**Université Paris 1 Panthéon Sorbonne**
**MAEF, MMEF, QEM, IMAEF, 2023-2024**

# Final Exam: Portfolio Theory

15th May 2024

- Documents, calculators and cell phone are prohibited.
- The duration of the exam is **2h00**.
- It is forbidden to leave without returning the copy of your exam.
- The exam is composed of **3 exercices** which can be treated independently. In an exercise, you can use the results from the previous questions. We freely use the notations introduced in the course.

**Exercice 1** (Questions on the lectures)

1. The market consists of $N$ risky assets and all investors are assumed to have mean-variance preferences. We denote by $R$ the random vector of the returns of the $N$ assets and by $\Sigma$ its covariance matrix which is assumed to be invertible. The weights of any portfolio is assumed to sum to 1.
   What is the definition of the GMV portfolio in this market? It is not asked to provide the explicit weights of its composition.
2. What are the four properties a risk measure $\rho$ must satisfy to be called a coherent risk measure?
3. For a given loss $L$ and confidence level $\alpha \in (0,1)$, recall the definitions of the Value-at-Risk at level $\alpha$ of $L$ denoted by $\mathrm{VaR}_\alpha(L)$ and the Expected Shortfall at level $\alpha$ of $L$ denoted by $\mathrm{ES}_\alpha(L)$.
4. Is the Value-at-Risk a coherent risk measure? Why?
5. Prove that for $\alpha \in (0,1)$, one has
$$\mathrm{VaR}_\alpha(L) \leqslant \mathrm{ES}_\alpha(L).$$
6. Prove that if $L_1 \leqslant L_2$ then
$$\mathrm{VaR}_\alpha(L_1) \leqslant \mathrm{VaR}_\alpha(L_2), \quad \alpha \in (0,1).$$
7. Provide the definition of a copula of dimension $N$.

**Exercice 2** (Markowitz portfolio optimization)

In this exercise, all investors are assumed to have mean-variance preferences. The market consists of $N$ risky assets. We denote by $R$ the random vector of the returns of the $N$ assets and by $\Sigma$ its covariance matrix. The matrix $\Sigma$ is assumed to be invertible. The weights of any portfolio is assumed to sum to 1.

We consider the minimization program

$$\min \left\{ \frac{1}{2} X^T \Sigma X - \lambda \mathbb{E}[X^T R] \right\}$$
$$\text{such that } \mathbf{1}^T X = 1 \tag{0.1}$$

where $\lambda \geqslant 0$ is a parameter and $\mathbf{1} = (1, \cdots, 1)$ with $N$ components.

1. Write the Lagrangian of the optimization program and the optimality conditions, using the notation $\gamma$ for the Lagrange multiplier associated to the constraint (0.1).
2. Show that the Lagrange multiplier satisfies $\gamma = \frac{1-\lambda a}{c}$ where $a := \mathbb{E}[R]^T \Sigma^{-1} \mathbf{1}$ and $c := \mathbf{1}^T \Sigma^{-1} \mathbf{1}$.
3. Determine the optimal portfolio weight $X(\lambda)$ as a function of $\lambda$.
4. Deduce that any solution $X(\lambda)$ can be written as a linear combination of the GMV portfolio and another portfolio $\bar{X}$:
$$X(\lambda) = \alpha \bar{X} + (1-\alpha) X_{GMV}.$$
   What is the value of $\alpha$ and the expression of the weights $\bar{X}$?

5. Using the characterization established in the course, show that the other portfolio $\bar{X}$ is also a frontier portfolio. Under which condition on $a$ is it efficient? *Hint: One can use the expression of efficient portfolio $X(\mu)$ as a function of $\mu \in \mathbb{R}$ as established in the course.*

6. Study the expected return of $X(\lambda)$, as $\lambda$ goes from 0 to $\infty$ depending on the values of the parameters $a, b$ and $c$ and explain which portfolio we buy and short sell respectively to generate the efficient frontier.

**Exercice 3** (Characterization of the Expected Shortfall)

For $x$, we let $x_+ = \max(x, 0)$. For a fixed $\alpha \in (0,1)$, we define the map $\rho : L^1(\mathbb{P}) \to \mathbb{R}$ by

$$\rho(X) = \inf_{\xi \in \mathbb{R}} \left\{ \xi + \frac{1}{1-\alpha} \mathbb{E}[(X-\xi)_+] \right\}. \tag{0.2}$$

1. Prove that the function $\xi \mapsto \xi + \frac{1}{1-\alpha}\mathbb{E}[(X-\xi)_+]$ is convex.

2. Using Jensen's inequality for the convex function $x_+$, prove that

$$\lim_{\xi \mapsto +\infty} \left\{ \xi + \frac{1}{1-\alpha} \mathbb{E}[(X-\xi)_+] \right\} = \lim_{\xi \mapsto -\infty} \left\{ \xi + \frac{1}{1-\alpha} \mathbb{E}[(X-\xi)_+] \right\} = +\infty.$$

3. Prove that if the cdf of $X$ is continuous then $\xi \mapsto \xi + \frac{1}{1-\alpha}\mathbb{E}[(X-\xi)_+]$ is continuously differentiable on $\mathbb{R}$ with a derivative given by

$$1 - \frac{1}{1-\alpha} \mathbb{E}[\mathbf{1}_{X \geqslant \xi}].$$

4. Conclude that the optimization problem (0.2) admits a solution and that any minimizer $\xi^\star$ satisfying

$$\rho(X) = \xi^\star + \frac{1}{1-\alpha} \mathbb{E}[(X-\xi^\star)_+]$$

   is a Value-at-Risk of $X$ at level $\alpha$.

5. Prove that $\rho(X) = \mathrm{ES}_\alpha(X)$.