion of the Solow Model is that at some point there is a steady state at which the economy doesn't grow anymore per capita, so capital accumulation doesn't explain economic growth over a long period of time. What explains long term economic growth is technological progress, which enhances the level of productivity.

The conclusion of the Ramsey model is quite similar to the one of the Solow Model, except that in the Ramsey model, the saving rate is not exogenous anymore but is endogenous, that is we will need to solve the problem of the consumer to determine the optimal saving decisions at each period.

The model:

Capital Accumulation:

The equation of capital accumulation now writes in the Ramsey Model:

$$K_{t+1} = (1-\delta)K_t + \underbrace{Y_t - C_t}_{=I_t}$$

In the Solow model, we had that $I_t = sY_t$, where s is exogenously set. This is not true anymore and to determine I_t we will need to find the optimal value of C_t by solving the consumer's problem.

To simplify the explanations, we are assuming a Coob-Douglas production function, in which labor is constant and equals 1, that is: $Y_t = K_t^\alpha L^{1-\alpha} = K_t^\alpha$. This implies that $c_t = \frac{C_t}{L} = C_t$, $k_t = \frac{K_t}{L} = K_t$ and $y_t = \frac{Y_t}{L} = Y_t$.

Consumer's Problem:

In order to solve the consumer's problem, let's first define the instantaneous utility function by $u(c_t)$, which captures the utility that the representative household derives from consumption at time t. Now that this is defined, the Central Planner seeks to maximize the discounted sum of total utilities, defined as:

$$U = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where $\beta \in (0,1)$ is the discount factor. This functional form implies that the weight given to tomorrow's utility is a fraction β of today's utility, and the weight given to utility the day after tomorrow is a faction β^2 of today's utility and so on.

Solving the Model:

Solving the Model using the Lagrangian:

The Central's Planner problems consists in maximizing the discounted sum of total utilities, under the constraint that the capital accumulation equation is satisfied:

$$\max_{\{k_{t+1},c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{such that: } k_{t+1} = (1-\delta)k_t + \underbrace{y_t - c_t}_{=I_t}$$

$$c_t \geq 0, \quad k_{t+1} \geq 0 \quad \forall t \quad \text{and } k_0 \text{ is given.}$$

We can solve this problem as usual using a Lagrangian[5].

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \mu_t \left[(1-\delta)k_t + k_t^{\alpha} - k_{t+1} - c_t\right]$$

Let $\lambda_t = \frac{\mu_t}{\beta^t}$. We have:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) + \lambda_t \left[(1-\delta)k_t + k_t^{\alpha} - k_{t+1} - c_t\right]\right]$$

As we are used to, we now compute the First Order Conditions, by deriving the Lagrangian with respect to c_t and k_{t+1} and by equalizing it to zero. We get:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} = 0 &\iff \beta^t u'(c_t) - \beta^t \lambda_t = 0 \\
&\iff u'(c_t) = \lambda_t \quad (*) \\
\frac{\partial \mathcal{L}}{\partial k_{t+1}} = 0 &\iff -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left[(1-\delta) + f'(k_{t+1})\right] = 0 \\
&\iff \beta^{t+1} \lambda_{t+1} \left[(1-\delta) + f'(k_{t+1})\right] = \beta^t \lambda_t \\
&\iff \beta \lambda_{t+1} \left[(1-\delta) + f'(k_{t+1})\right] = \lambda_t \quad (**)
\end{aligned}$$

By knowing from $(*)$ that $\lambda_t = u'(c_t)$ and $\lambda_{t+1} = u'(c_{t+1})$, and plugging it in $(**)$, we get the Euler equation:

$$u'(c_t) = \beta u'(c_{t+1}) \left[(1-\delta) + f'(k_{t+1})\right]$$

The Euler equation in the Ramsey growth model represents how individuals balance consumption over time. It tells us that the marginal utility of consumption today should equal the discounted value of tomorrow's marginal utility, adjusted for the return on capital. Essentially, individuals will save and invest as long as the return on capital (after accounting for depreciation) is greater than the immediate satisfaction from consuming today. Remember that in case of perfect competition, inputs earn there marginal return, so the interest rate, which is the return for capital without depreciation, is equal to the marginal product of capital, that is $r_t = f'(k_t)$. $(1-\delta) + f'(k_t)$ is the money that I would get the next period if I were investing one unit of consumption in capital accumulation.

[5] *Mathematical refreseher on the Lagrangian Method:* *We have $f, g : \mathbb{R}^2 \to \mathbb{R}$. We aim to maximize $f(x,y)$, under the constraint that $g(x,y) = a$. To solve this, we can write the Lagrangian in the following way:*

$$\mathcal{L}(x,y,\lambda) = f(x,y) + \lambda \left[g(x,y) - a\right]$$

To find the candidates for the point at which the maximization under constraint problem is solved, we can compute the following first-order-conditions:

$$\frac{\partial \mathcal{L}}{\partial x}(x,y,\lambda) = 0 \qquad \frac{\partial \mathcal{L}}{\partial y}(x,y,\lambda) = 0 \qquad \frac{\partial \mathcal{L}}{\partial \lambda}(x,y,\lambda) = 0 \iff g(x,y) = a$$

We thus have an initial value for capital and a system of two equations:

Equation 1 $\quad \frac{u'(c_t)}{\beta u'(c_{t+1})} = \big(f'(k_{t+1}) + 1 - \delta\big)$ (Euler) $\forall t = 0, ..., T-1$ T equations, T+1 unknowns

Equation 2: $\quad c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$ (Resource Constraint) $\forall t = 0, ..., T$ T+1 equations, T+2 unknowns

It turns out that this is not enough to solve the system, currently there are multiple solutions to the problem and we need to impose an additional condition to have a unique solution, because without the initial condition k_0, we have 2 more unknowns than equations.

If time is finite in the model, meaning there exists a certain date T, above which the economy shuts down. If that is the case, then there is no incentive for people to save following that period, optimally the household should consume everything so that the economy should shut down with no capital left. Indeed, recall that our goal is to solve the Planner's problem, which seeks to maximize the intertemporal utility of the representative household, if it were that $K_{T+1} > 0$, then it would be a waste of utility, because the consumer could consume until $k_{T+1} = 0$, and this would lead to a net increase in utility. That is why, if time is finite in the model, to get a unique solution, we add a terminal condition that states that:
$k_{T+1} = 0$.

If the problem is not finite, we impose the transversality condition, that states that the discounted value of capital far away in the future goes to 0:

$$\lim_{t \to +\infty} \beta^t u'(c_t) k_{t+1} = 0$$

In words, this condition forces that the product of the marginal return of capital times the value of this state variable does not increase too fast, at a rate asymptotically faster than $1/\beta$. Economically, the transversality condition ensures that the individual would never want to have positive wealth in the very very long-run, because the discounted value of capital far away in the future goes to 0. Typically, you might give value to wealth in 5, 10 years, even 50 years, maybe even in 75 years, because you think of your children, but you don't give any value at all to your wealth in 1000 years.

Steady State:

We look for the steady state in that economy, that is the state in which per capita consumption and capital remain constant, that is: $c_t = c_{t+1} = c^*$ and $k_t = k_{t+1} = k^*$. First the Euler equation becomes at Steady State:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \big(f'(k_{t+1}) + 1 - \delta\big) \qquad \Rightarrow \qquad 1 = \beta\big(1 - \delta + f'(k)\big)$$

The steady state value of capital is thus obtained by:

$$\begin{aligned} f'(k^*) - \delta &= \frac{1}{\beta} - 1 \\ \iff f'(k^*) &= \frac{1 - \beta + \beta\delta}{\beta} \\ \iff k^* &= \left(f'\right)^{-1}\left(\frac{1 - \beta + \beta\delta}{\beta}\right) \end{aligned}$$

Once we know the steady state value of capital, k^*, we use the ressource constraint to get the steady state value of consumption, that is:

$$\begin{aligned} c^* &= f(k^*) + (1-\delta)k^* - k^* \\ \iff c^* &= f(k^*) - \delta k^* \end{aligned}$$

Transitional dynamics:

Assume for the rest of this section that the instantaneous utility is the log function, that is: $u(c_t) = \ln(c_t)$. The Euler equation becomes:

$$\frac{c_{t+1}}{c_t} = \beta\big(f'(k_{t+1}) + 1 - \delta\big)$$

In logs, we get:

$$\ln c_{t+1} - \ln c_t = \ln \beta + \ln\left(f'(k_{t+1}) + 1 - \delta\right)$$

We define a new parameter ρ, that measures the impatience of an agent. The higher it is, the more impatient is that agent. We define the following relationship between ρ and β: $\beta = \frac{1}{1+\rho}$. Now using the fact that $\ln(1 + x) \approx x$ for x close to 0[6], we have:

$$\begin{aligned}
\ln c_{t+1} - \ln c_t &= \ln(1) - \ln(1+\rho) + \ln\left(f'(k_{t+1}) + 1 - \delta\right) \\
\underbrace{\ln c_{t+1} - \ln c_t}_{\Delta \ln c_t} &= f'(k_{t+1}) - \delta - \rho
\end{aligned}$$

Now using that $\frac{\Delta c_t}{c_t} = \Delta \ln c_t$, we get:

$$\frac{\Delta c_t}{c_t} = f'(k_{t+1}) - \delta - \rho$$

Finally, we can rewrite the resource constraint as:

$$\begin{aligned}
k_{t+1} - k_t &= f(k_t) - \delta k_t - c_t \\
\Delta k_t &= f(k_t) - \delta k_t - c_t
\end{aligned}$$

We can thus determine the transition dynamics of the model, by graphing a phase diagram. Using the system of two equations:

$$\begin{aligned}
\frac{\Delta c_t}{c_t} &= f'(k_{t+1}) - \delta - \rho \\
\Delta k_t &= f(k_t) - \delta k_t - c_t
\end{aligned}$$

We are going to graph two curves, a curve in which consumption is constant, that is $\Delta c_t = 0$ and a second curve in which capital remains constant, that is $\Delta k_t = 0$.

First notice that $\Delta c_t = 0$ implies that: $f'(k_{t+1}) - \delta - \rho$. This is a straight line, because it holds for only one single value of per capita capital. And knowing that $f''(.) < 0$, if capital is above this threshold level, consumption will decrease because $f'(k_{t+1}) < \delta + \rho \Rightarrow \Delta c_t < 0$.

On the other hand, capital is constant, that is $\Delta k_t = 0$, when $c_t = f(k_t) - \delta k_t$, and capital decreases when consumption is too high (because people do not save enough): $\Delta k_t < 0 \Rightarrow c_t > f(k_t) - \delta k_t$.

We now have everything we need to graph the phase diagram of the Ramsey Growth Model. A phase diagram is simply a graphical representation of dynamic system. In our case, we are interested in the dynamics of capital and consumption. So we will on a (k, c) plane, graph the dynamics of both of these variables. In practice what we do is that we start by graphing all the points at which a given variable is stable. In a two-variable case as we are, the intersection of the two curves will be our steady state. We then dissociate each zone, and think of where a given variable is growing or decreasing.

Notice how the initial value of consumption c_0 has a huge impact on the path of the economy. Indeed, the initial value of capital stock, k_0, is an exogenous parameter of the model, but the initial value of c must be optimally chosen:

[6] Taylor expansion at order one: $f(x+h) = f(x) + (x+h-x)f'(x) + o(\varepsilon), here : ln(1+x) = ln(1) + (1+x-1)\frac{1}{1} + o(\varepsilon) \approx x$

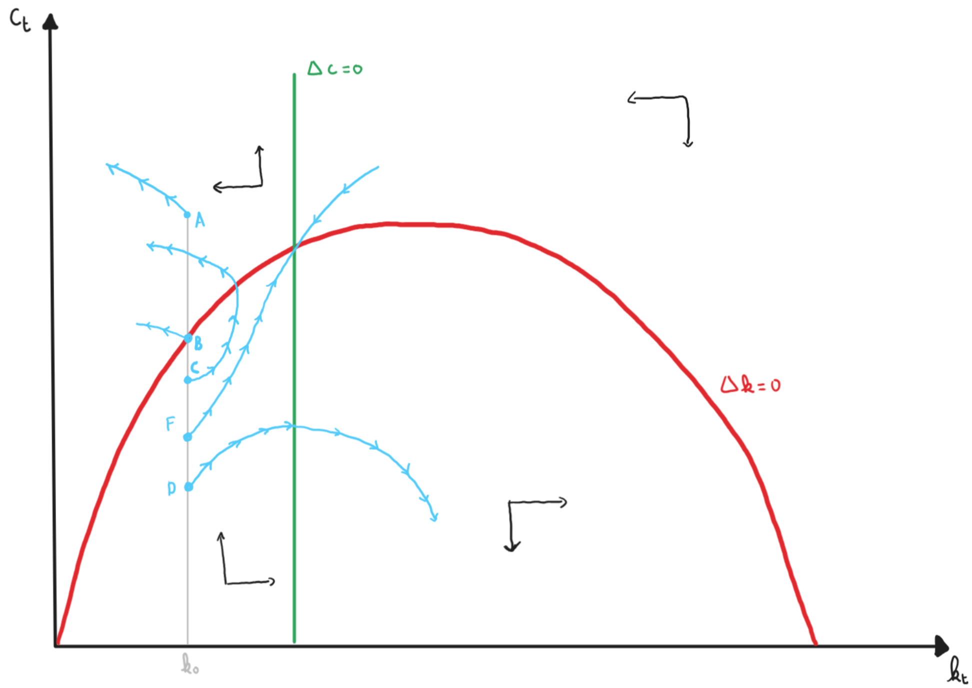

Figure: The phase diagram of the Ramsey Growth Model

- Point A: If c_0 is above the $\Delta k = 0$ curve, at point A, then $\Delta c > 0$ but $\Delta k < 0$. Thus the eocnomy mooves towards the upper left corner of the diagram.
- Point B: If c_0 is such that Δk is initially equal to 0, point B, the economy begins by moving directly up, then $\Delta c > 0$ and $\Delta k < 0$, and so the economy again moves at the upper left corner of the diagram.
- Point C: If the economy begins slightly below the $\Delta k = 0$ curve, point C, Δk is initially positive but small (since k is a continuous function of c) and $\Delta c > 0$. Thus, in this case, the economy moves up and slightly to the right initially, after it crosses the $\Delta k = 0$ curve, Δk becomes negative ad once again the economy is on a path of a rising c and a failing k, to the upper left corner.
- Point D: if the economy starts at a very low level of consumption, then $\Delta c > 0$ and $\Delta k > 0$. But Δc is proportional to c, so when c is small, Δc is small too. Thus c remains low and so the economy eventually crosses the $\Delta c = 0$ curve. After this, $\Delta c < 0$ and $\Delta k > 0$. The economy moves towards the down right corner.
- Point F: It is the only level of c_0, c_0^*, that allows the economy to converge towards the steady state of the economy, where $\Delta c = 0$ and $\Delta k = 0$. For any level of consumption above this critical level, the $\Delta k = 0$ curve is eventually crossed before the $\Delta c = 0$ curve and so the economy ends up on a path of perpetually rising consumption and failing capital (over-consumption). And if consumption is lower than this critical level, the $\Delta c = 0$ line is reached first, and so the economy embarks on a path of failing consumption and rising capital (over-accumulation). That is why, initial consumption, for a given k_0, must be precisely equal to k_0^* for the economy to converge to the Steady State.