Université Paris 1 Panthéon Sorbonne
MAEF, MMEF, QEM, IMAEF, 2023-2024

Mid Term Exam: Portfolio Theory

13th March 2024

- This is a closed book exam, no documents or electronic devices of any kind.
- The duration of the exam is **1h00**.
- It is forbidden to leave without returning the copy of your exam sheet.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions. We freely use the notations introduced in the course.

Exercice 1 (Questions on the lectures) (7 pts)

1. (2 pts) Let R be the N-dimensional random vector representing of the assets returns. If Σ is the covariance matrix of R, prove that
$$\text{Var}(X^T R) = X^T \Sigma X.$$
2. (2 pts) Which property of Σ ensures that the assets cannot be combined into a risk-free portfolio?
3. (1 pt) In this setting, what is the definition of an efficient portfolio?
4. (2 pts) What optimization problem is X_{GMV} (the weights of the Global Minimum Variance portfolio) the solution to?

Exercice 2 (Optimal portfolios with two correlated assets) (8 pts)

We assume that the market contains only two risky assets with $\mathbb{E}[R_1] = 1$, $\mathbb{E}[R_2] = 2$, standard deviations $\sigma_1 = 1$, $\sigma_2 = 3$ and $\text{Cov}(R_1, R_2) = -\frac{3}{4}$. The vector of portfolio weight is given by $X = (x, 1-x)^T$.

1. (1 pt) Determine the covariance matrix Σ?
2. (1 pt) What is the correlation coefficient between R_1 and R_2?
3. (2 pts) Determine the variance of portfolio's return as a function of x, where x stands for the proportion of asset 1.
4. (2 pts) Compute the vector of weights X_{GMV} of the GMV portfolio and its return.
5. (2 pts) In this situation, what is the form of the efficient frontier in the standard deviation/mean plane? Justify your answer.

Exercice 3 (Markowitz optimization problem with $N+1$ assets) (12 pts)

We consider N risky assets and denote by $R = (R_1, \cdots, R_N)$ the vector of assets return. Σ is its covariance matrix which is assumed to be invertible throughout the exercise. We denote by r the return of the risk-free asset. Let $\mu \in \mathbb{R}$ and $\mathcal{L} = X^T \Sigma X + \lambda(\mu - r(1 - X^T \mathbf{1}) - \mathbb{E}[X^T R])$ for some constant λ and where $\mathbf{1}$ denotes the N-dimensional vector with all components being equal to 1.

1. (1.5 pts) For which optimisation problem is $\mathcal{L}$ the Lagrangian?
2. (1.5 pts) What is the financial interpretation of this optimisation problem?
3. (2 pts) Prove that when X is a solution to this problem, we have $X = \frac{\lambda}{2}\Sigma^{-1}(\mathbb{E}[R] - r\mathbf{1})$.
4. (2 pts) Prove that λ satisfies $\lambda = 2\frac{\mu - r}{f^2}$, with $f^2 = b - 2ar + cr^2$ where a, b, c are three constants to be specified. Prove that $f^2 > 0$.
5. (2 pts) Compute the variance of any optimal portfolio.
6. (3 pts) Prove that, in the standard deviation/mean plane, the efficient frontier is composed of one line. What are its slope and intercept? What is the GMV portfolio?