Université Paris 1 Panthéon Sorbonne
MAEF, MMEF, QEM, IMAEF, 2023-2024

Mid Term Exam: Portfolio Theory

13th March 2024

- This is a closed book exam, no documents or electronic devices of any kind.
- The duration of the exam is **1h00**.
- It is forbidden to leave without returning the copy of your exam sheet.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions. We freely use the notations introduced in the course.

Exercice 1 (Questions on the lectures) (7 pts)

1. (2 pts) Let R be a N-dimensional random vector representing the returns of N assets. If Σ is the covariance matrix of R, prove that

$$\text{Var}(X^T R) = X^T \Sigma X.$$

Answer: $\text{Var}(X^T R) = \mathbb{E}[(X^T R - X^T \mathbb{E}[R])(X^T R - X^T \mathbb{E}[R])^T] = X^T \Sigma X$. Refer to the course for more details.

2. (2 pts) Which property of Σ ensures that the assets cannot be combined into a risk-free portfolio?

Answer: Σ must be invertible in order to ensure that the risky assets cannot be combined into a risk-free asset.

3. (1 pt) In this setting, what is the definition of an efficient portfolio?

Answer: A portfolio is said to be efficient if: there is no portfolio with the same risk and a higher return, there is no portfolio with the same expected return and a lower risk.

4. (2 pts) What optimization problem is X_{GMV} (the weights of the Global Minimum Variance portfolio) the solution to?

Answer: $X_{GMV} := \arg\min_{X \in \Delta_N} X^T \Sigma X$ where $\Delta_N := \left\{ X \in \mathbb{R}^N : \sum_{i=1}^N X_i = 1 \right\}$.

Exercice 2 (Optimal portfolios with two correlated assets) (8 pts)

We assume that the market contains only two risky assets with $\mathbb{E}[R_1] = 1$, $\mathbb{E}[R_2] = 2$, standard deviations $\sigma_1 = 1$, $\sigma_2 = 3$ and $\text{Cov}(R_1, R_2) = -\frac{3}{4}$. The vector of portfolio weight is given by $X = (x, 1-x)^T$.

1. (1 pt) Determine the covariance matrix Σ?

Answer:

$$\Sigma = \begin{pmatrix} 1 & -\frac{3}{4} \\ -\frac{3}{4} & 9 \end{pmatrix}.$$

2. (1 pt) What is the correlation coefficient between R_1 and R_2?

Answer: $\rho = \frac{Cov(R_1,R_2)}{\sigma_1 \sigma_2} = -\frac{1}{4}$.

3. (2 pts) Determine the variance of portfolio's return as a function of x, the proportion of asset 1.

Answer: With only two assets, we compute the variance

$$Var(X^T R) = x^2 \sigma_1^2 + 2Cov(R_1, R_2)x(1-x) + (1-x)^2 \sigma_2^2 = \frac{23}{2}x^2 - \frac{39}{2}x + 9$$

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4. (2 pts) Compute the vector of weights X_{GMV} of the GMV portfolio and its return. The GMV portfolio is the one that minimizes the variance which is a strongly convex function of x. We obtain $x^* = \frac{39}{46}$. The GMV portfolio is given by its weight $X_{GMV} = (\frac{39}{46}, \frac{7}{46})^T$, its return $\mu_{GMV} = x^* + 2(1 - x^*) = 2 - x^* = \frac{53}{46}$.

5. (2 pts) In this situation, what is the form of the efficient frontier in the standard deviation/mean plane?

   In this situation, the efficient frontier is an hyperbola since the variance of the return of the portfolio is a parabola as a function of the return.

Exercice 3 (12 pts)

We consider N risky assets and denote by $R = (R_1, \cdots, R_N)$ the vector of assets return. Σ is its covariance matrix which is assumed to be invertible throughout the exercise. We denote by r the return of the risk-free asset. Let $\mu \in \mathbb{R}$ and $\mathcal{L} = X^T\Sigma X + \lambda(\mu - r(1 - X^T\mathbf{1}) - \mathbb{E}[X^T R])$ for some constant λ and where $\mathbf{1}$ denotes the N-dimensional vector with all components being equal to 1.

1. (1 pt) For which optimisation problem is $\mathcal{L}$ the Lagrangian?

   Answer: $\min X^T\Sigma X$ under the constraint is $(1 - X^T\mathbf{1})r + \mathbb{E}[X^T R] = \mu$.

2. (2 pt) What is the financial interpretation of this optimisation problem?

   Answer: Minimize the risk of the portfolio under the constraint that the return of the portfolio composed of N risky assets and one risk-free asset with weight $(1, X)$ is μ.

3. (2 pts) Prove that when X is a solution to this problem, we have $X = \frac{\lambda}{2}\Sigma^{-1}(\mathbb{E}[R] - r\mathbf{1})$.

   Answer: it suffices to write $\nabla_X \mathcal{L} = 0$ to obtain the expression of X.

4. (2 pts) Prove that λ satisfies $\lambda = 2\frac{\mu - r}{f^2}$, with $f^2 = b - 2ar + cr^2$ where a, b, c are three constants to be specified.

   Answer: it suffices to write that $\nabla_\lambda \mathcal{L} = 0$ and to use the expression of X obtained in the previous question so that $\mu - r = \frac{\lambda}{2}(\mathbb{E}[R] - r\mathbf{1})^T\Sigma^{-1}(\mathbb{E}[R] - r\mathbf{1}) = \frac{\lambda}{2}f^2$.

5. (2 pts) Compute the variance of any optimal portfolio.

   Answer: One has $\mathrm{Var}(X^T R) = X^T\Sigma X = \frac{(\mu - r)^2}{f^2}$.

6. (3 pts) Prove that, in the standard deviation/mean plane, the efficient frontier is composed of one line. What are its slope and intercept? What is the GMV portfolio?

   Answer: For $\mu \geqslant r$, one has $\sigma(X^T R) = (\mu - r)/f$ so that the slope is f and intercept is r. The GMV is the risk-free asset.